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Some considerations on the usefulness to approximate a Markov chain by a solution of a stochastic differential equation

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Abstract. In this paper some problems regarding to the Markov property are discussed. Among other things, it is presented the extended Markov property, as it has been synthetized by Kiyosi Itô. Also, it will be emphasized the possibility to approximate a Markov chain by a solution of a stochastic differential equation in a problem of financial risk.
Keywords: Brownian motion, stochastic differential equations, Markov process, transition probabilities .

1 Introduction

As it can be observed, in the last time, a great interest has been shown to some topics relating to stochastic approximation procedures and their applications.

Generally speaking, it can be considered a problem where computation is split among several processors, operating and transmitting data to one another asynchronously. Such algorithms are only being to come into prominence, due to both the developments of decentralized processing and applications where each of several locations migth control or adjusted *local variable* but the criterion of concern is global. For example a current *decentralized* application is in Q-learning where the component update at any time depends on the state of a Markov process.

After Robbins & Monro laid the foundations of the stochastic approximations procedures, several problems have been developed especially by Z. Schuss, H.J. Kushner, K. Itô, H.P. McKean Jr., M.T. Wasan, B. Øksendal, N. Ikeda, S. Watanabe. Results on almost sure convergence of stochastic approximation processes are often proved by a separation of deterministic (pathwise) and stochastic considerations. The basic idea is to show that a "distance" between estimate and solution itself has the tendency to become smaller.

In the last decades a great interest has been shown to the investigations of applications in many diverse areas, and this has accelerated in the last time, with new applications. Shortly speaking, the basic stochastic approximation

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algorithm is nothing but a stochastic difference equation with a small step size, and the basic questions for analysis concern its qualitative behaviour over a long time interval, such as convergence and rate of convergence.

When a stochatic differential equation is considered if it is allowed for some randomness in some of its coefficients, it will be often obtained a so-called *stochastic differential equation* which is a more realistic mathematical model of the considered situation. Many practical problems conduct us to the following notion: *the equation obtained by allowing randomness in the coefficients of a differential equation is called a "stochastic differential equation".* Thus, it is clear that any solution of a stochastic differential equation must involve some randomness. In other words one can hope to be able to say something about the probability distribution of the solutions.

On the other hand, as it is known, a precise definition of the Brownian motion involves a measure on the path space, such that it is possible to put the Brownian motion on a firm mathematical foundation. Numerous scientific works has been done on its applications in diverse areas including among other things stability of structures, solid-state theory, population genetics, communications, and many other branches of the natural sciences, social sciences and engineering. We emphasize here many contributions due to P. Lévy, K. Itô, H.P. McKean, Jr., S. Kakutani, H.J. Kushner, A.T. Bharucha-Reid and other.

If we refer, for example, to some aspects in genetics, as the approximation of Markov chains by solutions of some stochastic differential equations to determine the probability of extinction of a genotype, then the Markovian nature of the problem will be pointed out, and we think that this is a very important aspect.

In this paper we shall discuss firstly some aspects relating to the approximation in the study of Markov processes and Brownian motion. Such problems were developed particularly by Z. Schuss, H.J. Kushner, K. Itô, H.P. McKean Jr., B. Øksendal, M.T. Wasan.

Then we refer to some aspects regarding to the Markov property in a vision of K. Itô. And finally, as an application, a problem of stochastic approximation in the risk analysis, based on a study of of Hu Yaozhong, in connection with a stochastic differential equation, is considered.

2 In short about stochastic differential equations

We know that to describe the motion of a particle driven by a *white noise* type of force (due to the collision with the smaller molecules of the fluid) the following equation is used

$$\frac{d\mathbf{v}(t)}{dt} = -\beta\mathbf{v}(t) + \mathbf{f}(t) \tag{1}$$

where $\mathbf{f}(t)$ is the white noise term.

The equation (1) is referred to as the *Langevin's equation* . Its solution is the following

$$y(t) = y_0 e^{-\beta t} + e^{-\beta t}\int_0^t e^{-\beta t}\mathbf{f}(s)ds. \tag{2}$$

If we denote by $\mathbf{w}(t)$ the Brownian motion (see the next section), then it is given by

$$\mathbf{w}(t) = \frac{1}{q}\int_0^t \mathbf{f}(s)ds, \tag{3}$$

so that $\mathbf{f}(s) = \dfrac{qd\mathbf{w}(s)}{ds}$. But $\mathbf{w}(t)$ is nowhere differentiable, such that $\mathbf{f}(s)$ is not a function. Therefore, the solution (2), of Langevin's equation, is not a well-defined function. This difficulty can be overcome, in the simple case, as follows. Integrating (2) by parts, and using (3), it results

$$y(t) = y_0 e^{-\beta t} + q\mathbf{w}(t) - \beta q \int_0^t e^{-\beta(t-s)}\mathbf{w}(s)ds. \tag{4}$$

