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BetaBoop Brings in Chaos

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Abstract. The Verhults differential equation $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,(1-N(t))$ and its logistic parabola difference equation counterpart $x_{t+1} = \alpha x_t\,(1-x_t)\,\mathrm{I}_{(0,1)}(x_t)$, $\alpha \in [0,4]$, are tied to sustainable growth. We investigate the implications of considering $1-N(t)$, the linear truncation of the MacLaurin expansion of $-\ln N(t)$, or $N(t)$, the linear truncation of $-\ln(1-N(t))$, i.e. of curbing down either the retroaction factor $1-N(t)$ or the growing factor $N(t)$, which leads to Gumbel extreme value population for maxima or minima, respectively. More generally, we consider $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,(-\ln N(t))^{1+\gamma^*}$ — or, alternatively, $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,(-\ln(1-N(t)))^{1+\gamma^*}\,(1-N(t))$ —- and its difference equation counterpart. Simple extensions of the beta densities arise naturally in this context, and we discuss a $BetaBoop(p,q,P,Q)$, $p,q,P,Q>0$ family of probability density functions, that for $P=Q=1$ reduces to the usual $Beta(p,q)$ family.
Keywords: Population dynamics and chaos, extremal models, beta family.

1 Introduction

The rationale of the Verhulst population dynamics model

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,(1-N(t)) \tag{1}$$

is well-known: due to the malthusian reproduction rate $r>0$, $r\,N(t)$ implies growth, but on the other hand the retroaction term $-r\,N^2(t)$ slows down the growth impetus, and ultimately dominates, an action that is often interpreted in terms of sustainability. Hence the logistic solution of (1), $N(t) = 1/(1+\mathrm{e}^{-rt})$ (normalized so that $N(t)$ is a probability distribution function), is often tied to the idea of sustainable population dynamics growth.

Using Euler's algorithm, with an appropriate factor s, the equation (1) can be rewritten as

$$N(t+1) = N(t) + sr\,N(t)\,(1-N(t)) \quad \Longleftrightarrow \quad x_{t+1} = \alpha x_t\,(1-x_t) \tag{2}$$

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where $x_t = sr\,N(t)/(sr+1)$, $\alpha = 1 + sr$; if $\alpha \in (0,4)$, $x_t \in (0,1) \Longrightarrow x_{t+1} \in (0,1)$.

Due to its connection to the logistic curve, $\alpha\, x\,(1-x)\mathrm{I}_{(0,1)}(x)$ is sometimes referred to as logistic parabola. Observe that, with the notation $X_{p,q} \frown Beta(p,q)$, $\alpha\, x\,(1-x)\,\mathrm{I}_{(0,1)}(x) = \frac{\alpha}{6}\, f_{X_{2,2}}(x)$, where $f_{X_{2,2}}(x) = 6\,x\,(1-x)\,\mathrm{I}_{(0,1)}(x)$ is the probability density function of $X_{2,2} \frown Beta(2,2)$.

The fact that Euler's algorithm transforms the logistic differential equation in the difference equation model $x_{n+1} = \alpha\, x_n\,(1-x_n)$ had an important impact in the recognition that bifurcations, fractality, and ultimate chaos were indeed important tools in modeling population dynamics, when the reproduction rate r is explosive and sustainability fails.

As the Verhulst model is closely tied to the $Beta(2,2)$ probability density function, Aleixo *et al.* [1], [2], investigated the population dynamics of its natural extensions tied to general $Beta(p,q)$ models. Explicit solutions of the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N^{p-1}(t)\,(1-N(t))^{q-1}$ exist only for some (p,q) other than $(2,2)$ — for instance, $4\,\mathrm{e}^{rt}/\,(1+\mathrm{e}^{rt})^2$ is the solution of $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,\sqrt{1-N(t)}$ — but using appropriate software (we used *Mathematica* 7) numerical approximations of the solutions of practical problems are easily worked out.

As $\ln N(t) = -\sum_{k=1}^{\infty} (1-N(t))^k/k$, the factor $1-N(t)$ in (1) may be looked at as the linear truncation of $-\ln N(t)$. In the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,(-\ln N(t)), \tag{3}$$

the retroaction factor $-\ln N(t)$ is much lighter than $1-N(t)$, and hence it is not surprising that the solution of (3), $N(t) = \mathrm{e}^{-\mathrm{e}^{-rt}}$ (once again normalized to be a probability distribution function) is one of the extreme value laws for maxima, namely the Gumbel law.

