Chaotic Modeling and Simulation (CMSIM) 1: 77-90, 2011

Sir Pinski Rides Again

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Abstract. The iterative procedure of removing "almost everything" from a triangle ultimately leading to the Sierpinski's gasket $\mathcal{S}$ is well-known. But what is in fact left when almost everything has been taken out? Using the Sir Pinski's game described by Schroeder [4], we identify two dual sets of invariant points in this exquisite game, and from these we identify points left over in Sierpinski gasket. Our discussion also shows that the chaos game does not generate the Sierpinski gasket. It generates an approximation or, at most, a subset of $\mathcal{S}$.

Keywords: Sierpinski gasket, Sierpinski points, fractals, Sir Pinski game, chaos game, self-similarity, periodicity.

1 Introduction

Let $\mathcal{T}$ be a triangle with vertices A, B, C, and denote a, b, c the corresponding opposite sides.

The first step of the classical iterative construction of the Sierpinski gasket is to remove the middle trianle $\mathcal{M}_1$ with vertices A', B', C', the middle points of a, b, c, respectively. In this first step we obtain

$$\mathcal{S}_1 = \mathcal{T} - \mathcal{M}_1 = T_1 \cup R_1 \cup L_1,$$

where T_1 is the 'top triangle', R_1 is the 'right triangle', and L_1 is the 'left triangle'. Observe that T_1, R_1 and L_1 are similar to $\mathcal{T}$.

In the second step we repeat the above procedure, removing $\mathcal{M}_{2,T_1}$ in T_1, removing $\mathcal{M}_{2,R_1}$ in R_1, and removing $\mathcal{M}_{2,L_1}$ in L_1. With the notation $\mathcal{M}_2 = \mathcal{M}_{2,T_1} \cup \mathcal{M}_{2,R_1} \cup \mathcal{M}_{2,L_1}$, in this second step we obtain

$$\mathcal{S}_2 = \mathcal{S}_1 - \mathcal{M}_2 = \mathcal{T} - (\mathcal{M}_1 \cup \mathcal{M}_2).$$

$\mathcal{S}_2$ is the union of 3^2 triangles similar to $\mathcal{T}$. Each of them is easily identified using self-explanatory notations such as $\overrightarrow{T_1 R_2}$.

Received: 21 July 2011 / Accepted: 28 October 2011

ISSN 2241-0503

A similar procedure is indefinitely repeated: in step k we obtain $\mathcal{S}_k$ by removing the middle triangles from each of the 3^{k-1} triangles whose union is $\mathcal{S}_{k-1}$; we denote $\mathcal{M}_k$ the union of the middle triangles removed from $\mathcal{S}_{k-1}$ to obtain $\mathcal{S}_k$.

Notations such as $\overrightarrow{R_1T_2T_3L_4T_5}$ indicate in $\mathcal{S}_5$ that we are considering the triangle obtained when in the 1st, 2nd, 3rd, 4th and 5th steps we go respectively to the right, to the top, to the top, to the left and to the top triangles of the one obtained in the previous step.

The Sierpinski gasket is

$$\mathcal{S} = \lim_{k\to\infty} \mathcal{S}_k = \bigcap_{k=1}^{\infty} \mathcal{S}_k.$$

From Banach's contractive mapping fixed point theorem it follows that the Sierpinski gasket

$$\mathcal{S} = \mathcal{T} - \bigcup_{k=1}^{\infty} \mathcal{M}_k = \psi_A(\mathcal{S}) \cup \psi_B(\mathcal{S}) \cup \psi_C(\mathcal{S}),$$

where $\psi_A(\cdot)$ is the dilation of ratio $1/2$ towards the top vertex A, $\psi_B(\cdot)$ is the dilation of ratio $r = 1/2$ towards the left vertex B, and $\psi_C(\cdot)$ is the dilation of ratio $1/2$ towards the right vertex C. In other words, $\mathcal{S}$ is the unique non-empty fixed point of the corresponding Hutchinson [2] operator ψ, where $\psi(\mathcal{A}) = \psi_A(\mathcal{A}) \cup \psi_B(\mathcal{A}) \cup \psi_C(\mathcal{A})$, i.e. $\psi(\mathcal{A}) = \mathcal{A}$ if and only if $\mathcal{A} = \mathcal{S}$.

Thus, the use of the contracting ratio $r = 1/2$ or of the doubling scale factor $s = 1/r = 2$ can provide some structural information on the Sierpinski gasket.

1.1 The Sir Pinski Game

Let $\mathcal{T}$ be a triangle. A player chooses a point P_0 inside the triangle. Sir Pinski game consists of iteratively jumping to the points $\{P_1, P_2, \dots\}$, where P_{k+1} doubles the distance of P_k to its nearest vertex. The player looses at step n if $P_0, P_1, P_2, \dots, P_{n-1} \in \mathcal{T}$ and $P_n \notin \mathcal{T}$.

