Projective synchronization of different chaotic discrete-time neural networks with delays, based on impulsive controllers

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Abstract. In this paper, an impulsive control approach is presented for the projective synchronization of two different chaotic Hopfield-type discrete-time neural networks with delays. The global asymptotic stability of the error dynamical system is studied, using linear matrix inequalities, vector Lyapunov functions and the stability theory of impulsive systems. Simulation examples are given to illustrate the feasibility and effectiveness of the proposed approach.
Keywords: projective synchronization, impulsive control, neural network.

1 Introduction

In recent years, we have seen a rapid growth of theoretical and experimental studies on chaos synchronization due to potential applications in secure communication, information processing, pattern formation, etc. One of the chaos synchronization methods that is most often discussed is the master-slave scheme, introduced by Pecora and Carroll [1,2].

Impulsive synchronization allows the synchronization of the master and slave systems using small impulses generated by samples (called synchronizing impulses) of the state variables of the master system, only at discrete time instances. This drastically reduces the amount of information transmitted from the master system to the slave system, making this method more efficient and useful in a great number of real-life applications. After a finite period of time, the two systems behave in accordance with each other and synchronization is achieved. In other words, this is equivalent to the attractivity of the null solution of the error dynamics between the master and slave systems. Therefore, the qualitative theory of impulsive dynamical systems and impulsive control [3–5] plays a fundamental role in impulsive synchronization.

Impulsive synchronization has been applied to a number of chaos-based secure communication systems, exhibiting good performance and proving to be more robust

Received: 20 July 2011 / Accepted: 30 October 2011

ISSN 2241-0503

than continuous synchronization [6–9]. Recently, a large number of papers discussed impulsive synchronization in continuous-time neural networks [10–15] and dynamical networks [16–21], but very few refer to discrete-time networks [22–24].

Another interesting research problem is projective synchronization, characterized by a scaling factor with respect to which two systems synchronize proportionally. Applied to secure communication, this feature can be used to M-nary digital communication for achieving fast communication. Projective synchronization phenomena were first reported by Gonzalez-Miranda [25] and Mainieri and Rehacek [26], who observed that when chaotic systems exhibit invariance properties under a special type of continuous transformation, amplification and displacement of the attractor occurs. Recently, several projective synchronization results have been presented for neural networks using the adaptive approach [27], integral sliding mode controllers [28], impulsive controllers [29] and the observer-based approach [30].

In this paper, projective synchronization results based on impulsive controllers are developed for two different chaotic Hopfield-type discrete-time neural networks with delays. The main synchronization result obtained here are based on linear matrix inequalities and the vector Lyapunov function method [31]. An illustrative example is also given, along with computer simulation results, with the aim of visualizing the satisfactory control performance.

2 Main results

In the following, we denote $\mathbb{Z}_{-\tau} = \{-\tau+1, -\tau+2..., 0\}$, $\mathbb{Z}^+ = \{1,2,3,...\}$ and $\mathbb{Z}_0^+ = \{0,1,2,...\}$.

Consider the general discrete-time delayed Hopfield-type neural network in the following matrix form:

$$\begin{cases} \mathbf{x}(t) = A\mathbf{x}(t-1) + Tg(\mathbf{x}(t-\tau)) \ , \ t \in \mathbb{Z}^+ \\ \mathbf{x}(s) = \phi_1(s) \qquad\qquad\qquad\qquad , \ s \in \mathbb{Z}_{-\tau} \end{cases} \tag{1}$$

where the matrix A is a diagonal matrix, the activation function g is of the form $g(\mathbf{x}) = (g_1(x_1), g_2(x_2), ..., g_n(x_n))^T$, $g(\mathbf{0}) = \mathbf{0}$, $T = (T_{ij})_{n\times n}$ is the interconnection matrix and $\phi_1(s)$, $s \in \mathbb{Z}_{-\tau}$ represent the initial conditions. For simplicity, we choose equal time delays, $\tau \in \mathbb{Z}^+$.