On the other hand $\ln(1-N(t)) = -\sum_{k=1}^{\infty} N^k(t)/k$, and considering that the growing factor $N(t)$ in (1) is the linear approximation of $-\ln(1-N(t))$, we may regard (1) as an approximation of

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,(-\ln(1-N(t)))\,(1-N(t)) \tag{4}$$

whose solution, once again normalized, is the Gumbel extreme value distribution for minima, $N(t) = 1 - \mathrm{e}^{-\mathrm{e}^{rt}}$, which makes sense since in this case we curbed down the growing factor.

Pestana *et al.* [9] investigated $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,(-\ln N(t))$ and its discretization counterpart $x_{t+1} = s\,r\,x_t(-\ln x_t)$ in modeling extremal growth rate, as observed in the dynamics of cancer cells populations.

The generalization

$$f_{p,Q}(x) = \frac{p^Q}{\Gamma(Q)}\, x^{p-1}(-\ln x)^{Q-1}\mathrm{I}_{(0,1)}(x)$$

of the beta densities, has been introduced by Brilhante *et al.* [3]. In Section 2 we discuss the behavior of $x_{t+1} = r\,x_t(-\ln x_t)\,\mathrm{I}_{(0,1)}(x)$, the more general differential equation $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,(-\ln N(t))^{1+\gamma}$ and its connection to extreme value laws, as well as the behavior of $x_{t+1} = s\,r\,x_t(-\ln x_t)^{1+\frac{1}{\gamma}}\mathrm{I}_{(0,1)}(x)$. In Section 3 we introduce a new extension of the beta densities, namely

$$f_{p,q,P,Q}(x) = c\,x^{p-1}(1-x)^{q-1}(-\ln(1-x))^{P-1}(-\ln x)^{Q-1}\mathrm{I}_{(0,1)}(x), \qquad (5)$$

$p, q, P, Q > 0$, and a general discussion on modeling population dynamics via differential equations/difference equations, questioning whether chaos is in fact an appropriate framework in the description of evolution of populations.

2 Extreme value laws and population dynamics

As observed in Section 1, the Gumbel distribution function for maxima, $N(t) = \mathrm{e}^{-\mathrm{e}^{-rt}}$, is a solution of the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,(-\ln N(t))$, and the Gumbel distribution function for minima, $N^*(t) = 1 - \mathrm{e}^{-\mathrm{e}^{rt}}$, is a solution of the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}N^*(t) = r\,(-\ln(1-N^*(t)))\,(1-N^*(t))$. We now consider difference equations closely tied to those differential equations, i.e., we assume that there exists an appropriate c such that

$$N(t+1) = N(t) + cN(t)\,(-\ln N(t)) \Longleftrightarrow N(t+1) = -c\,N(t)\,\ln\left(\frac{N(t)}{\mathrm{e}^{\frac{1}{c}}}\right),$$

and we obtain the difference equation,

$$x_{t+1} = c\,x_t\,(-\ln x_t), \qquad (6)$$

closely associated to (3). As long as $x_t \in (0,1)$, if $c \in (0,\mathrm{e})$ we also have $x_{t+1} \in (0,1)$. The stationary solutions of (6) are $x_{t+1} = x_t = x_0$ with $x_0 = 0$ or $x_0 = \mathrm{e}^{-\frac{1}{c}}$. In view of the stability criterion for the stationary solutions, $\left|c\,(-\ln x - 1)\right| < 1$, and hence the stationary solution $x_0 = \mathrm{e}^{-\frac{1}{c}}$ is stable for $0 < c < 2.$, cf. Fig. 1

Using in *Mathematica* 7 the output of the instructions

```
Clear[f, x]
f[c_][x_] := c x *(-Log[x]) // N
x[c_][n_] := x[c][n] = f[c][x[c][n - 1]] // N
x[c_][0] := 0 // N;
tb = Table[{c, x[c][n]}, {c, .1, Exp[1], .01}, {n, 1000, 1300}];
Short[tb]
```

as input for the instructions

```
tb2 = Flatten[tb, 1];
ListPlot[tb]
```

we obtain the graph in Fig. 2, exhibiting bifurcations for $c \geq 2$, and ultimately chaos, as expected from the observations above.