But all functions in (4) are well defined and continuous, such that the solution (3) can be interpreted by giving it the meaning of (4). Now, such a procedure can be generalized in the following way. Let us given two functions $f(t)$ and $g(t)$ that are considered to be defined for $a \leq t \leq b$. For any partition $P : a \leq t_0 < t_1 < \cdots < t_n$, we denote

$$S_P = \sum_{i=1}^{n} f(\xi_i)[g(t_i) - g(t_{i-1})],$$

where $t_{i-1} \leq \xi_i \leq t_i$. If a limit exists

$$\lim_{|P|\to 0} S_P = I$$

where $|P| = \max_{1\leq i\leq n}(t_i - t_{i-1})$, then it is said that I is the *Stieltjes integral* of $f(t)$ with respect to $g(t)$. It is denoted

$$I = \int_a^b f(t)dg(t).$$

Now the stochastic differential equation

$$\begin{aligned} &dx(t = a(x(t),t)dt + b(x(t),t)dw(t) \\ &x(0) = x_0 \end{aligned} \tag{5}$$

is defined by the Itô integral equation

$$x(t) = x_0 + \int_0^t a(x(s),s)ds + \int_0^t b(x(s),s)dw(s). \tag{6}$$

The simplest example of a stochastic differential equation is the following equation

$$\begin{aligned} &dx(t) = a(t)dt + b(t)dw(t) \\ &x(0) = x_0 \end{aligned} \tag{7}$$

which has the solution

$$x(t) = x_0 + \int_0^t a(s)ds + \int_0^t b(s)dw(s).$$

The *transition probability density* of $x(t)$ is a function $p(x, s; y, t)$ satisfying the condition

$$P(x(t) \in A \,|\, x(s) = x) = \int_A p(x, s; y, t)dy$$

for $t > s$ where A is any set in R. It is supposed that $a(t)$ and $b(t)$ are deterministic functions.

The stochastic integral

$$\chi(t) = \int_0^t b(s)dw(s)$$

is a limit of linear combinations of independent normal variables

$$\sum_i b(t_i)[w(t_{i+1}) - w(t_i)].$$

Thus, the integral is also a normal variable.

But, then

$$\chi(t) = x(t) - x_0 - \int_0^t a(s)ds$$

is a normal variable, and therefore

$$p(x, s; y, t) = \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{(y-m)^2}{2\sigma}}$$

where

$$m = E(x(t) \,|\, x(s) = x).$$

Now

$$E(x(t) \,|\, x(s) = x) = x + \int_s^t a(u)du$$

is the expectation of the stochastic integral vanishes.

And the variance is given by the relation

$$\sigma = Var\, x(t) = E\left[\int_s^t b(u)dw(u)\right]^2 = \int_s^t b^2(u)du.$$

In conclusion, $p(x, s; y, t)$ is given by the following equation

$$p(x, s; y, t) = \left[2\pi \int_s^t b^2(u)du\right]^{-\frac{1}{2}} \cdot e^{-\frac{\left(y - x - \int_s^t a(u)du\right)^2}{2\int_s^t b^2(u)du}}.$$

[For more details and proofs see, for example: G. Da Prato and J. Zabczyk[3], G. Da Prato[4], N. Ikeda and S. Watanabe[6], K. Itô and H. P. McKean Jr.[8], B. Øksendal[13], Z. Schuss[26]].

3 Brownian motion

Brownian motion, used especially in Physics, is of ever increasing importance not only in Probability theory but also in classical Analysis. Its fascinating properties and its far-reaching extension of the simplest normal limit theorems to functional limit distributions acted, and continue to act, as a catalyst in random Analysis. It is probable the most important stochastic process. As some authors remarks too, the Brownian motion reflects a perfection that seems closer to a law of nature than to a human invention.