Consider that system (1) is the master system, while the slave system is:

$$\begin{cases} \mathbf{y}(t) = A'\mathbf{y}(t-1) + T'g'(\mathbf{y}(t-\tau)) \ , \ t \in \mathbb{Z}^+ \\ \mathbf{y}(t_k^+) = \mathbf{y}(t_k) + J_k(\mathbf{x},\mathbf{y}) \qquad\quad , \ k \in \mathbb{Z}^+ \\ \mathbf{y}(s) = \phi_2(s) \qquad\qquad\qquad\qquad\quad , \ s \in \mathbb{Z}_{-\tau} \end{cases} \tag{2}$$

where A' is a diagonal matrix, $T' = (T'_{ij})_{n\times n}$ is the interconnection matrix, the function g' is of the form $g'(\mathbf{x}) = (g'_1(x_1), g'_2(x_2), ..., g'_n(x_n))^T$, $g'(\mathbf{0}) = \mathbf{0}$, J_k is the jump operator, $\phi_2(s)$, $s \in \mathbb{Z}_{-\tau}$ represent the initial conditions, and the impulse times

$t_k \in \mathbb{Z}^+$ are such that

$$0 = t_0 < t_1 < ... < t_k < ... < \lim_{k\to\infty} t_k = \infty$$
$$1 \le \tau \le \inf_{k\in\mathbb{Z}^+}(t_{k+1} - t_k) = \delta$$
$$\Delta = \sup_{k\in\mathbb{Z}^+}(t_{k+1} - t_k) < \infty$$

In the slave system (2), the meaning of $y(t_k^+)$ is the following: $y(t_k)$ is computed from the first equation and then replaced by $y(t_k^+)$ according to the second equation (impulse effect).

Projective synchronization is achieved between the master and slave systems (1) and (2) if and only if

$$\lim_{t\to\infty} \|\rho\mathbf{x}(t) - \mathbf{y}(t)\| = 0 \tag{3}$$

where $\rho \in \mathbb{R}^\star$ is the scaling factor which characterizes the projective synchronization.

In what follows, we will consider that the jump operator J_k takes the form

$$J_k(\mathbf{x}, \mathbf{y}) = A_k(\rho\mathbf{x}(t_k) - \mathbf{y}(t_k))$$

where $A_k \in \mathbb{R}^{n\times n}$.

Let $\mathbf{e} = \rho\mathbf{x} - \mathbf{y}$ be the error of the systems (1) and (2). It satisfies:

$$\begin{cases} \mathbf{e}(t) = A\mathbf{e}(t-1) + (A - A')\mathbf{y}(t-1) + \\ \quad +\Phi_\rho(\mathbf{x}(t-\tau), \mathbf{y}(t-\tau)) & , t \in \mathbb{Z}^+ \\ \mathbf{e}(t_k^+) = (I - A_k)\mathbf{e}(t_k) & , k \in \mathbb{Z}^+ \\ \mathbf{e}(s) = \rho\phi_1(s) - \phi_2(s) = \phi_\rho(s) & , t \in \mathbb{Z}_{-\tau} \end{cases} \tag{4}$$

where $\Phi_\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\Phi_\rho(\mathbf{x}, \mathbf{y}) = \rho T g(\mathbf{x}) - T' g'(\mathbf{y}).$$

In the following, consider $h_i : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$h_i(x_i, y_i) = \begin{cases} \dfrac{g_i(x_i) - g_i(y_i)}{x_i - y_i} & \text{if } x_i \neq y_i \\ 0 & \text{if } x_i = y_i \end{cases}$$

Let be the matrix function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ by

$$H(\mathbf{x}, \mathbf{y}) = \text{diag}(h_1(x_1, y_1), h_2(x_2, y_2), ..., h_n(x_n, y_n)).$$

Since $g(\mathbf{x}) = (g_1(x_1), g_2(x_2), ..., g_n(x_n))^T$, it follows that

$$g(\mathbf{x}) - g(\mathbf{y}) = H(\mathbf{x}, \mathbf{y})(\mathbf{x} - \mathbf{y}).$$

On the other hand, let $k_i : \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$k_i(y_i) = \begin{cases} \dfrac{g_i(y_i)}{y_i} & \text{if } y_i \neq 0 \\ 0 & \text{if } y_i = 0 \end{cases}$$

Considering the matrix function $K : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ given by

$$K(\mathbf{y}) = \text{diag}(k_1(y_1), k_2(y_2), ..., k_n(y_n)),$$

it follows that

$$g(\mathbf{y}) = K(\mathbf{y})\mathbf{y}.$$

In the same way, we can express:

$$g'(\mathbf{y}) = K'(\mathbf{y})\mathbf{y}.$$

Therefore we obtain

$$\begin{aligned}\Phi_\rho(\mathbf{x},\mathbf{y}) &= \rho T\left[g(\mathbf{x}) - g\left(\rho^{-1}\mathbf{y}\right)\right] + \rho T g\left(\rho^{-1}\mathbf{y}\right) - T'g'(\mathbf{y}) \\ &= \rho T H\left(\mathbf{x},\rho^{-1}\mathbf{y}\right)\left(\mathbf{x}-\rho^{-1}\mathbf{y}\right) + TK(\rho^{-1}\mathbf{y})\mathbf{y} - T'K'(\mathbf{y})\mathbf{y} \\ &= TH(\mathbf{x},\rho^{-1}\mathbf{y})\mathbf{e} + \left[TK(\rho^{-1}\mathbf{y}) - T'K'(\mathbf{y})\right]\mathbf{y} \\ &= \Psi_\rho(\mathbf{x},\mathbf{y})\mathbf{e} + \Omega_\rho(\mathbf{y})\mathbf{y}, \end{aligned} \tag{5}$$

where $\Psi_\rho(\mathbf{x},\mathbf{y}) = TH(\mathbf{x},\rho^{-1}\mathbf{y})$ and $\Omega_\rho(\mathbf{y}) = TK(\rho^{-1}\mathbf{y}) - T'K'(\mathbf{y})$.

In what follows, we will use the following notations:

- For $u, v \in \mathbb{R}$, $\mathbf{c} = [c_1, c_2, ..., c_\tau]^T \in (0,\infty)^\tau$, we define $Q(u; v; \mathbf{c}) \in \mathbb{R}^{\tau\times\tau}$ as

$$Q(u;v;\mathbf{c}) = \begin{cases} u+v & ,\text{if } \tau = 1 \\ \begin{bmatrix} u & 0 & 0 & \dots & 0 & v\frac{c_1}{c_\tau} \\ \frac{c_2}{c_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{c_3}{c_2} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{c_\tau}{c_{\tau-1}} & 0 \end{bmatrix} & ,\text{if } \tau \geq 2 \end{cases} \tag{6}$$

- $\|\cdot\|_\tau$ the maximum norm defined on $\mathbb{R}^\tau$:

$$\|\mathbf{z}\|_\tau = \max\{|z_1|, |z_2|, ..., |z_\tau|\}$$

- $|||\cdot|||_\tau$ the matrix norm induced by $\|\cdot\|_\tau$:

$$|||M|||_\tau = \max_{1\leq i\leq\tau}\sum_{j=1}^{\tau}|m_{ij}|$$

The following theorem represents the main impulsive projective synchronization result for discrete-time delayed Hopfield neural networks.

Theorem 1. *Let be a positive-definite matrix* P*, the dot product* $\langle\mathbf{x},\mathbf{y}\rangle = \mathbf{x}^T P\mathbf{y}$ *and the corresponding vector norm* $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T P\mathbf{x}}$*. Assume that:*
(i) There exist positive constants α_1*,* α_2 *such that the matrix*

$$M(\mathbf{x},\mathbf{y}) = \begin{bmatrix} APA-\alpha_1 P & AP\Psi & AP(A-A') & AP\Omega \\ \Psi^T PA & \Psi^T P\Psi - \alpha_2 P & \Psi^T P(A-A') & \Psi^T P\Omega \\ (A-A')PA & (A-A')P\Psi & (A-A')P(A-A') & (A-A')P\Omega \\ \Omega^T PA & \Omega^T P\Psi & \Omega^T P(A-A') & \Omega^T P\Omega \end{bmatrix} \tag{7}$$

is negative semi-definite, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
(ii) For any $k \in \mathbb{Z}^+$*, there exist positive constants* β_k *such that the matrix*

$$M(k) = (I - A_k)^T P (I - A_k) - \beta_k P \tag{8}$$

is negative semi-definite.
(iii) There exists $\mathbf{c} = [c_1, c_2, ..., c_\tau] \in (0, \infty)^\tau$ *such that, defining the matrices* $Q_k \in \mathbb{R}^{\tau \times \tau}$ *(*$k \in \mathbb{Z}$*) as*

$$Q_k = Q(\beta_k \alpha_1; \beta_k \alpha_2; \mathbf{c}) Q(\alpha_1; \alpha_2; \mathbf{c})^{t_k - t_{k-1} - 1} \tag{9}$$

with $Q(u; v; \mathbf{c})$ *given by (6),* α_1*,* α_2*, and* β_k *given by (i) and (ii), the following holds:*

$$\lim_{k \to \infty} \left(\prod_{i=1}^{k} Q_i \right) \mathbf{c} = 0. \tag{10}$$

Then the null solution of system (4) is globally attractive, i.e. the master system (1) and the slave system (2) are globally synchronized with scaling factor ρ.