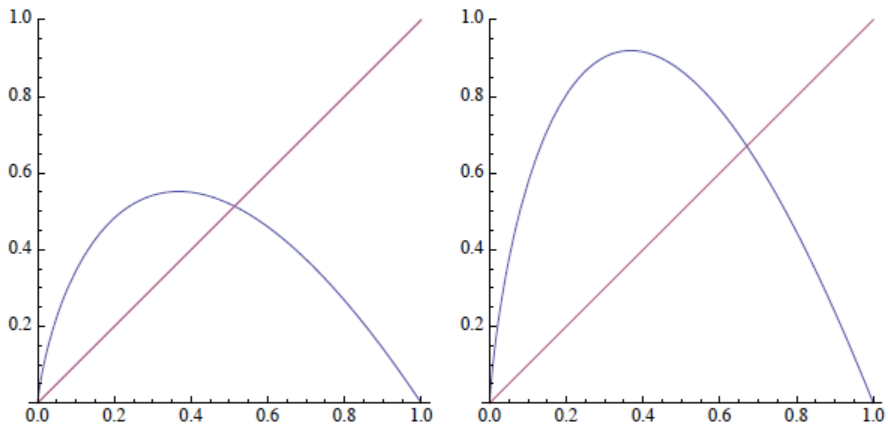

Fig. 1. Left: $1.5\,x_t\,(-\ln x_t)$; right: $2.5\,x_t\,(-\ln x_t)$.

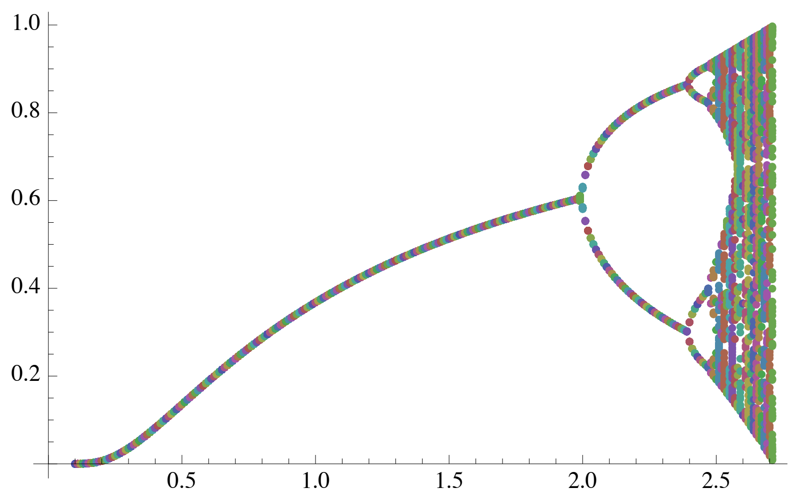

Fig. 2. Bifurcation diagram, solving $x = f(c,x) = c\,x\,(-\ln x)$, $c \in (0, \mathrm{e})$, using the fixed point method.

As we have discussed previously, the Gumbel distribution for minima $N(t) = 1 - \mathrm{e}^{-\mathrm{e}^{rt}}$ is a solution for the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}N^*(t) = r\,(-\ln(1 - N^*(t)))\,(1 - N^*(t))$, which is tied to the difference equation $x_{t+1} = c(-\ln(1 - N(t)))\,(1 - N(t))$. Fig. 3 is the simile of Fig. 2 for this case.

A more general situation involves the study of the differential equations

- $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,(-\ln N(t))^{1+\frac{1}{\gamma}}$, whose solution for $\gamma > 0$ is (again in standardized form) the Fréchet distribution function for maxima $N(t) = \mathrm{e}^{-(\frac{r}{\gamma}x)^{-\gamma}}\,\mathrm{I}_{[0,\infty)}(t)$, and whose solution for $\gamma < 0$ is the Weibull distribution function for maxima $N(t) = \mathrm{e}^{-(-\frac{r}{\gamma}t)^{\gamma}}\,\mathrm{I}_{(-\infty,0)}(t) + 1\,\mathrm{I}_{[0,\infty)}(t)$.

