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Dissipative Solitons: Structural Chaos and Chaos of Destruction

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Abstract. Dissipative soliton, that is a localized and self-preserving structure, develops as a result of two types of balances: self-phase modulation vs. dispersion and dissipation vs. gain. The contribution of dissipative, i.e. environmental, effects causes the complex "far from equilibrium" dynamics of a soliton: it can develop in a localized structure, which behaves chaotically. In this work, the chaotic laser solitons are considered in the framework of the generalized complex nonlinear Ginzburg-Landau model. For the first time to our knowledge, the model of a femtosecond pulse laser taking into account the dynamic gain saturation covering a whole resonator period is analyzed. Two main scenarios of chaotization are revealed: i) multipusing with both short- and long-range forces between the solitons, and ii) noiselike pulse generation resulting from a parametrical interaction of the dissipative soliton with the linear dispersive waves. Both scenarios of chaotization are associated with the resonant and nonresonant interactions with the continuum (i.e. vacuum) excitations.
Keywords: Dissipative soliton, Complex nonlinear Ginzburg-Landau equation, Chaotic soliton dynamics.

1 Introduction

The nonlinear complex Ginzburg-Landau equation (NCGLE) has a lot of applications in quantum optics, modeling of Bose-Einstein condensation, condensate-matter physics, study of non-equilibrium phenomena, and nonlinear dynamics, quantum mechanics of self-organizing dissipative systems, and quantum field theory [1]. In particular, this equation being a generalized form of the so-called master equation provides an adequate description of pulses generated by a mode-locked laser [2]. Such pulses can be treated as the dissipative solitons (DSs), that are the localized solutions of the NCGLE [3]. It was found, that the DS can demonstrate a highly non-trivial dynamics including formation of multi-soliton complexes [4], soliton explosions [5], noise-like solitons [6], etc. The resulting structures can be very complicated and consist of strongly or weakly interacting solitons (so-called soliton molecules and gas) [7] as well as the short-range noise-like oscillations inside a larger wave-packet [8]. The nonlinear dynamics of these structures can cause both regular and chaotic-like behavior.

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In this article, the different scenarios of the DS structural chaos will be considered. The first scenario is an appearance of the chaotic fine graining of DS. For such a structure, the mechanism of formation is identified with the parametric instability caused by the resonant interaction of DS with the continuum. The second scenario is formation of the multi-soliton complexes governed by both short-range forces (due to solitons overlapping) and long-range forces (due to gain dynamics). The underlying mechanism of formation is the continuum amplification, which results in the soliton production or/and the dynamical coexistence of DSs with the continuum.

2 Dissipative solitons of the NCGLE

Formally, the NCGLE consists of the nondissipative (hamiltonian) and dissipative parts. The nondissipative part can be obtained from variation of the Lagrangian [9]:

$$\mathscr{L} = \frac{i}{2}\left[A^*(x,t)\frac{\partial A(x,t)}{\partial t} - A(x,t)\frac{\partial A^*(x,t)}{\partial t}\right] + \\ +\frac{\beta}{2}\frac{\partial A(x,t)}{\partial t}\frac{\partial A^*(x,t)}{\partial t} - \frac{\gamma}{2}\left|A(x,t)\right|^2, \qquad (1)$$

where $A(x,t)$ is the field envelope depending on the propagation distance x and the "transverse" coordinate t (that is the local time in our case), β is the group-delay dispersion (GDD) coefficient (negative/positive for the normal/anomalous dispersion), and γ is the self-phase modulation (SPM) coefficient [10]. The dissipative part is described by the driving force:

$$\mathscr{Q} = -i\Gamma A(x,t) + i\frac{\rho_0}{1+\sigma\int_{-\infty}^{\infty}|A|^2\,dt'}\left[A(x,t) + \tau\frac{\partial^2}{\partial t^2}A(x,t)\right] + \\ +i\kappa\left[|A(x,t)|^2 - \zeta|A(x,t)|^4\right]A(x,t), \qquad (2)$$

where Γ is the net-dissipation (loss) coefficient, ρ_0 is the saturable gain (σ is the inverse gain saturation energy if the energy E is defined as $E \equiv \int_{-\infty}^{\infty}|A|^2\,dt'$), τ is the parameter of spectral dissipation (so-called squared inverse gainband width), and κ is the parameter of self-amplitude modulation (SAM). The SAM is assumed to be saturable with the corresponding parameter ζ.