It is obvious that $\mathcal{M}_1$ is the set of loosing points at step 1, $\mathcal{M}_2$ is the set of loosing points at step 2, and in general $\mathcal{M}_k$ is the set of loosing points at step k. Loosing points are illustrated in Figure 1, that also clarifies the connection of loosing points at step k with middle triangles removed at step k in the classical iterative construction of the Sierpinski gasket.

Schroeder [4] characterizes Sierpinski's gasket as the set of winning points $\mathcal{S} = \mathcal{T} - \bigcup_{k=1}^{\infty} \mathcal{M}_k$ of Sir Pinski game.

In other words, the Sierpinski points $S \in \mathcal{S}$ can be characterized as the set of points $S \in \mathcal{T}$ such that $\frac{S+A}{2}, \frac{S+B}{2}, \frac{S+C}{2}, \in \mathcal{S}$. So, starting from whatever point $P \in \mathcal{T}$, iteratively jumping for a point halving the distance to any of the vertices of the triangle $\mathcal{T}$ creates an infinite sequence of points in a straight line that ultimately converges to the vertex considered. Observe however that

- if $P \in \mathcal{S}$, all the iterates are Sierpinski points; but, on the other hand,

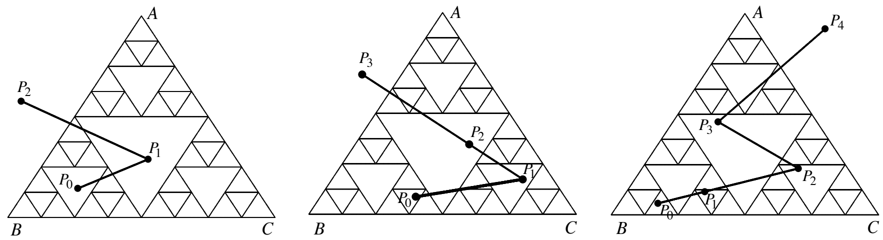

Fig. 1. Loosing points at steps 2 (*left*), 3 (*center*) and 4 (*right*).

- if $P \notin \mathcal{S}$, none of the iterates is a Sierpinski point.

In fact, the halving contractions ψ_i generate points that are nearer and nearer to Sierpinski points, but as the Sir Pinski game clearly shows, doubling the distance towards the nearest vertex ultimately leaves $\mathcal{T}$ unless the starting point is itself a Sierpinski point.

Observe that iteratively halving (or, alternatively, doubling) the distance to a fixed vertex of the triangle $\mathcal{T}$ creates an infinite sequence of colinear points. Hence we need some rule to use in turn, either deterministically or randomly, the different vertices in order to approximate the Sierpinski gasket $\mathcal{S}$. Sir Pinski game uses the deterministic rule: take the nearest vertex to the starting point/iterate, and double the distance. If the starting point is a Sierpinski point, this deterministic rule implies that we are not using a fixed vertex, and hence colinearity is broken up.

1.2 The Chaos Game

Barnsley [1] devised a *chaos game*, using randomness to generate subsets of the three sets $\psi_A(\mathcal{T}), \psi_B(\mathcal{T}), \psi_C(\mathcal{T})$: pick a starting point P_0, and generate iterates $\{P_1, P_2, \dots\}$, such that P_k is the midpoint of the segment whose endpoints are P_{k-1} and one of the vertices $v_L = B, v_R = C, v_T = A$ of $\mathcal{T}$, randomly chosen using the discrete uniform law

$$X = \left\{ \begin{array}{ccc} v_L & v_R & v_T \\ 1/3 & 1/3 & 1/3 \; . \end{array} \right.$$

This chaos game is generally presented as a device to generate the Sierpinski gasket $\mathcal{S}$, but in view of the above observations it produces in general an approximation of the Sierpinski gasket, since in general $P_0 \notin \mathcal{S}$. Observe also that even starting from a Sierpinski point, what we obtain is a subset of the Sierpinski gasket — for example, choosing P_0 as the top vertex of the equilateral triangle, as in [3], page 306, will generate as iterates only vertex points of the triangles left out when middle triangles are removed, in the classical deterministic iterative construction of $\mathcal{S}$. This issue will be discussed later in further detail.

2 Invariant Sets of Points in the Sir Pinski Game and the Sierpinski Gasket

As seen in the introduction, the points $S \in \mathcal{S}$ are easily described using the concept of self-similarity and its far-reaching consequences.

Using translation and rotation, if needed, we assume that the vertices of $\mathcal{T}$ are $v_{_L} = (0,0)$, $v_{_R} = (a,0)$, $a > 0$, and $v_{_T} = (c,d)$, $d > 0$. Different characterizations of the Sierpinski set arise with different choices of a, c, d.