Proof. Let $\mathbf{e}(t) = \mathbf{e}(t; t_0, \phi_\rho)$ be the solution of (4) with the initial condition $\phi_\rho(s)$, $s \in \mathbb{Z}_{-\tau}$. Consider the functional $u(t) = \|\mathbf{e}(t)\|^2 = \mathbf{e}(t)^T P \mathbf{e}(t)$.

Using Ψ and Ω to denote $\Psi(\mathbf{x}(t-\tau), \mathbf{y}(t-\tau))$ and $\Omega(\mathbf{x}(t-\tau), \mathbf{y}(t-\tau))$, and $B = A - A'$, from (i) we obtain:

$$\begin{aligned}
u(t) = & \begin{bmatrix} \mathbf{e}(t-1) \\ \mathbf{e}(t-\tau) \\ \mathbf{y}(t-1) \\ \mathbf{y}(t-\tau) \end{bmatrix}^T \begin{bmatrix} APA & AP\Psi & APB & AP\Omega \\ \Psi^T PA & \Psi^T P\Psi & \Psi^T PB & \Psi^T P\Omega \\ BPA & BP\Psi & BPB & BP\Omega \\ \Omega^T PA & \Omega^T P\Psi & \Omega^T PB & \Omega^T P\Omega \end{bmatrix} \begin{bmatrix} \mathbf{e}(t-1) \\ \mathbf{e}(t-\tau) \\ \mathbf{y}(t-1) \\ \mathbf{y}(t-\tau) \end{bmatrix} \\
\leq & \begin{bmatrix} \mathbf{e}(t-1) \\ \mathbf{e}(t-\tau) \\ \mathbf{y}(t-1) \\ \mathbf{y}(t-\tau) \end{bmatrix}^T \begin{bmatrix} \alpha_1 P & 0 & 0 & 0 \\ 0 & \alpha_2 P & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}(t-1) \\ \mathbf{e}(t-\tau) \\ \mathbf{y}(t-1) \\ \mathbf{y}(t-\tau) \end{bmatrix} \\
= & \alpha_1 \mathbf{e}(t-1)^T P \mathbf{e}(t-1) + \alpha_2 \mathbf{e}(t-\tau)^T P \mathbf{e}(t-\tau) \\
= & \alpha_1 u(t-1) + \alpha_2 u(t-\tau)
\end{aligned}$$

Moreover, using (ii) we have:

$$\begin{aligned}
u(t_k^+) &= \|\mathbf{e}(t_k^+)\|^2 = \|(I - A_k)\mathbf{e}(t_k)\|^2 = \mathbf{e}(t_k)^T (I - A_k)^T P (I - A_k) \mathbf{e}(t_k) \\
&\leq \beta_k \mathbf{e}(t_k)^T P \mathbf{e}(t_k) = \beta_k u(t_k)
\end{aligned}$$

Hence, $u(t)$ satisfies the following inequalities:

$$\begin{cases} u(t) \leq \alpha_1 u(t-1) + \alpha_2 u(t-\tau) \ , & t \in \mathbb{Z}^+ \\ u(t_k^+) \leq \beta_k u(t_k) & , \ k \in \mathbb{Z}^+ \end{cases} \tag{11}$$

In the rest of the proof, we will rely on the fact that if Q is a matrix with positive elements, the linear mapping $w \mapsto Qw$ is non-decreasing, i.e. for any two vectors

w_1, w_2 satisfying $w_1 \leq w_2$ we have $Qw_1 \leq Qw_2$ (the vector inequalities are considered component-wise).

We define the vector Lyapunov function $V : \mathbb{Z}_0^+ \to \mathbb{R}^\tau$

$$V(t) = \left[\, c_1 u(t) \; c_2 u(t-1) \; ... \; c_\tau u(t-\tau+1) \,\right]^T \tag{12}$$

From the first inequality from (11) it is easy to see that

$$V(t) \leq Q(\alpha_1; \alpha_2; \mathbf{c}) V(t-1) \quad , \forall t \in \mathbb{Z}^+$$

Hence:

$$V(t) \leq Q(\alpha_1; \alpha_2; \mathbf{c})^{t-t_k} V(t_k^+) \quad , \forall t \in (t_k, t_{k+1}] \cap \mathbb{Z} \tag{13}$$

The two inequalities from (11) imply

$$u(t_k^+) \leq \beta_k \alpha_1 u(t_k - 1) + \beta_k \alpha_2 u(t_k - \tau)$$

which leads to

$$V(t_k^+) \leq Q(\beta_k \alpha_1; \beta_k \alpha_2; \mathbf{c}) V(t_k - 1) \quad , \forall k \in \mathbb{Z}^+$$