Then, the desired CNGLE can be written as

$$i\frac{\partial A(x,t)}{\partial x} - \frac{\beta}{2}\frac{\partial^2}{\partial t^2}A(x,t) - \gamma|A(x,t)|^2A(x,t) = \\ = -i\Gamma A(x,t) + i\frac{\rho_0}{1+\sigma\int_{-\infty}^{\infty}|A|^2\,dt'}\left[A(x,t) + \tau\frac{\partial^2}{\partial t^2}A(x,t)\right] + \\ +i\kappa\left[|A(x,t)|^2 - \zeta|A(x,t)|^4\right]A(x,t). \qquad (3)$$

Eq. (3) is not integrable and only sole exact soliton-like solution is known for it [10,11]. Nevertheless, the so-called variational method [9] allows exploring the solitonic sector of (3). The force-driven Lagrange-Euler equations

$$\frac{\partial \int_{-\infty}^{\infty} \mathscr{L} dt}{\partial \mathbf{f}} - \frac{\partial}{\partial x}\frac{\partial \int_{-\infty}^{\infty} \mathscr{L} dt}{\partial \mathbf{f}} = 2\Re \int_{-\infty}^{\infty} \mathscr{Q}\frac{\partial A^*}{\partial \mathbf{f}} dt \tag{4}$$

allow obtaining a set of the ordinary first-order differential equations for a set $\mathbf{f}$ of the soliton parameters if one assumes the soliton shape in the form of some trial function $A(x,t) \approx \mathscr{F}(t,\mathbf{f})$. One may chose [14]

$$\mathscr{F} = a\left(x\right) \operatorname{sech}\left(\frac{t}{T(x)}\right) \exp\left[i\left(\phi(x) + \psi(x)\ln\left(\operatorname{sech}(\frac{t}{T(x)})\right)\right)\right], \tag{5}$$

with $\mathbf{f} = \{a(x), T(x), \phi(x), \psi(x)\}$ describing amplitude, width, phase, and chirp ("squeezing parameter" in other words) of DS, respectively.

Substitution of (5) into (4) results in four equations for the soliton parameters. These equations are completely solvable for a steady-state propagation (i.e. when $\partial_x a = \partial_x T = \partial_x \psi = 0, \partial_x \phi \neq 0$). The analysis demonstrates that the solitonic sector can be completely characterized by two-dimensional master diagram, that is the DS is two-parametrical and the corresponding dimensionless parameters are: $c \equiv \tau\gamma/|\beta|\kappa$, the dimensionless energy is of $\mathscr{E} \equiv E\sqrt{\kappa\zeta/\tau}$ for the anomalous GDD and is of $\mathscr{E} \equiv Eb^{-1}\sqrt{\kappa\zeta/\tau}$ for the normal GDD (here $b \equiv \gamma/\kappa$).

The master diagrams are shown in Fig. 1. The solid curves correspond to the stability thresholds defined as $\varGamma - \rho_0 \Big/ \left(1 + \sigma \int_{-\infty}^{\infty} |A|^2 dt\right) = 0$. Positivity of this value provides the vacuum stability. As will be shown, the vacuum destabilization is main source of the soliton instability causing, in particular, the chaotic dynamics.