If $\mathcal{T}$ is the triangle with vertices $v_{_L} = (0,0)$, $v_{_R} = (1,0)$, and $v_{_T} = (0,1)$ its Sierpinski points are, in dyadic notation, $s = (x, 1-x)$, i.e. if the abcissa is $x = 0.\nu_1\nu_2\nu_3\cdots$, the k-th digit of the ordinate is $1-\nu_k$ — for instance, $s = (0.11001011101\ldots, 0.00110100010\ldots)$, cf. Peitgen *et al.* [3], p. 173.

Let $\mathcal{T}$ be the equilateral triangle with unit height, vertices $v_{_L} = (0,0)$, $v_{_R} = (2\sqrt{3}/3, 0)$, and $v_{_T} = (\sqrt{3}/3, 1)$. Schroeder [4], pp. 22–24, used a sophisticated redundant three-coordinates points affixation to show that the Sierpinski points are those with coordinates (in dyadic expansion) $x = 0.a_1a_2a_3\cdots$, $y = 0.b_1b_2b_3\cdots$, $z = 0.c_1c_2c_3\cdots$, such that $(a_k, b_k, c_k) \in \{(1,0,0), (0,1,0), (0,0,1)\}$, $k = 1, 2, \ldots$

For our purposes it is more convenient to consider that $\mathcal{T}$ is the equilateral triangle with unit sides, with top vertex $A = (1/2, \sqrt{3}/2)$, left vertex $B = (0,0)$, and right vertex $C = (1,0)$. Project A in the point $A' = (1/3, 0)$, B in $B' = (5/6, \sqrt{3}/6)$, and C in $C' = (1/3, \sqrt{3}/3)$.

We claim that the points

- $V_1 = (3/7, 2\sqrt{3}/7)$, intersection of $\overline{AA'}$ with $\overline{CC'}$,
- $V_2 = (5/14, \sqrt{3}/14)$, intersection of $\overline{AA'}$ with $\overline{BB'}$, and
- $V_3 = (5/7, \sqrt{3}/7)$, intersection of $\overline{BB'}$ with $\overline{CC'}$,

are Sierpinski points, cf. Figure 2.

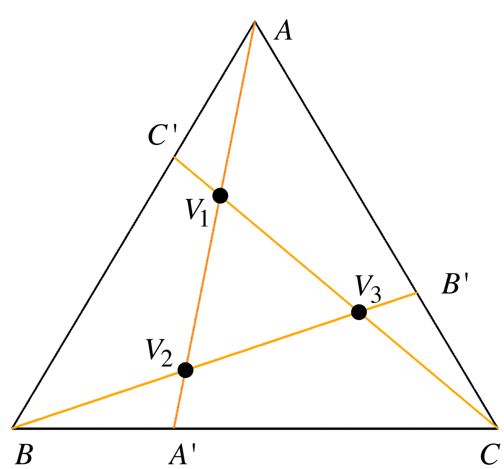

Fig. 2. Period-3 invariant Sir Pinski $\{V_1, V_2, V_3\}$ attractor.

In fact, V_1 is the midpoint of $\overline{AV_2}$, V_2 is the midpoint of $\overline{BV_3}$, V_3 is the midpoint of $\overline{CV_1}$, and therefore those points are winning points in the Sir Pinski game, i.e. $\{V_1, V_2, V_3\}$ is an invariant cycle-3 attractor of Sierpinski points.

Project A in the point $A'' = (2/3, 0)$, B in $B'' = (2/3, \sqrt{3}/3)$, and C in $C'' = (1/6, \sqrt{3}/6)$. Obviously, intersecting $\overline{AA''}$ with $\overline{BB''}$ we obtain $W_1 = (4/7, 2\sqrt{3}/7)$, intersecting $\overline{AA''}$ with $\overline{CC''}$ we obtain $W_2 = (9/14, \sqrt{3}/14)$, and intersecting $\overline{BB''}$ with $\overline{CC''}$ we obtain $W_3 = (2/7, \sqrt{3}/7)$. As it happens with $\{V_1, V_2, V_3\}$, for similar reasons, $\{W_1, W_2, W_3\}$ is an invariant cycle-3 attractor of Sierpinski points, cf. Figure 3.

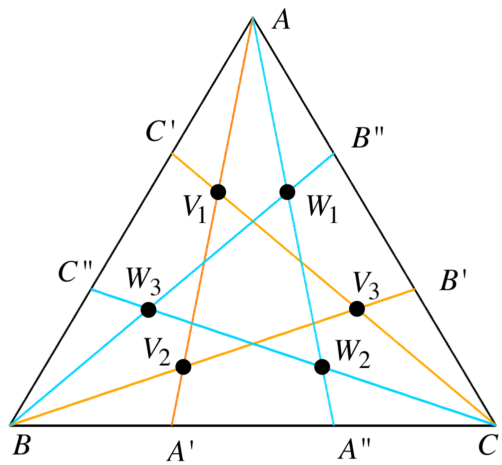

Fig. 3. Period-3 $\{V_1, V_2, V_3\}$ and $\{W_1, W_2, W_3\}$ invariant Sir Pinski points attractors. $\{A\}$, $\{B\}$ and $\{C\}$ are invariant in Sir Pinski game; $\{A', A''\}$, $\{B', B''\}$ and $\{C', C''\}$ are period-2 invariant sets in Sir Pinski game.