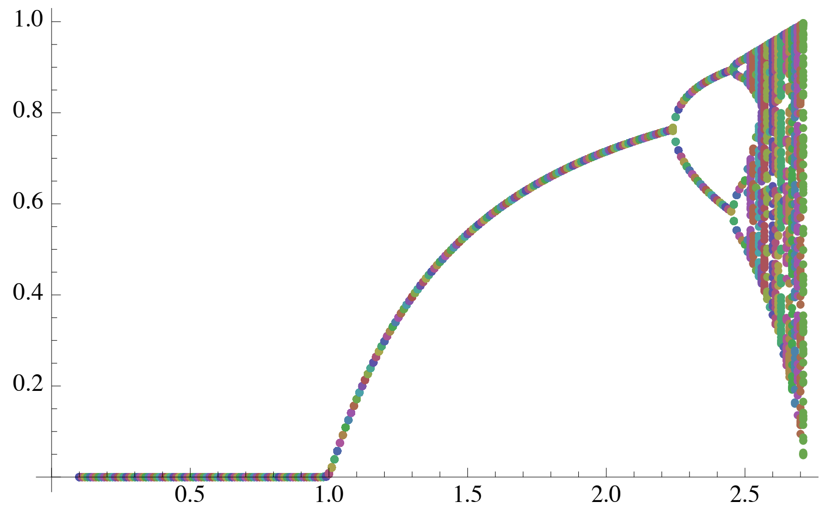

Fig. 3. Bifurcation diagram, solving $x = f(c, x) = c\,(-\ln(1-x))\,(1-x)$, $c \in (0, \mathrm{e})$, using the fixed point method.

- $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,(-\ln N(t))^{1+\frac{1}{\gamma}}$, whose solution for $\gamma > 0$ is the Fréchet distribution for minima, and for $\gamma < 0$ is the Weibull distribution function for minima.

Fig. 4 and Fig. 5 illustrate the dynamical behavior when solving by the fixed point method the difference equations closely associated to the above differential equations, namely $x_{t+1} = c\,x_t\,(-\ln x_t)^{1+\frac{1}{\gamma}}$ for $\gamma = 1$ (Fréchet-1) and $\gamma = -2$ (Weibull-0.5).

Remark 1. Considering the *G*eneral *E*xtreme *V*alue (*GEV*) distribution for maxima, $G_{\gamma^*}(t) = \mathrm{e}^{-(1+\gamma^* t)^{-1/\gamma^*}}$, $1 + \gamma^* t > 0$, it is obvious, from

$(1 + \gamma^* t)^{-1/\gamma^* - 1} = ((1 + \gamma^* t)^{-1/\gamma^*})^{\frac{1/\gamma^*+1}{1/\gamma^*}} = (-\ln G_{\gamma^*}(t))^{1+\gamma^*}$, that G_{γ^*} satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}G_\gamma(t) = G_\gamma(t)\,(-\ln G_\gamma(t))^{1+\frac{1}{\gamma}},\ \gamma = \frac{1}{\gamma^*}.$$

In the *GEV* representation, a shape parameter $\gamma^* > 0$ corresponds to the Fréchet-$\frac{1}{\gamma^*}$, $\gamma^* < 0$ corresponds to the Weibull-$\frac{1}{|\gamma^*|}$, and $\gamma^* \to 0$ corresponds to the Gumbel.

The similarity of

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,(-\ln N(t))^{1+\frac{1}{\gamma}}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,(-\ln(1 - N(t)))^{1+\frac{1}{\gamma}}\,(1 - N(t))$$

comes from the fact that stable distributions G for maxima (either Gumbel, or Fréchet or Weibull) and the corresponding stable distributions G^* for minima are tied through the relationship $G^*(x) = 1 - G(-x)$.

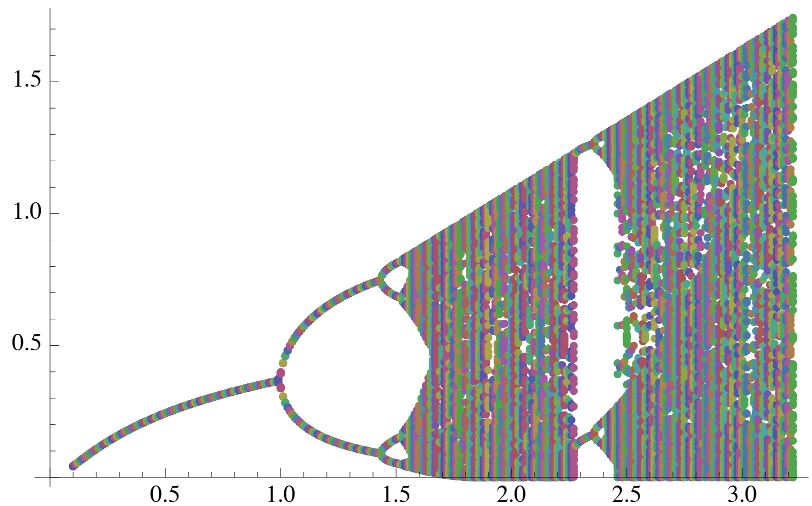

Fig. 4. Bifurcation diagram, solving $x = f(c, x) = c\,x\,(-\ln x)^2$ using the fixed point method.