In 1828 the English botanist Robert Brown observed that pollen grains suspended in water perform a continual swarming motion. The chaotic motion of such a particle is called *Brownian motion* and a particle performing such a motion is called a *Brownian particle.*

The first important applications of Brownian motion were made by L. Bachélier and A. Einstein. L. Bachélier derived (1900) the law governing the position of a single grain performing a 1-dimensional Brownian motion starting at $a \in R^1$ at time $t = 0$

$$P_a[x(t) \in db] = g(t, a, b)db \tag{8}$$

where $(t, a, b) \in (0, +\infty) \times R^2$ and g is the Green (or the *source*) function

$$g(t, a, b) = \frac{1}{t\sqrt{2\pi}}\, e^{-\frac{(b-a)^2}{2t^2}}$$

of the problem of heat flow

$$\frac{\partial u}{\partial t} = \frac{1}{2}\,\frac{\partial^2 u}{\partial^2 a}\,, \qquad (t > 0).$$

Bachélier also pointed out the Markovian nature of the Brownian path but he was unable to obtain a clear picture of the Brownian motion and his ideas were unappreciated at that time. This because a precise definition of the Brownian motion involves a measure on the path space, and it was not until 1908-1909 when the works of É. Borel and H. Lebesgue have been appeared. Beginning with this moment was possible to put the Brownian motion on a firm mathematical foundation and this was achived by N. Wiener in 1923.

It is very interesting that A. Einstein also derived (8) in 1905 from statistical mechanical considerations and applied it to the determination of molecular diameters. He wanted also to model the movement of a particle suspended in a liquid. Einstein's aim was to provide a means of measuring Avogadro's number, the number of molecules in a mole of gas, and experiments suggested by Einstein proved to be consistent with his predictions.

We remind, for example, the following aspect. Let us consider that $\mathbf{x}(t)$ is the notation for the displacement of the Brownian particle. Then, the probability density of this displacement, for sufficiently large values of t, is as follows

$$p(\mathbf{x}, t, \mathbf{x}_0, \mathbf{v}_0) \approx \frac{1}{(4\pi Dt)^{\frac{3}{2}}}\, e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{4Dt}} \tag{9}$$

where D is

$$D = \frac{kT}{m\beta} = \frac{kT}{6\pi a\eta} \tag{10}$$

and is referred to as the *diffusion coefficient*.

Furthermore it results that $p(\mathbf{x}, t, \mathbf{x}_0, \mathbf{v}_0)$ satisfies the diffusion equation given below

$$\frac{\partial p(\mathbf{x}, t, \mathbf{x}_0, \mathbf{v}_0)}{\partial t} = D\Delta p(\mathbf{x}, t, \mathbf{x}_0, \mathbf{v}_0). \tag{11}$$

The expression of D in (10) was obtained by A. Einstein.

Remark 1. From physics it is known the following result due to Maxwell: Let us suppose that the energy is proportional to the number of particles in a gas and let us denoted $E = \gamma n$, where γ is a constant independent of n. Then,

$$P\{a < v_i^1 < b\} = \frac{\int\limits_a^b \left(1 - \frac{x^2 m}{2\gamma n}\right)^{\frac{3n-3}{2}} dx}{\int\limits_{-\left(\frac{2\gamma n}{m}\right)^{\frac{1}{2}}}^{+\left(\frac{2\gamma n}{m}\right)^{\frac{1}{2}}} \left(\frac{1 - x^2 m}{2\gamma n}\right)^{\frac{3n-3}{2}} dx} \to$$

$$\to \left(\frac{3m}{4\pi\gamma}\right)^{\frac{1}{2}} \int\limits_a^b e^{-\frac{3mx^2}{4\gamma}} dx.$$

Now, for $\gamma = \dfrac{3kT}{2}$ the following Maxwell's result is found

$$\lim_{n\to\infty} P\{a < v_i^1 < b\} = \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \int\limits_a^b e^{-\frac{mx^2}{2kT}} dx.$$

T is called the "absolute temperature", while k is the "Boltzmann's constant".

[For details and proofs see K. Itô and H. P. McKean Jr.[8], Z. Schuss[26], D. W. Stroock[27], G. V. Orman[20]].

4 On Markov property

In some previous papers we have dicussed on Markov processes in a vision of Kiyosi Itô. In this section we shall continue this discussion by considering the extended Markov property.

More details and other aspects can be found in K. Itô[7],[9], K. Itô and H. P. McKean Jr.[8], D. W. Stroock[27], A. T. Bharucha-Reid[2].

Let S be a *state space* and consider a particle which moves in S. Also, suppose that the particle starting at x at the present moment will move into the set $A \subset S$ with probability $p_t(x, A)$ after t units of time, “irrespectively of its past motion", that is to say, this motion is considered to have a *Markovian character*.

The *transition probabilities* of this motion are $\{p_t(x, A)\}_{t,x,A}$ and is considered that the time parameter $t \in T = [0, +\infty)$.