Remark 1. If we re-scale multiplying by $2/\sqrt{3}$ in order to have unit heights (i.e., each vertex is at distance 1 from the opposite side), the ordinates of the transformed V_1^* and W_1^* become $4/7$, the ordinates of the transformed V_2^* and W_2^* become $1/7$, and the ordinates of the transformed V_3^* and W_3^* become $2/7$.

Hence, if we adhere to Schroeder [4] three-coordinates system (x, y, z), where x is the distance from the bottom side, y the distance from the left side, and z the distance from the right side, we see that the period-3 invariant points must have x-coordinate $4/7$, $1/7$ or $2/7$.

From the $(2\pi/3)$-rotational symmetry of $\mathcal{T}$, it follows that in Schroeder's three coordinates system

$V_1^* = (4/7, 1/7, 2/7)$, $V_2^* = (1/7, 2/7, 4/7)$, $V_3^* = (2/7, 4/7, 1/7)$, $W_1^* = (4/7, 2/7, 1/7)$, $W_2^* = (1/7, 4/7, 2/7)$, and $W_3^* = (2/7, 1/7, 4/7)$. $\square$

Remark 2. The points $V_1, V_2, V_3, W_1, W_2, W_3$ lie on a circumference of radius $\sqrt{21}/21$ centered at the barycenter $(1/2, \sqrt{3}/6)$ of $\mathcal{T}$. $\square$

Remark 3. Each vertex of $\mathcal{T}$ is invariant in Sir Pinski game. Hence $A, B, C \in \mathcal{S}$. On the other hand, in Sir Pinski game, the image of A' is A'' and vice-versa, i.e. $\{A', A''\}$ is a period-2 invariant set, and the same holds for $\{B', B''\}$ and $\{C', C''\}$. $\mathcal{V} = \{V_1, V_2, V_3\}$ and $\mathcal{W} = \{W_1, W_2, W_3\}$ are period-3 invariant sets (attractors) in Sir Pinski game.

Higher order periodic invariant sets do exist. For instance, using conditions $(a{-}1/2)^2{+}(b{-}\sqrt{3}/2)^2 = 4[(2a{-}1/2)^2{+}(2b{-}\sqrt{3}/2)^2]$ and $(2b - \sqrt{3}/2)/(2a - 1/2) = (\sqrt{3}/2 - b)/(a - 1/2)$ on the set of points $\{(a, b), (2a, 2b), (1 - a, b), (1 - 2a, 2b)\}$, so that $(a, b) = (0.3, 0.288675)$, we obtain the period-4 invariant set $\{(0.3, 0.288675), (0.6, 0.636194), (0.7, 0.288675), (0.4, 0.636194)\}$, cf. Figure 4.

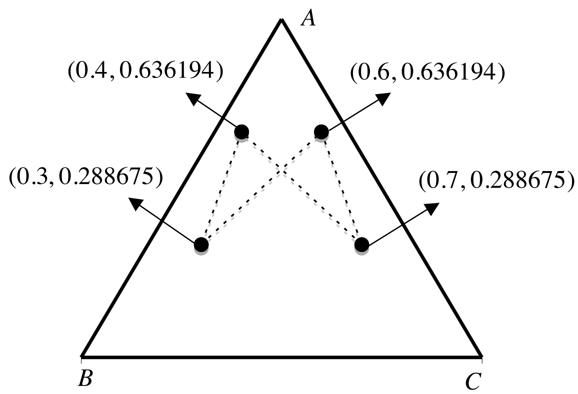

Fig. 4. A period-4 invariant Sir Pinski set.

Using the $(2\pi/3)$-rotational symmetry of $\mathcal{T}$, two other period-4 invariant sets are readily obtained.

More complex conditions may be used to generate other periodicity invariant sets. $\square$

Now we perform the same construction in the T_1 (Top), L_1 (Left) and R_1 (Right) triangles remaining once the middle triangle of $\mathcal{T}$ is removed in step 1 of the classical construction of the Sierpinski gasket, obtaining 2×3^2 points — 3^2 Vs and 3^2 Ws — , as shown in Figure 5. With the self-explaining addressing and notations $V_{i,\overrightarrow{L_1}}, W_{i,\overrightarrow{L_1}}$, $i = 1, 2, 3$, it is obvious that $V_{i,\overrightarrow{L_1}} = \frac{1}{2} V_i$ and $W_{i,\overrightarrow{L_1}} = \frac{1}{2} W_i$ — for instance, $V_{2,\overrightarrow{L_1}} = (5/28, \sqrt{3}/28)$, $V_{1,\overrightarrow{L_1}} = (4/14, 2\sqrt{3}/14)$.