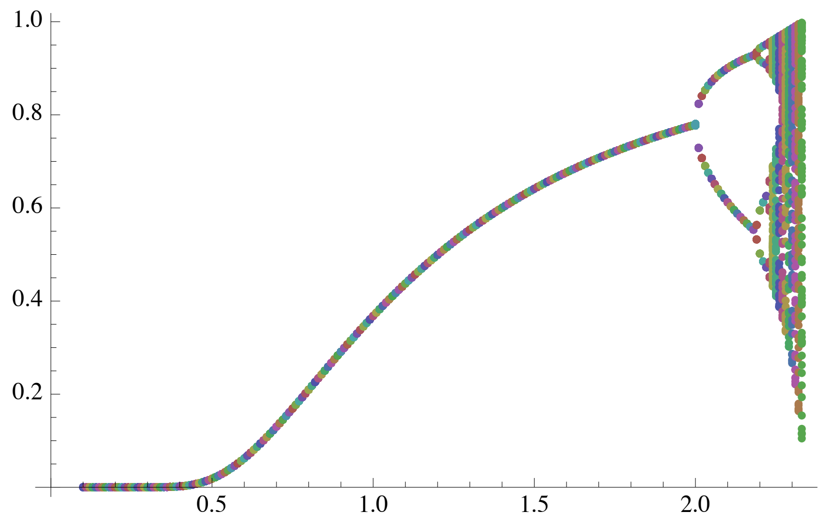

Fig. 5. Bifurcation diagram, solving $x = f(c, x) = c\,x\,(-\ln x)^{0.5}$ using the fixed point method.

3 The BetaBoop family

Brilhante *et al.* [3] extensively studied the family of probability density functions

$$f_{p,Q}(x) = \frac{p^Q}{\Gamma(Q)}\, x^{p-1}(-\ln\, x)^{Q-1}\, \mathrm{I}_{(0,1)}(x),$$

$p, Q > 0$, and their relevance in population studies.

Denote $X_{p,Q} \frown Betinha(p,Q)$, $p,Q > 0$, the random variable whose probability density function is $f_{p,Q}$, given above.

In fact, $4\,x\,(-\ln x)\,\mathrm{I}_{(0,1)}(x)$, tied to the Gumbel model, is the case $p = Q = 2$ in this family, just as $6\,x\,(1-x)\,\mathrm{I}_{(0,1)}(x)$, tied to the logistic parabola and the Verhulst population model, is the case $p = q = 2$ of the $Beta(p,q)$ family of probability density functions, whose dynamical behavior has been studied in depth in Aleixo *et al.* [1], [2], and references therein. This new family provides difference models whose associated differential models have as solution, among others, the stable distributions for maxima.

In the previous section we have seen that the probability density function of random variables $Y_{P,q} = 1 - X_{q,P}$, with $q = 2$, are connected to difference equations associated to differentail equations having as solutions the stable distributions for minima.

In fact, in view of Hölder's inequality, the function

$$x^{p-1}(1-x)^{q-1}(-\ln(1-x))^{P-1}(-\ln x)^{Q-1}\,\mathrm{I}_{(0,1)}(x)$$

is integrable for every $p,q,P,Q > 0$, and hence there exists $c \in (0,\infty)$ such that $f_{p,q,P,Q}$, in (5), is a probability density function. We denote the corresponding random variable $X_{p,q,P,Q} \frown BetaBoop(p,q,P,Q)$. Observe that $BetaBoop(p,q,1,1)$ is the same as $Beta(p,q)$, and $BetaBoop(p,1,P,1)$ is the same as $Betinha(p,P)$.

Betty Boop brought in chaos to the American Board of Censorship — sorry, we were dreaming of Betty Boop and Jessica Rabbit, and what we really meant to say is $BetaBoop(p,q,P,Q)$ brings in chaos, in the sense that the fixed point solution of equations of the type

$$x = c\,x^{p-1}(1-x)^{q-1}(-\ln(1-x))^{P-1}(-\ln x)^{Q-1}$$

exhibit all the problems first encountered in the numerical solution of the case $p = q = 2$, $P = Q = 1$. In Fig. 6 we illustrate this for $p = q = P = Q = 1.5$, and in Fig. 7 for $p = q = 1, P = Q = 3$.