The master diagram in Fig. 1, *a* (dashed curve) reveals a very simple asymptotic for the maximum energy of the chirped DS:

$$E \approx 17\,|\beta| \Big/ \sqrt{\kappa\zeta\tau}. \tag{6}$$

The continuum rises above this energy. The corresponding expression for the chirp-free DS developing in the anomalous GDD regime (see dashed curve in Fig. 1, *b*) is

$$E \approx \sqrt{5\beta/\zeta\gamma}. \tag{7}$$

Asymptotical (i.e. corresponding to $c \ll 1$) expressions for the widths of the chirped and chirp-free DSs are:

$$T \approx \frac{8}{|c|}\frac{\gamma}{\kappa}\sqrt{\frac{\tau\zeta}{\kappa}},$$
$$T \approx \frac{2}{\sqrt{5c}}\sqrt{\frac{\tau\zeta}{\kappa}}, \tag{8}$$

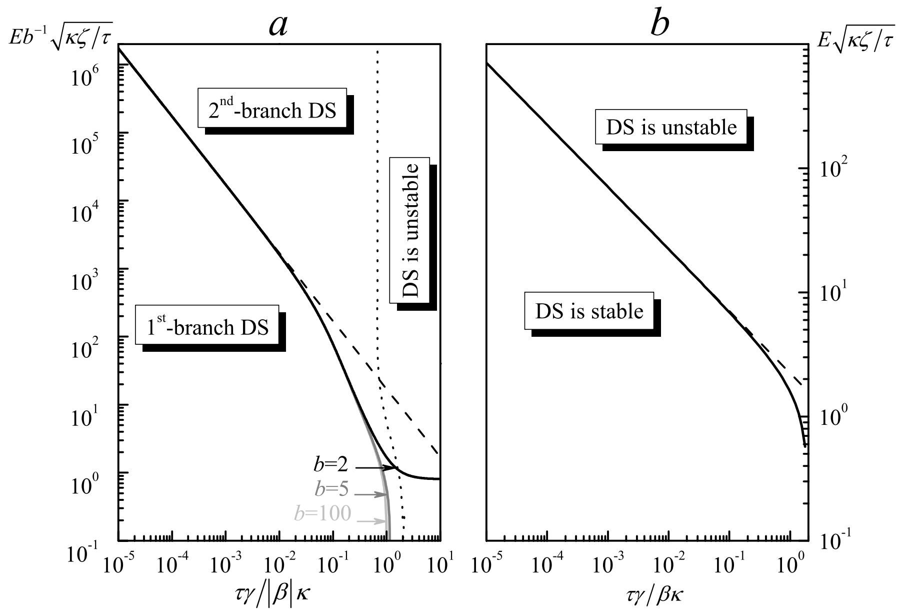

Fig. 1. Master diagrams of DS for the normal (a) and anomalous (b) GDDs. The solid curves correspond to the DS stability thresholds obtained from the variational approximation. The dashed curves correspond to the asymptotic $c \equiv \tau\gamma/|\beta|\kappa \ll 1$. The dotted curve in ($a$) corresponds to the stability threshold of the second DS branch with $\theta > 1$ in (9).

respectively. If the DS in the anomalous GDD is chirp-free by definition, the DS in the normal GDD has an asymptotical chirp $|\psi| \approx 4\kappa/\gamma\,|c|$. One has note, that the asymptotical DS spectral widths are $\propto \sqrt{c\kappa/\tau\zeta}$ for the chirp-free DS and $\propto \sqrt{\kappa/\tau\zeta}$ for the chirped DS.

3 Nonresonant excitation of continuum

The DS of unperturbed Eq. (3) does not interact directly with the continuum. An existence of the stability thresholds shown in Fig. 1 has a simple physical explanation: an approach to the stability threshold causes the DS spectral broadening that increases the spectral loss for the DS [12]. As a result, the DS energy decreases and the energy-dependent net-loss $\varGamma - \rho_0 \Big/ \left(1 + \sigma \int_{-\infty}^{\infty} |A|^2\, dt\right)$ crosses zero-level. Hence, the continuum rises and, in the anomalous GDD regime, the multiple-pulsing develops (Fig. 2, left). Strong interactions inside a multi-pulse complex lead to the structural chaotization. The field remains localized on a picosecond scale, but chaotically structured on a femtosecond one (Fig. 2, right). On the other hand, the solitonic "soup" (Fig. 2, right) can create spontaneously the stable soliton complexes (Fig. 3).