Analogously, the corresponding points in the Right triangle R_1 are $V_{i,\overrightarrow{R_1}} = (1/2, 0) + \frac{1}{2} V_i$ and $W_{i,\overrightarrow{R_1}} = (1/2, 0) + \frac{1}{2} W_i$, and the corresponding points in the Top triangle T_1 are $V_{i,\overrightarrow{T_1}} = (1/4, \sqrt{3}/4) + \frac{1}{2} V_i$ and $W_{i,\overrightarrow{T_1}} = (1/4, \sqrt{3}/4) + \frac{1}{2} W_i$. For instance, $V_{1,\overrightarrow{T_1}} = (13/28, 11\sqrt{3}/28)$.

The 3^2 V points in this second stage of the construction are, following the above algorithm, $(3/14, \sqrt{3}/7)$, $(5/7, \sqrt{3}/7)$, $(13/28, 11\sqrt{3}/28)$, $(5/28, \sqrt{3}/28)$, $(19/28, \sqrt{3}/28)$, $(3/7, 2\sqrt{3}/7)$, $(5/14, \sqrt{3}/14)$, $(6/7, \sqrt{3}/14)$, $(1/28, 9\sqrt{3}/28)$

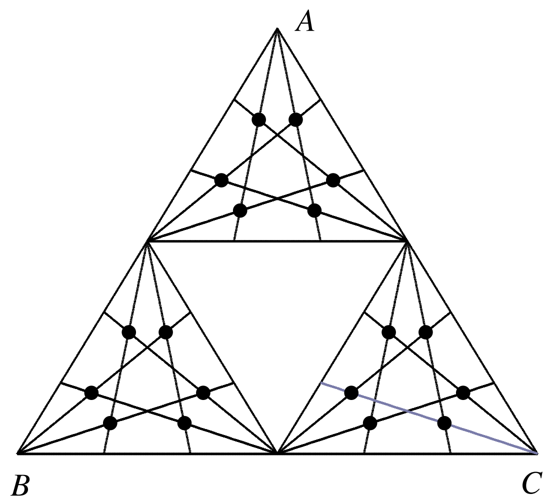

Fig. 5. More Sierpinski points, in T_1, in L_1 and in R_1.

— exactly the 9 points we obtain when we compute the middle point of the segments joining each of the $(3/7, 2\sqrt{3}/7)$, $(5/14, \sqrt{3}/14)$, $(5/7, \sqrt{3}/7)$ V points from stage one of the construction with each of the three vertices of $\mathcal{T}$. Similar results hold in what concerns W points.

Continuing the procedure, in step 3 of the iterative construction of Sierpinski's gasket we obtain 2×3^3 points as shown in Fig. 6. (We have included some extra segments connecting points to make clear that in Sir Pinski game whatever the initial V point [respectively, W point], in a few steps we shall land in the attractor $\mathcal{V} = \{V_1, V_2, V_3\}$ [respectively, in $\mathcal{W} = \{W_1, W_2, W_3\}$].)

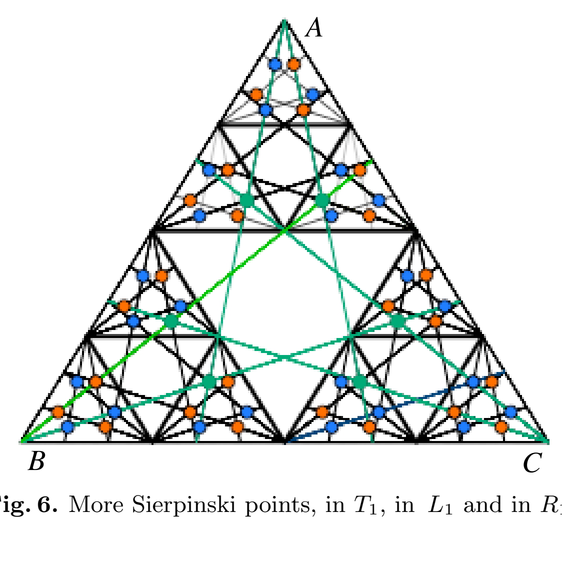

Fig. 6. More Sierpinski points, in T_1, in L_1 and in R_1.

Once again the coordinates of any V or W point are easy to compute. For instance $W_{1,\overrightarrow{L_1T_2}} = (1/8, \sqrt{3}/8) + (1/2)^2 \times (4/7,\ 2\sqrt{3}/7) = (9/56, 11\sqrt{3}/56)$, since the left vertex of the triangle whose address is $\overrightarrow{L_1T_2}$ is $(1/8, \sqrt{3}/8)$.