From (13) we obtain

$$\begin{aligned} V(t_k^+) \leq & Q(\beta_k \alpha_1; \beta_k \alpha_2; \mathbf{c}) Q(\alpha_1; \alpha_2; \mathbf{c})^{t_k - t_{k-1} - 1} V(t_{k-1}^+) \\ = & Q_k V(t_{k-1}^+) \quad , \forall k \in \mathbb{Z}^+ \end{aligned}$$

and hence

$$V(t_k^+) \leq \prod_{i=1}^{k} Q_i V(0) \quad , \forall k \in \mathbb{Z}^+ \tag{14}$$

We have

$$\begin{aligned} V(0) &= \left[\, c_1 u(0) \; c_2 u(-1) \; ... \; c_\tau u(-\tau+1) \,\right]^T \\ &\leq \max_{s \in \mathbb{Z}_{-\tau}} u(s) \, \mathbf{c} = \max_{s \in \mathbb{Z}_{-\tau}} \|\mathbf{e}(s)\|^2 \, \mathbf{c} = \max_{s \in \mathbb{Z}_{-\tau}} \|\phi_\rho(s)\|^2 \, \mathbf{c} \end{aligned}$$

From inequality (14) we obtain

$$V(t_k^+) \leq \max_{s \in \mathbb{Z}_{-\tau}} \|\phi_\rho(s)\|^2 \left(\prod_{i=1}^{k} Q_i \right) \mathbf{c}$$

Taking into account (iii), we obtain

$$\lim_{k \to \infty} V(t_k^+) = 0$$

From (13) we have

$$\begin{aligned} \|V(t)\|_\tau \leq & \|Q(\alpha_1; \alpha_2; \mathbf{c})^{t-t_k} V(t_k^+)\|_\tau \\ \leq & |||Q(\alpha_1; \alpha_2; \mathbf{c})^{t-t_k}|||_\tau \|V(t_k^+)\|_\tau \\ \leq & |||Q(\alpha_1; \alpha_2; \mathbf{c})|||_\tau^{t-t_k} \|V(t_k^+)\|_\tau \\ \leq & |||Q(\alpha_1; \alpha_2; \mathbf{c})|||_\tau^{\Delta} \|V(t_k^+)\|_\tau \quad , \forall t \in (t_k, t_{k+1}) \cap \mathbb{Z} \end{aligned}$$

which implies that $\lim_{t\to\infty} V(t) = 0$. As

$$\|\mathbf{e}(t)\|^2 = u(t) \le \frac{1}{c_1}\|V(t)\|_\tau$$

it follows that $\mathbf{e}(t)$ tends to 0 as t tends to infinity, which completes the proof. □

The Schur complement [32] is a useful tool for establishing wether a matrix is positive (or negative) (semi-)definite. We have the following result.

Proposition 1 (See [32].). *Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix partitioned as*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$$

in which M_{11} is square and nonsingular. Let

$$M/M_{11} = M_{22} - M_{12}^T M_{11}^{-1} M_{12}$$

denote the Schur complement of M_{11}. Then:
(a) $M > 0$ if and only if $M_{11} > 0$ and $M/M_{11} > 0$;
(b) $M \ge 0$ if and only if $M_{11} > 0$ and $M/M_{11} \ge 0$.
where " > 0" means positive definite and " ≥ 0" means positive semi-definite.

Remark 1.
(i) If $T'g'(\mathbf{y}) = \rho T g(\rho^{-1}\mathbf{y})$, it follows that $\Omega_\rho(\mathbf{y}) = \mathbf{0}$. Moreover, if $A = A'$, we can easily see that the matrix $M(\mathbf{x}, \mathbf{y})$ given by (7) becomes:

$$M(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} APA - \alpha_1 P & AP\Psi & \mathbf{0} & \mathbf{0} \\ \Psi^T PA & \Psi^T P\Psi - \alpha_2 P & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Therefore, the Schur complement $M/M_{11} = \mathbf{0}$, and the matrix $M(\mathbf{x}, \mathbf{y})$ is negative semi-definite if and only if

$$M_{11}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} APA - \alpha_1 P & AP\Psi \\ \Psi^T PA & \Psi^T P\Psi - \alpha_2 P \end{bmatrix} < 0$$

(ii) More, if $A = aI$, with $a \in (0, 1)$, the matrix $M_{11}(\mathbf{x}, \mathbf{y})$ given above becomes:

$$M_{11}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} (a^2 - \alpha_1)P & aP\Psi(\mathbf{x}, \mathbf{y}) \\ a\Psi(\mathbf{x}, \mathbf{y})^T P & \Psi(\mathbf{x}, \mathbf{y})^T P\Psi(\mathbf{x}, \mathbf{y}) - \alpha_2 P \end{bmatrix}$$

Since $P > 0$, we have that $(M_{11})_{11} = (a^2 - \alpha_1)P < 0$ if and only if $\alpha_1 > a^2$.