In fact, many other generalizations of the logistic parabola

$f_c(x) = cx(1-x)\,\mathrm{I}_{(0,1)}(x)$ are potentially interesting in modeling population dynamics, as far as they reflect recognizable characteristics. For instance, the linear truncation of $\mathrm{e}^{-x} \approx (1-x)$ shows that $c\,x\,\mathrm{e}^{-x}\,\mathrm{I}_{(0,1)}(x) \approx c^*\,x\,(1-x)\,\mathrm{I}_{(0,1)}(x)$. In Fig. 8 we represent the bifurcation diagram corresponding to the difference equation $x_{t+1} = c\,x\,\mathrm{e}^{-x}$, modeling extremely slow growth.

Tsoularis, [10], in his overview of extensions of the logistic growth model, describes a *hyper-Gompertz* class, introduced by Turner *et al.*, [11], which is a subclass of the BetaBoop family. Our approach, using retroaction factor functions whose linearization is $(1-x)$ (such as $-\ln x$ or e^{-x}) and/or growing factor functions for which x is the linear truncation (such as $-\ln(1-x)$), leads to a wider class of growth models. Knowledge of the biological population dynamics may serve as an educated guess guideline to choose appropriate growth and retroaction factors, and as a basis to choose among competing growth models.

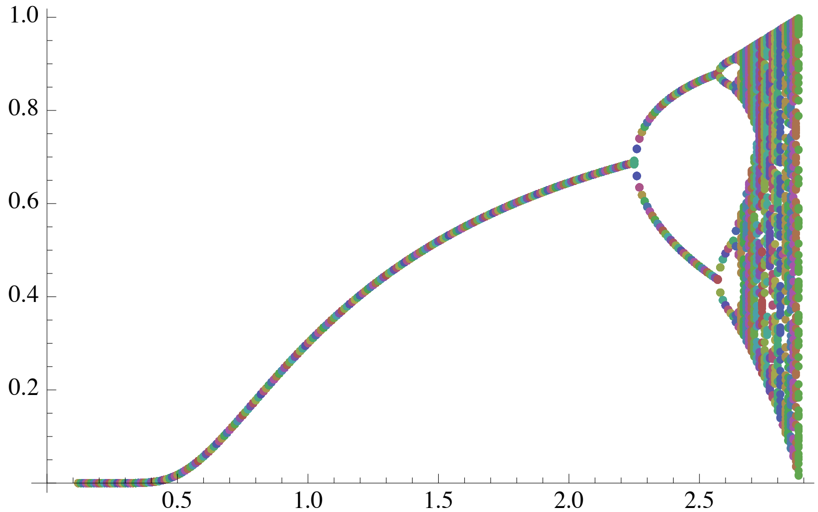

Fig. 6. Bifurcation diagram, solving $x = f(c,x) = c(x(1-x)(-\ln(1-x))(-\ln x))^{0.5}$ using the fixed point method.

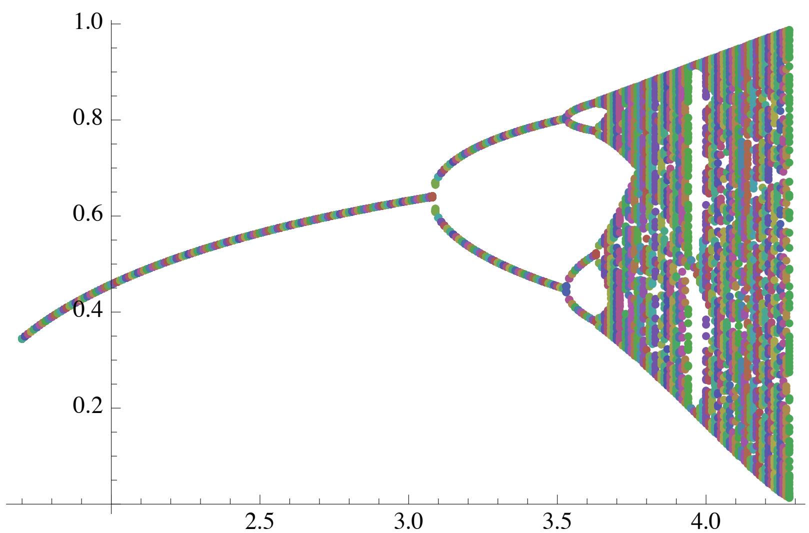

Fig. 7. Bifurcation diagram, solving $x = f(c,x) = c((-\ln x\,(-\ln(1-x)))^2$ using the fixed point method.