The state space S is assumed to be a *compact Hausdorff space with a countable open base*, so that it is homeomorphic with a compact separable metric space by the Urysohn's metrization theorem. The σ-field generated by the open space (the topological σ-field on S) is denoted by $K(S)$. Therefore, a *Borel set* is a set in $K(S)$.

The *mean value*

$$m = M(\mu) = \int_R x\, \mu(dx)$$

is used for the center and the scattering degree of an one-dimensional probability measure μ having the second order moment finite, and the *variance* of μ is defined by

$$\sigma^2 = \sigma^2(\mu) = \int_R (x - m)^2 \mu(dx).$$

On the other hand, from the Tchebychev's inequality, for any $t > 0$, we have

$$\mu(m - t\sigma, m + t\sigma) \leq \frac{1}{t^2},$$

so that several properties of 1-dimensional probability measures can be derived.

Note that in the case when the considered probability measure has no finite second order moment, σ becomes useless. In such a case one can introduce the central value and the dispersion that will play similar roles as m and σ for general 1-dimensional probability measures.

Remark 2. We recall that J. L. Doob defined the *central value* $\gamma = \gamma(\mu)$ as being the real number γ which verifies the following relation

$$\int_R arctg(x - \gamma)\mu(dx)) = 0.$$

Here, the existence and the uniqueness of γ follows from the fact that $arctg(x - \gamma)$ is continuous and decreases strictly from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$, for x fixed, as γ moves from $-\infty$ to $+\infty$.

The *dispersion* δ is defined as follows

$$\delta = \delta(\mu) = -\log \int\int_{R^2} e^{-|x-y|} \mu(dx)\mu(dy).$$

We will assume that the transition probabilities $\{p_t(x, A)\}_{t \in T, x \in S, A \in K(S)}$ satisfy the following conditions:

(1) for t and A fixed,
 a) the transition probabilities are Borel measurable in x;
 b) $p_t(x, A)$ is a probability measure in A;

(2) $p_0(x, A) = \delta_x(A)$ (i.e. the δ-measure concentrated at x);

(3) $p_t(x,\cdot) \overset{weak}{\longrightarrow} p_t(x_0,\cdot)$ as $x \to x_0$ for any t fixed, that is

$$\lim_{x\to x_0} \int f(y)p_t(x,dy) = \int f(y)p_t(x_0,dy)$$

for all continuous functions f on S;

(4) $p_t(x,U(x)) \longrightarrow 1$ as $t \searrow 0$, for any neighborhood $U(x)$ of x;

(5) the Chapman-Kolmogorov equation holds:

$$p_{s+t}(x,A) = \int_S p_t(x,dy)p_s(y,A).$$

The *transition operators* can be defined in a similar manner. Consider $C = C(S)$ to be the space of all continuous functions (it is a separable Banach space with the supremum norm).

The operators p_t, defined by

$$(p_t f)(x) = \int_S p_t(x,dy)f(y), \quad f \in C$$

are called *transition operators.*

The conditions for the transition probabilities can be adapted to the transition operators, but we do not insist here.

Remark 3. Let us observe that the conditions (1) – (5) above are satisfied for "Brownian transition probabilities". One can define

$$p_t(x,dy) = \frac{1}{t\sqrt{2\pi}}\, e^{-\frac{(y-x)^2}{2t^2}}\, dy \quad \text{in } R$$
$$p_t(\infty,A) = \delta_\infty A.$$

Now the Markov process can be defined.

Definition 1 *A Markov process is a system of stochastic processes*

$$\{X_t(\omega), t\in T, \omega \in (\Omega, K, P_a)\}_{a\in S},$$

that is for each $a \in S$, $\{X_t\}_{t\in S}$ is a stochastic process defined on the probability space (Ω, K, P_a).

The transition probabilities of a Markov process will be denoted by $\{p(t,a,B)\}$. Now let us denote by $\{H_t\}$ the transition semigroup and let R_α be the resolvent operator of $\{H_t\}$.

The next results shows that $p(t,a,B)$, H_t and R_α can be expressed in terms of the process as follows:

Theorem 1 *Let f be a function in $C(S)$. Then*

i) $p(t,a,B) = P_a(X_t \in B)$.

ii) For $E_a(\cdot) = \int_\Omega \cdot P_a(d\omega)$ *one has* $H_t f(a) = E_a(f(X_t))$.

iii) $R_\alpha f(a) = E_a\left(\int_0^\infty e^{-\alpha t} f(X_t)dt\right)$.