The Schur complement of $(M_{11})_{11}$ is

$$M_{11}/(M_{11})_{11} = \frac{\alpha_1}{\alpha_1 - a^2}\Psi(\mathbf{x}, \mathbf{y})^T P\Psi(\mathbf{x}, \mathbf{y}) - \alpha_2 P$$

and taking into account that $\Psi(\mathbf{x}, \mathbf{y}) = TH(\mathbf{x}, \mathbf{y})$ we obtain

$$M_{11}/(M_{11})_{11} = \frac{\alpha_1}{\alpha_1 - a^2}H(\mathbf{x}, \mathbf{y})T^T PTH(\mathbf{x}, \mathbf{y}) - \alpha_2 P$$

Assuming that the activation functions g_i are Lipschitz continuous, it follows that there exists $L > 0$ such that $|h_i(x, y)| \leq L$ for any $i = \overline{1, n}$. Therefore, we obtain that

$$M_{11}/(M_{11})_{11} \leq \frac{\alpha_1 L^2}{\alpha_1 - a^2} T^T PT - \alpha_2 P$$

If the right hand side of this inequality is negative definite, it follows that $M_{11}/(M_{11})_{11}$ is negative definite as well.

Remark 2. If $A_k = a_k I$, $a_k \in \mathbb{R}$, the matrix $M(k)$ given by (8) becomes:

$$M(k) = [(1 - a_k)^2 - \beta_k]P$$

and it is negative semi-definite if and only if $(1 - a_k)^2 \leq \beta_k$.

Remark 3. Condition (iii) of Theorem 1 means that the solution of the linear system

$$\mathbf{z}_{k+1} = Q_k \mathbf{z}_k \qquad (\text{in } \mathbb{R}^\tau) \tag{15}$$

with the initial condition $\mathbf{z}_0 = \mathbf{c}$, converges to 0.

If for any $k \in \mathbb{Z}$ we have $\beta_k = \beta$, the matrices Q_k defined by (9) become:

$$Q_k = Q(\beta\alpha_1; \beta\alpha_2; \mathbf{c}) Q(\alpha_1; \alpha_2; \mathbf{c})^{t_k - t_{k-1} - 1}$$

Since $\delta \leq t_k - t_{k-1} \leq \Delta$, it follows that for any $k \in \mathbb{Z}$, the matrix Q_k belongs to the finite set of matrices

$$\mathcal{Q} = \{Q(\beta\alpha_1; \beta\alpha_2; \mathbf{c}) Q(\alpha_1; \alpha_2; \mathbf{c})^{m-1},\ m \in \{\delta, \delta + 1, ..., \Delta\}\}.$$

Therefore, the system (15) is a switching system.

Moreover, if $t_k - t_{k-1} = \delta = \Delta$ for any $k \in \mathbb{Z}$, then the system (15) reduces to an autonomous linear system, with the matrix $Q = Q(\beta\alpha_1; \beta\alpha_2; \mathbf{c}) Q(\alpha_1; \alpha_2; \mathbf{c})^{\Delta - 1}$. In this case, condition (iii) of Theorem 1 is equivalent to $\mathbf{c}$ belonging to the stable eigenspace of the matrix Q.

In the following, based on the results obtained in Theorem 1, sufficient conditions for the impulsive projective synchronization of non-delayed neural networks will be given. With this aim, we will consider $\tau = 1$ in systems (1), (2) and (4).

Proposition 2. *Assume that $\tau = 1$. Let be a positive-definite matrix P, the dot product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T P \mathbf{y}$ and the corresponding vector norm $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T P \mathbf{x}}$. Assume that:*
(i) there exist positive constants α_1, α_2 and β_k such that conditions (i) and (ii) of Theorem 1 holds.
(ii) there exists $\epsilon \in (0, 1)$ such that

$$\beta_k(\alpha_1 + \alpha_2)^{t_k - t_{k-1}} \leq \epsilon \quad , \forall k \geq \tilde{k}.$$

Then the null solution of system (4) is globally attractive, i.e. the master system (1) and the slave system (2) are globally synchronized with scaling factor ρ.