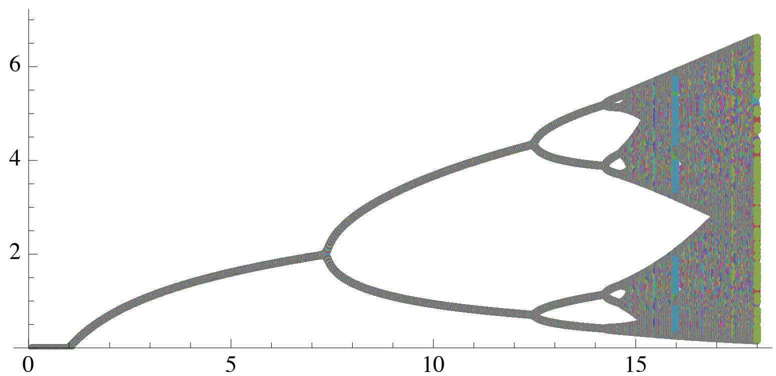

Fig. 8. Bifurcation diagram, solving $x = f(c, x) = c\,x\,\mathrm{e}^{-x}$ using the fixed point method.

4 Avoiding Chaos and a Class of New Non-Stable Extreme Value Laws

Finally, let us remark that there are grounds to argue that the chaos map $\big($for instance $x_{t+1} = c\,x_t\,(-\ln x_t)\big)$ is not an appropriate discrete equivalent of the original differential equation — for that example, $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,(-\ln N(t))$ —, inasmuch as the chaos map implies bifurcations and ultimately chaos, inexistent in the original differential equation.

An interesting point is that if we consider that the retroaction acts at time $t+1$, we obtain a difference equation $x_{t+1} = c\,x_t\,(-\ln x_{t+1})$, that has the same stationary solutions as the chaos map $x_{t+1} = c\,x_t\,(-\ln x_t)$, but does not exhibit bifurcation and chaos. In fact, from $x_{t+1} = c\,x_t\,(-\ln x_{t+1})$ we get a solution $f_c(x) = cx\,W\left(\frac{1}{cx}\right)$, where $W()$ is the Logarithmic Product function, a function taking on real values for $x > -0.5$.

Fig. 9 below shows that $cx\,W\left(\frac{1}{cx}\right)\,\mathrm{I}_{(0,\infty)}(x)$ is a distribution function, that may serve as a non-stable extreme value law, but that definitively is not a good approximation to the Gumbel distribution. (We used $c = 2$, and computed the Gumbel scale parameter 0.52688, so that the the lines cross at the 0.9 quantile.)

This is patently a rather poor approximation, even for quite large values. In fact, investigating this approximation has been motivated solely from the fact that this is a non-chaotic solution of a modified difference equation approximation to the differential equation whose solution is the Gompertz curve, i.e. the Gumbel distribution, when properly normalized.

In fact

$$\lim_{x\to\infty} \frac{1 - ctx\,W\left(1/(ctx)\right)}{1 - cx\,W\left(1/(cx)\right)} = t^{-1},$$

showing that this new law is in the domain of attraction of the Fréchet with shape parameter 1, whatever the value $c > 0$.

Fig. 10, comparing the distribution function $2x\,W\left(\frac{1}{2x}\right)\,\mathrm{I}_{(0,\infty)}(x)$ and the Fréchet distribution $\mathrm{e}^{-\frac{0.44995}{x}}\,\mathrm{I}_{(0,\infty)}(x)$, shows that this approximation is quite

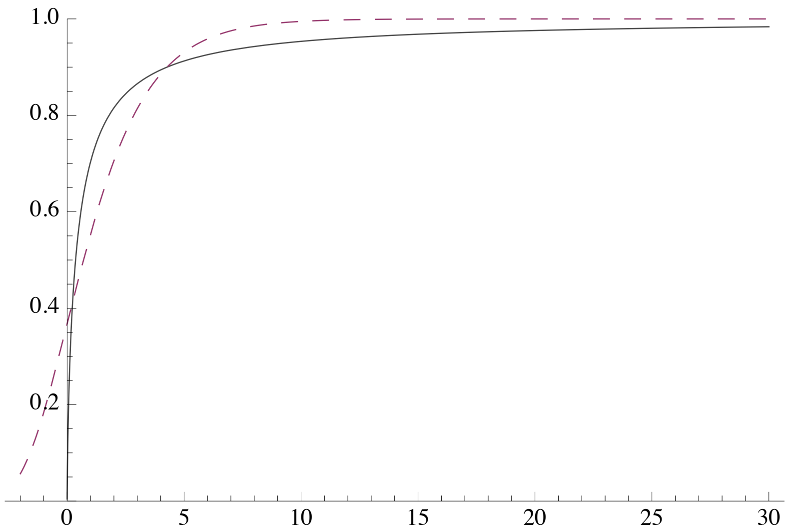

Fig. 9. $2x\, W\left(\frac{1}{2x}\right)\, \mathrm{I}_{(-0,\infty)}(x)$ (solid line) approximation of $\mathrm{e}^{-\mathrm{e}^{-0.52688\,x}}$ (dashed line).

good. Once again, the scale parameter of the Fréchet distribution has been chosen so that the lines cross at the common 0.9 quantile.