Using the same line of reasoning, the $V_{i,\overrightarrow{R_1T_2R_3}}$ points of $\overrightarrow{R_1T_2R_3}$ have coordinates $(3/4, \sqrt{3}/8) + (1/2)^3\, V_i$, the $W_{i,\overrightarrow{R_1L_2T_3T_4}}$ points of $\overrightarrow{R_1L_2T_3T_4}$ have coordinates $(13/16, (13/16)(\sqrt{3}/2)) + (1/2)^4\, W_i$. More generally,

- in step n, the coordinates of the original Vs and Ws are scaled by a factor $(1/2)^n$;
- the address determines the left vertex of the triangle: a $\overrightarrow{L_k}$ does not affect neither the abcissa nor the ordinate, a $\overrightarrow{R_k}$ shifts the left corner $(1/2)^k$ and does not affect the ordinate, and a $\overrightarrow{T_k}$ adds $(1/4)^k$ to the abcissa and $(1/2)^k\sqrt{3}/2$ to the ordinate.

For instance, the left corner of $\overrightarrow{T_1L_2L_3R_4R_5T_6}$ is $(1/4 + (1/2)^4 + (1/2)^5 + (1/4)^6, (1/2 + (1/2)^6)\,(\sqrt{3}/2)) = (1409/4096, 33\sqrt{3}/128)$. Hence, the Sierpinski point $W_{3,\overrightarrow{T_1L_2L_3R_4R_5T_6}}$ is $(1409/4096, 33\sqrt{3}/128) + (1/2)^6(2/7, \sqrt{3}/7) = (10119/28672, 233\sqrt{3}/896)$.

Remark 4. Suppose that in the k-th step of the iterative deterministic construction of the Sierpinski gasket we focus our attention in one of the remaining triangles, for instance $\overrightarrow{T_1R_2R_3T_4\cdots L_k}$.

- The midpoints of the segments whose endpoints are the vertex A and the points of $\overrightarrow{T_1R_2R_3T_4\cdots L_k}$ are the points of $\overrightarrow{T_1T_2R_3R_4T_5\cdots L_{k+1}}$.
- The midpoints of the segments whose endpoints are the vertex B and the points of $\overrightarrow{T_1R_2R_3T_4\cdots L_k}$ are the points of $\overrightarrow{L_1T_2R_3R_4T_5\cdots L_{k+1}}$.
- The midpoints of the segments whose endpoints are the vertex C and the points of $\overrightarrow{T_1R_2R_3T_4\cdots L_k}$ are the points of $\overrightarrow{R_1T_2R_3R_4T_5\cdots L_{k+1}}$.

Hence, the chaos game transforms the V points [respectively, the W points] of $\overrightarrow{T_1R_2R_3T_4\cdots L_k}$ in V points [respectively, W points] of either $\overrightarrow{T_1T_2R_3R_4T_5\cdots L_{k+1}}$, or $\overrightarrow{L_1T_2R_3R_4T_5\cdots L_{k+1}}$ or $\overrightarrow{R_1T_2R_3R_4T_5\cdots L_{k+1}}$. □

It seems useless to elaborate more on this matter to conclude that:

- In the k-th step of the classical construction of the Sierpinski gasket we may explicitly compute the coordinates of 3 V points and of 3 W points in each remaining triangle.
- The midpoint of any V point [respectively, W point] and any vertex of $\mathcal{T}$ is a V point [respectively, a W point]. In other words, in the chaos game the set of V points and the set of W points do not communicate.
- In Sir Pinski game, a V starting point generates iterates that ultimately will land in $\mathcal{V}$, and a W starting point generates iterates that ultimately will land in $\mathcal{W}$. Hence all V and W points are winning points of the Sir Pinski game, i.e. they lie in $\mathcal{S}$. We say that V points [respectively, W points] are in the attraction domain of $\mathcal{V}$ [respectively, of $\mathcal{W}$], or that $\mathcal{V}$ and $\mathcal{W}$ are invariant periodicity-3 attractors in Sir Pinski game.

Remark 5. We also observe that subsets of 3 V points and 3 W points lie in circumferences centered at the barycenter of $\mathcal{T}$, cf. Fig. 7. □

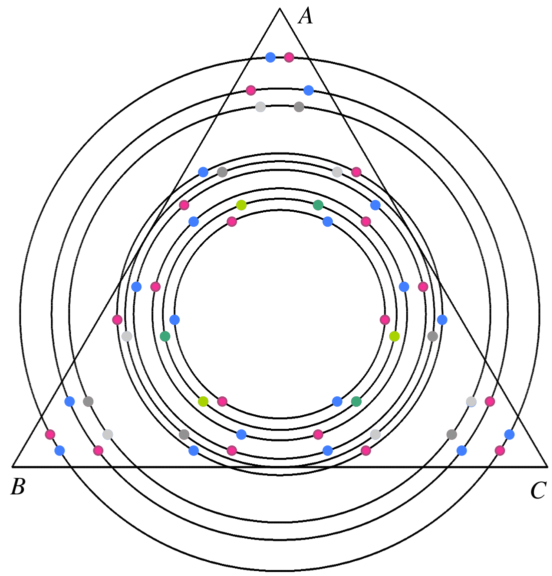

Fig. 7. A consequence of the $\frac{2\pi}{3}$-rotational symmetry of $\mathcal{S}$

3 Concluding Remarks

3.1 The Chaos Game Does Not Generate the Sierpinski Gasket

Under the heading “Randomness Creates Deterministic Shapes”, Peitgen *et al.* [3], p. 299, raise some interesting questions. The discussion in the previous section patently shows that the chaos game does not generate the Sierpinski gasket.