Proof. To verify that condition (iii) of Theorem 1 is also fulfilled, it is enough to notice that since $\tau = 1$, we have:

$$Q_k = \beta_k(\alpha_1 + \alpha_2)^{t_k - t_{k-1}} \leq \epsilon \quad , \forall k \geq \tilde{k}.$$

Hence for any $k \geq \tilde{k}$ we have

$$\prod_{i=1}^{k} Q_i \leq \epsilon^{k-\tilde{k}} \prod_{i=1}^{\tilde{k}} Q_i \overset{k\to\infty}{\longrightarrow} 0$$

and based on Theorem 1, the proof is complete. □

3 Example

We consider the following master system

$$\begin{cases} x_1(t) = a\,x_1(t-1) + T\tanh(x_3(t-\tau)) \\ x_2(t) = a\,x_2(t-1) + T\sin(x_1(t-\tau)) \\ x_3(t) = a\,x_3(t-1) + T\tanh(x_2(t-\tau)) \end{cases} \quad , \; t \in \mathbb{Z}^+ \tag{16}$$

where $a = 0.5$ and $\tau = 3$.

Based on the theoretical results obtained in [33], it can be shown that the null solution of (16) is asymptotically stable if and only if $T \in (-0.534, 0.5)$. At $T = 0.5$ a Cusp bifurcation takes place in system (16) and at $T = -0.534$ a supercritical Neirmark-Sacker bifurcation occurs. If $|T|$ is sufficiently large, the system (16) will exhibit chaotic behavior.

The Lyapunov characteristic exponents for $T \in (-2, 0)$ are presented in Fig. 1. The collision of the two largest Lyapunov exponents for $T = -0.534$ corresponds to the supercritical Neimark-Sacker bifurcation. For $T \in (-1.52, -0.534)$, the largest Lyapunov exponent is null, which corresponds to the existence of an asymptotically stable limit cycle. As T decreases bellow -1.52, the largest Lyapunov exponent becomes positive, suggesting chaotic behavior in system (16).

For example, for $T = -1.7$ we obtain that the two largest Lyapunov exponents are positive: 0.03 and 0.002 respectively. Hence, the system (16) is hyperchaotic in a neighborhood of the null solution. Indeed, considering the initial conditions:

$$\begin{aligned} \mathbf{x}(-2) &= (-2.86584, 0.438859, 1.73719) \\ \mathbf{x}(-1) &= (-3.12197, 1.91191, 0.620975) \\ \mathbf{x}(0) &= (-3.24297, 2.1437, 1.88233) \end{aligned} \tag{17}$$

the trajectory of (16) is shown in Fig. 2.

The slave system that we consider for projective synchronization is

$$\begin{cases} y_1(t) = a\,y_1(t-1) + T\rho\tanh(\rho^{-1}y_3(t-\tau)) \\ y_2(t) = a\,y_2(t-1) + T\rho\sin(\rho^{-1}y_1(t-\tau)) \quad , \; t \in \mathbb{Z}^+ \\ y_3(t) = a\,y_3(t-1) + T\rho\tanh(\rho^{-1}y_2(t-\tau)) \\ \mathbf{y}(t_k^+) = \mathbf{y}(t_k) + a_k(\rho\mathbf{x}(t_k) - \mathbf{y}(t_k)) \quad , \; k \in \mathbb{Z}^+ \end{cases} \tag{18}$$

with $\rho = 0.5$, $a_k = 0.7$ and $t_k = 8k$.

Null initial conditions have been considered for the slave system. Figures 3-5 show that projective synchronization is achieved with scaling factor $\rho = 0.5$.

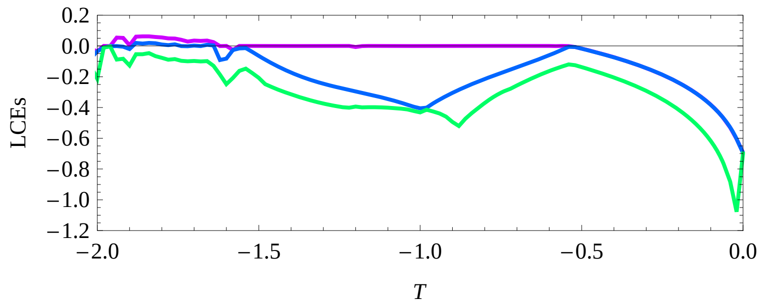

Fig. 1. Lyapunov characteristic exponents for $T \in (-2, 0)$.