However in the normal GDD, the chirped DS is so adaptable that the spectral loss growth with an approaching to the stability threshold leads to an

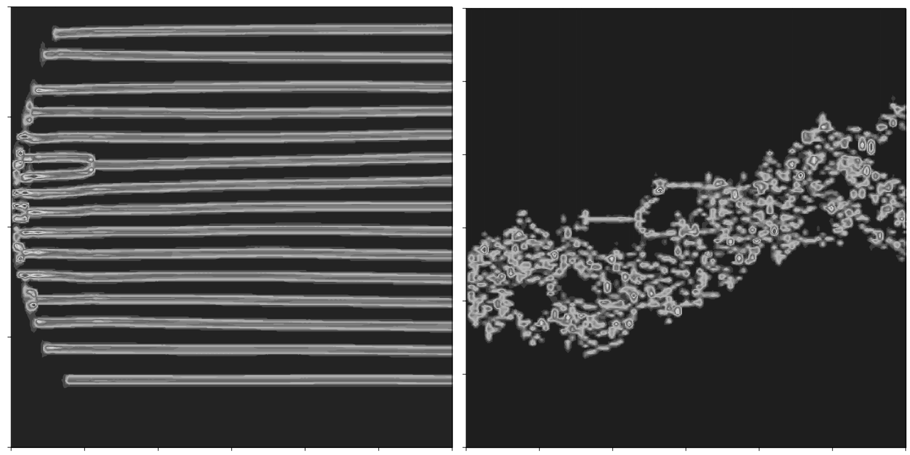

Fig. 2. Multiple-pulse complexes (contour plot with brighter regions corresponding to higher powers) in the anomalous GDD regime. Regular multi-pulsing (left) and structural chaotization (right) with approaching the GDD to zero are shown. The axes are t (vertical) and z (horizontal).

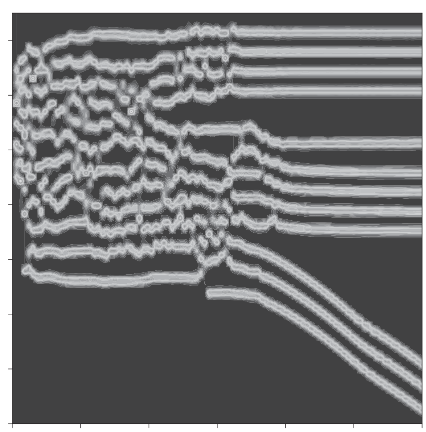

Fig. 3. Spontaneous self-ordering from a solitonic "soup".

appearance of a new DS branch. The stability border corresponding to this branch can be found from the approximated theory of [15] and is shown by the dotted curve in Fig. 1, *a*. Such a branch corresponds to the DS with a so-called "finger-like" spectrum [13,15]. This spectrum has a main part of power in the vicinity of the spectrum center. As a result, a spectral loss decreases that leads to an energy growth close to the boundary of the DS stability. Such a chirped DS provides a perfect energy scalability (dotted curve in Fig. 1, *a*) with $E \xrightarrow{c \to const} \infty$ and can be described by

$$\mathscr{F} = \frac{a(x)}{\sqrt{\theta(x) + \cosh\left(\frac{t}{T(x)}\right)}} \exp\left[i\left(\phi(x) + \psi(x)\ln\left(\theta(x) + \cosh\left(\frac{t}{T(x)}\right)\right)\right)\right] \tag{9}$$

with $\theta(x) > 1$.