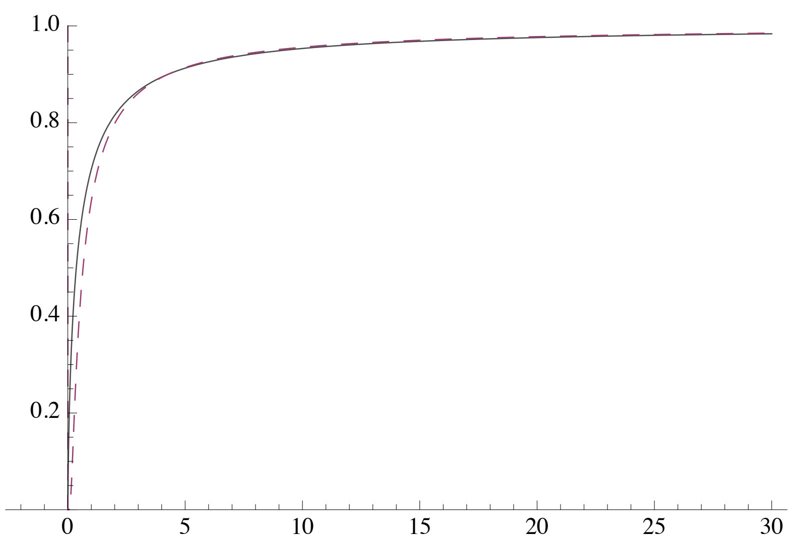

Fig. 10. $2x\, W\left(\frac{1}{2x}\right)\, \mathrm{I}_{(-0,\infty)}(x)$ (solid line) approximation of $\mathrm{e}^{-\frac{0.44995}{x}}\, \mathrm{I}_{(0,\infty)}(x)$ (dashed line).

Below, in Fig. 11, we plot the second derivative of $2x\, W\left(\frac{1}{2x}\right)\, \mathrm{I}_{(-0,\infty)}(x)$. Observe that Mejzler [4], [5], [6], [7], [8] developed an interesting $\mathcal{M}$-class of "self-decomposable" extreme value laws that arise as limit of suitably consistent sequences of independent — but not necessarily identically distributed — random variables, that is in the extreme values scheme a simile of Khinchine's $\mathcal{L}$-class in the asymptotic additive theory. Mejzler's characterization is done in terms of log concavity.

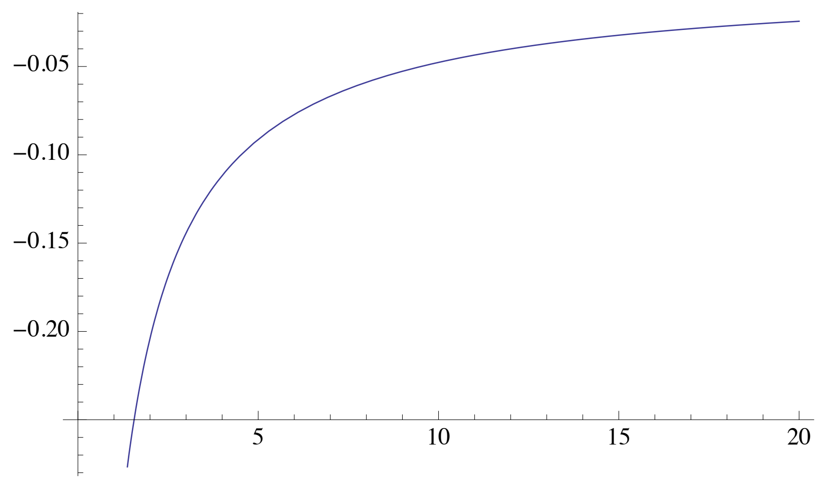

Fig. 11. Log-concavity of $2x\,W\left(\frac{1}{2x}\right)\,\mathrm{I}_{(-0,\infty)}(x)$.

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