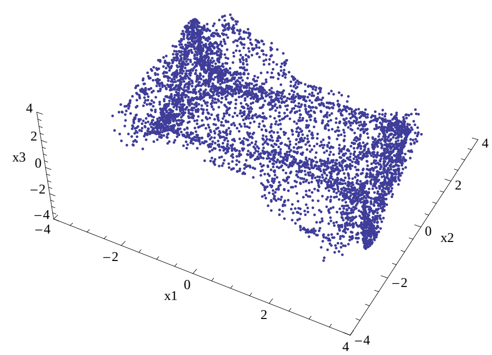

Fig. 2. Trajectory of the master system (16) with $T = -1.7$ and initial conditions (17) (5000 iterations have been plotted).

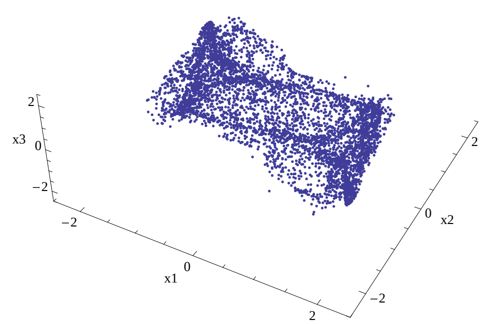

Fig. 3. Trajectory of the slave system (18) with $T = -1.7$ and null initial conditions (5000 iterations have been plotted).

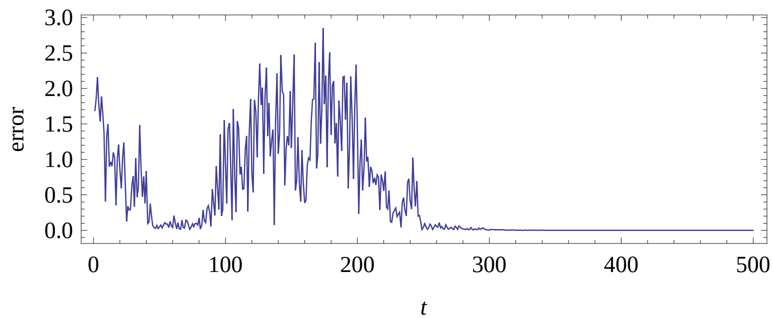

Fig. 4. Norm of the synchronization error $\mathbf{e}(t) = \rho\mathbf{x}(t) - \mathbf{y}(t)$.

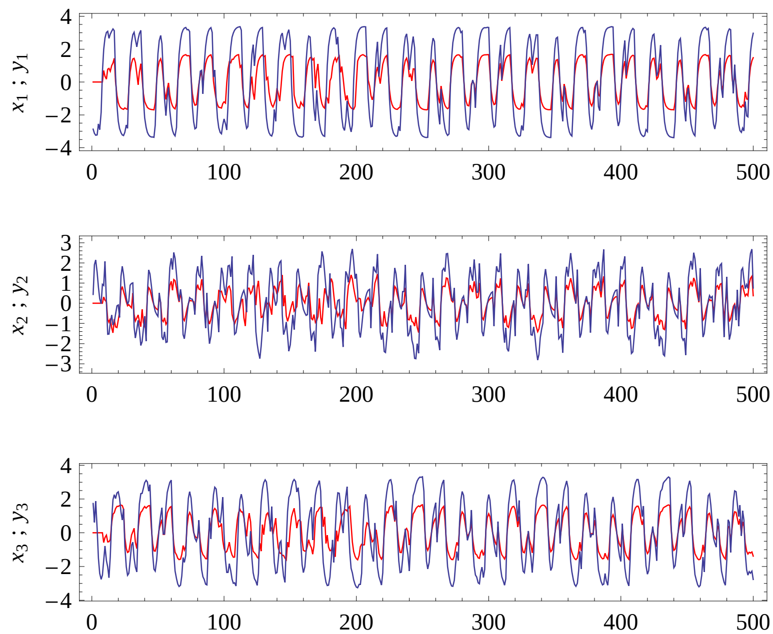

Fig. 5. Evolution of x_i (blue - master system) versus y_i (red - slave system), $i = 1, 2, 3$.

4 Conclusions

In this paper, sufficient conditions for the projective synchronization by impulsive controllers of general delayed discrete-time neural networks have been given. Numerical results show good agreement with the theoretical findings. Extending these results to more complicated neural network models with different types of time-delays may constitute a direction for future research.

Acknowledgements

This work was supported by CNCSIS-UEFISCSU, project number PN-II-RU-PD-145/2010 (Advanced impulsive and fractional-order neural network models).

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