More precisely, if the starting point P_0 is not a Sierpinski point, its descendants are not Sierpinski points, and eventually some of them computed in the initial steps are clearly spurious specks observed upon close scrutiny of the images. The set looks like the Sierpinski gasket, because the composition of contractions creates something that is very close to the Sierpinski gasket, but its intersection with the Sierpinski gasket $\mathcal{S}$ is void.

On the other hand, our discussion shows that sets generated by the chaos game starting with a Sierpinski V point and with a Sierpinski W point are disjoint. Moreover, any of them leaves out points in the domain of attraction of invariant attractors with periodicities other than 3.

So, even with a carefully selected Sierpinski point in any of those invariant sets, the best we can get applying the chaos game is a rarefied pale image of the rich complexity of the Sierpinski gasket. The gross imperfection of the representation of points and our eyes trick us in believing we are generating the Sierpinski gasket. In fact, the representation we get is as innacurate as the representation we get after a finite number of steps of removal of middle triangles, in the classical deterministic construction.

3.2 The Chaos Game and Sierpinski Polygons

The Sierpinski gasket is the sole fixed point resulting from $\frac{1}{2}$ contractions towards the vertices A, B and C of a triangle; for aesthetic reasons, in most situations it is worked out using an equilateral triangle.

We now consider contractions on regular polygons with $n > 3$ vertices. Pick a point at random inside the polygon, and then draw the next point a fraction of the distance between it and a polygon vertex picked at random. Continue the process (after eventually throwing out the first few points). The result of this "chaos game" is sometimes, but not always, a fractal. In Fig. 8 we show the result of the contractions $r_1 = \frac{1}{3}$ and $r_1 = \frac{3}{8}$ towards the vertices of a regular pentagon.

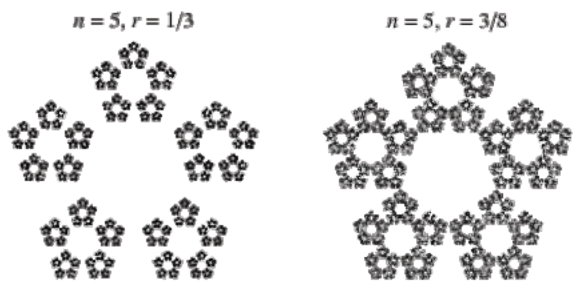

Fig. 8. Contractions $r_1 = \frac{1}{3}$ and $r_1 = \frac{3}{8}$ towards the vertices of a regular pentagon.

It is obvious that greater scaling factors $s = \frac{1}{r}$ will originate "islands", and smaller scaling factors can create overcrowded sets, with overlapping. Consequently we must refine our original definition, so that the union of contractions creates the richest fixed point without overlapping. Looking at what happens in what regards hexagons and decagons, see Fig. 9, it is easy to conclude that the ideal scaling factor for a n vertices regular polygon is

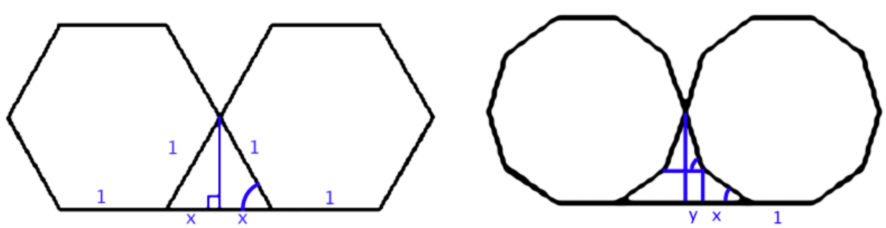

Fig. 9. Geometric rationale for computing the appropriate scaling factor.

$$s = 2 \sum_{k=0}^{\left[\frac{n}{4}\right]} \cos \frac{2\,\pi\,k}{n}.$$

(where $\left[\frac{n}{4}\right]$ denotes the integer part of $\frac{n}{4}$) — in particular, for the regular pentagon, $s = 2\,(1 + \cos\frac{4\,\pi}{5}) \approx 2.618033989$, and hence the ideal contraction is $r = \frac{1}{s} \approx 0.381966011$.

Once again the chaos game — generating the next point as the weighted mean of the current point and a vertex selected at random (i.e., using a discrete uniform law), with weights $\frac{1}{r}$ and $1-\frac{1}{r}$ — gives a hint of aspect of the resulting fractal.