An additional source of the DS destabilization is a long range "force" caused by the dynamic gain saturation:

$$\frac{\partial \rho}{\partial t} = P(\rho_0 - \rho) - \sigma\rho|a|^2 - \frac{\rho}{T_r}. \tag{10}$$

Here ρ is the time-dependent gain, P is the pump-rate, and T_r is the gain relaxation time.

The gain dynamics can result in the nonresonant vacuum excitation far from the DS. As a result, a large-scale solitonic (multi-soliton) structure appears (Fig. 4) and, the satellites appear both nearby (few picosecond) the main pulse and far (nanoseconds) from it. Strong interaction between the pulses with the contribution from a gain dynamics results in a chaotic behavior. For the chirped DS, the dynamic gain saturation can result in a parametric resonance and the DS becomes finely structured [18].

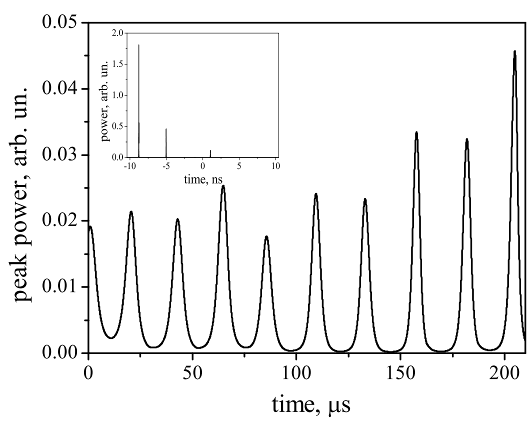

Fig. 4. Multiple DS evolution in the presence of the dynamic gain saturation.

4 Resonant excitation of continuum

The stability threshold shown in Fig. 1 corresponds to an unperturbed DS of (3). The physically meaningful perturbation results from a higher-order dispersion correction to the Lagrangian: $\mathscr{L} = \mathscr{L}_0 + \frac{i\delta}{2}\frac{\partial^2 A}{\partial t^2}\frac{\partial A^*}{\partial t}$, where $\mathscr{L}_0$ is the unperturbed Lagrangian (1) and δ is the third-order dispersion (TOD) parameter. As a result of such perturbation, the resonant interaction of DS with the continuum at some frequency ω_r can appear [16]. The resonance appears if the wave-number of a linear wave (i.e. a vacuum excitation) equals to the DS wave-number q: $\beta\omega^2 + \delta\omega^3 = q$.

Hence, the stability threshold becomes lower (i.e. shifts in the direction of lower E and c) than that shown in Fig. 1. As the resonance occurs in the spectral domain, an exploration of the DS spectrum is most informative in this case. The numerical results corresponding to a mode-locked Cr:ZnSe oscillator [17] are shown in Fig. 5. Non-zero δ can be treated as a frequency-dependence of net-GDD. As a result of such dependence, the zero GDD shifts towards the DS spectrum with the growing $|\delta|$. Increasing TOD transforms the initially rectangular spectrum (curve 1 in Fig. 5) to trapezoid (curve 2) and then to triangular (curve 3) ones. Simultaneously, an dispersive spectral component appears within the anomalous GDD region. The DS spectrum acquires a strong modulation (curve 4), when the resonant frequency shift inside the DS spectrum: $|\omega_r| < \sqrt{\zeta a_0^2/|\beta|} \approx |\beta/\delta|$. Finally, the spectrum becomes completely fragmented with a further TOD growth.