For instance, Fig. 10 exhibits the result of 25,000 runs of the chaos game associated with the Sierpinski octogon, generated using in R the source file

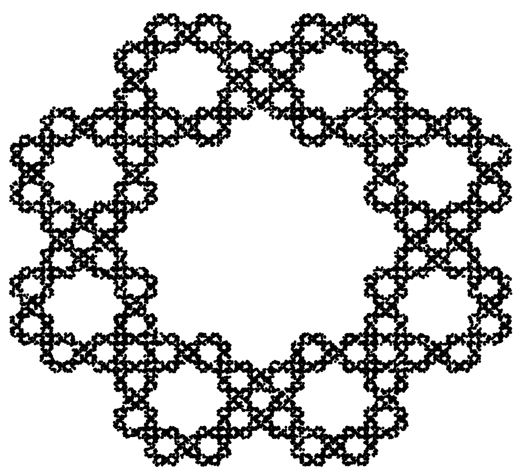

Fig. 10. Chaos game associated with Sierpinski octogon.

with the instructions

```
   #   Sierpinski  Octogon
# 2011/01/14

######################
#######################

cat("Number of runs?")
nruns<-scan(n=1)
cat("Initial point abcissa?")
x<-scan(n=1)
cat("initial point ordinate?")
y<-scan(n=1)

u<-runif(1)
if (u <= 1/8)
{
x<-0.29289322*x
```

```
y<-0.29289322*y
}
if (u>1/8 & u<=1/4)
{
x<-(0.29289322*x+0.707106781)
y<-(0.29289322*y)
}
if (u>1/4 & u<=3/8)
{
x<-(0.29289322*x+0.707106781*1.70710678 )
y<-(0.29289322*y+0.707106781*0.70710678)
}
if (u>3/8 & u<=1/2)
{
x<-(0.29289322*x+0.707106781*1.70710678 )
y<-(0.29289322*y+0.707106781*1.70710678)
}
if (u>1/2 & u<=5/8)
{
x<-(0.29289322*x+ 0.707106781 )
y<-(0.29289322*y+0.707106781*2.41421356)
}
if (u>5/8 & u<=3/4)
{
x<-(0.29289322*x  )
y<-(0.29289322*y+0.707106781*2.41421356)
}
if (u>3/4 & u<=7/8)
{
x<-(0.29289322*x-0.707106781*0.70710678 )
y<-(0.29289322*y+0.707106781*1.70710678)
}
if (u>7/8)
{
x<-(0.29289322*x-0.707106781*0.70710678 )
y<-(0.29289322*y+0.707106781*0.70710678)
}

plot(x,y,xlim=c(-1,2),ylim=c(0,3),pch=20, cex=0.2,
xaxt="n",yaxt="n",xlab="",ylab="",bty="n")

for(i in 1:nruns)
{
u<-runif(1)

if (u <= 1/8)
{
```

```
x<-0.29289322*x
y<-0.29289322*y
}
if (u>1/8 & u<=1/4)
{
x<-(0.29289322*x+0.707106781)
y<-(0.29289322*y)
}
if (u>1/4 & u<=3/8)
{
x<-(0.29289322*x+0.707106781*1.70710678 )
y<-(0.29289322*y+0.707106781*0.70710678)
}
if (u>3/8 & u<=1/2)
{
x<-(0.29289322*x+0.707106781*1.70710678 )
y<-(0.29289322*y+0.707106781*1.70710678)
}
if (u>1/2 & u<=5/8)
{
x<-(0.29289322*x+ 0.707106781 )
y<-(0.29289322*y+0.707106781*2.41421356)
}
if (u>5/8 & u<=3/4)
{
x<-(0.29289322*x  )
y<-(0.29289322*y+0.707106781*2.41421356)
}
if (u>3/4 & u<=7/8)
{
x<-(0.29289322*x-0.707106781*0.70710678 )
y<-(0.29289322*y+0.707106781*1.70710678)
}
if (u>7/8)
{
x<-(0.29289322*x-0.707106781*0.70710678 )
y<-(0.29289322*y+0.707106781*0.70710678)
}

points(x,y,pch=20,cex=0.25)
for (j in 1:25000) a=1
}
```

4e (we used the approximation $r = 2 + \sqrt{2} \approx 3.414213562$ for the scaling factor, and the weights 0.292893219 for the current point and 0.707106781 for the randomly chosen vertex in order to compute weighted means at each step). This code is easily modified for any n, using the appropriate scaling factor.

Once again, for any n this generates a subset or an approximation of the fixed point of an Hutchinson operator which is the union of contractions towards each of the vertices of the polygon, with appropriate scaling factor, but is not, in fact, the (extended) Sierpinski fixed point.

References

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FCT *This reseach has been supported by National Funds through FCT — Fundação para a Ciência e a Tecnologia, project PEst-OE/MAT/UI0006/2011, and PTDC/FEDER.*