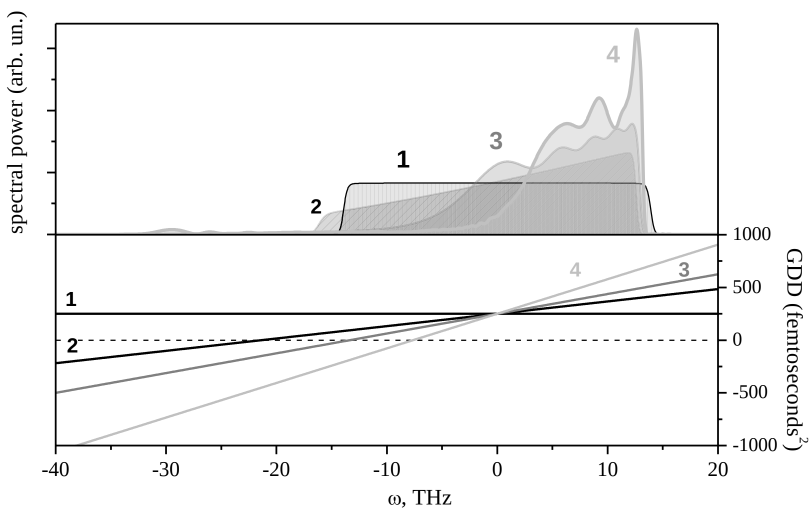

Fig. 5. Spectra of the chirped DSs corresponding to the different net-GDD. The GDD slope depends on the TOD value. Here, the positive GDDs correspond to the normal dispersions.

Resonant interaction with the dispersive wave perturbs strongly the DS spectrum, causes the spectrum structurization and the central frequency jitter. As a result, the soliton behaves chaotically (Fig. 6) although its energy remains almost constant.

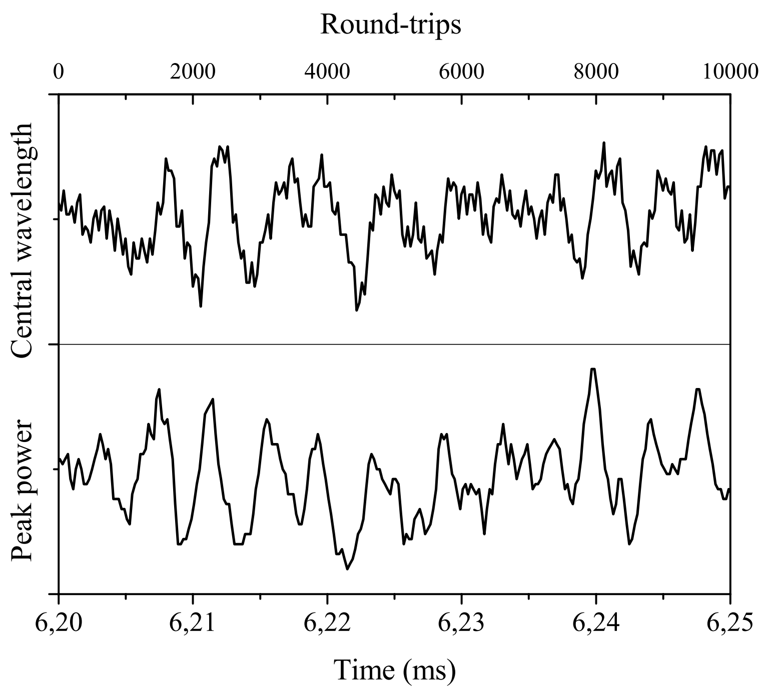

Fig. 6. Jitter of the DS central frequency and its peak power due to resonant interaction with the continuum.

5 Conclusion

Unlike a classical soliton, a DS posses nontrivial dynamics, which can be very complicated. In particular, a chaotic interaction with the excited vacuum (continuum) develops. Such an interaction can be both nonresonant and resonant. A nonresonant excitation of the vacuum forms the multi-soliton complexes. Strong interactions inside such complexes cause the structural chaos. Long-range interactions in a system can be additional source of the nonresonant vacuum excitation that leads to the macro-structural solitonic chaos. A resonant excitation of the vacuum causes the DS spectral jitter with a subsequent chaotic dynamics and even a soliton destruction. One has to note, that a strong localization of the chirped DS does not prevent the soliton traceability in an even chaotic regime. Such a traceability promises a lot of applications in the spectroscopy, for instance.

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