Proof. One can observe that i) and ii) follow immediately.

To prove iii), we will use the following equality:

$$R_\alpha f(a) = \int_0^\infty e^{-\alpha t} H_t f(a) dt = \int_0^\infty e^{-\alpha t} E_a(f(H_t))dt.$$

Since $f(X_t(\omega))$ is right continuous in t for ω fixed, and measurable in ω for t fixed, it is therefore measurable in the pair (t, ω). Thus, we can use Fubini's theorem and therefore we obtain

$$R_\alpha f(a) = E_a \left(\int_0^\infty e^{-\alpha t} f(X_t) dt \right),$$

which proves iii).

Definition 2 *The operator $\theta_t : \Omega \to \Omega$ defined by*

$$(\theta_t \omega)(s) = \omega(s + t)$$

for every $s \in T$ is called the "shift operator".

Obviously, the operator θ_t satisfies the property

$$\theta_{t+s} = \theta_t \theta_s,$$

called the *semigroup property.*

For $\mathcal{C}$ a σ-field on Ω, the space of all bounded $\mathcal{C}$-measurable functions will be denoted by $\mathbf{B}(\Omega, \mathcal{C})$, or simple $\mathbf{B}(\mathcal{C})$.

4.1 The classical and the extended Markov property

Now the *Markov property* is expressed in the theorem below.

Theorem 2 *Let be given $\Gamma \in K$. The following is true*

$$P_a(\theta_t \omega \in \Gamma | K_t) = P_{X_t(\omega)}(\Gamma) \quad a.s.(P_a);$$

that is to say

$$P_a(\theta_t^{-1} \Gamma | K_t) = P_{X_t(\omega)}(\Gamma).$$

Remark 4. The following notation can be used

$$P_{X_t(\omega)}(\Gamma) = P_b(\Gamma)|_{b = X_t(\omega)}.$$

Now, to prove the theorem, it will be suffice to show that

$$P_a(\theta_t^{-1} \Gamma \cap D) = E_a(P_{X_t}(\Gamma), D) \tag{12}$$

for $\Gamma \in K$ and $D \in K_t$.

Three cases can be distinguished.

1). Let us consider Γ and D as follows:

$$\Gamma = \{X_{s_1} \in B_1\} \bigcap \{X_{s_2} \in B_2\} \bigcap \cdots \bigcap \{X_{s_n} \in B_n\},$$

and

$$D = \{X_{t_1} \in A_1\} \bigcap \{X_{t_2} \in A_2\} \bigcap \cdots \bigcap \{X_{t_m \in A_m}\}$$

with

$$0 \le s_1 < s_2 < \cdots < s_n$$
$$0 \le t_1 < t_2 < \cdots < t_m \le t$$

and $B_i, A_j \in K(S)$.

Now it will be observed that the both sides in (12) are expressed as integrals on S^{m+n} in terms of transition probabilities. Thus, one can see that they are equal.

2). Let now be Γ as in the case **1)** and let us denote by D a general member of K_t. For Γ fixed the family $\mathcal{D}$ of all D's satisfying (12) is a Dynkin class. If $\mathcal{M}$ is the family of all M's in the case **1)** then, this family is multiplicative and $\mathcal{M} \subset \mathcal{D}$. In this way it follows

$$\mathcal{D}(\mathcal{M}) \subset \mathcal{D} = K(\mathcal{M}) = K_t$$

and one can conclude that, for Γ in the case **1)** and for D general in K_t, the equality (12) holds.

3). (*General case.*) This case can be obtained in a same manner from **2)** by fixing an arbitrary $D \in K_t$.

It will be obtained that $P_a(\Gamma)$ is Borel measurable in a for any $\Gamma \in K$.

Corollaire 1

$$E_a(G \circ \theta_t, D) = E_a(E_{X_t}(G), D) \quad \textit{for } G \in \mathbf{B}(K), D \in K_t,$$
$$E_a(F \cdot (G \circ \theta_t)) = Ea(F \cdot E_{X_t}(G)) \quad \textit{for } G \in \mathbf{B}(K), F \in \mathbf{B}(K_t),$$
$$E_a(G \circ \theta_t | K_t) = E_{X_t}(G) \quad (a.s.)(P_a) \ \textit{for } G \in \mathbf{B}(K).$$

But it is interesting to see that the Markov property can be extended, as it is given in the following theorem, according to K. Itô:

Theorem 3 *(The extended Markov property).*

$$P_a(\theta_t \omega \in \Gamma | K_{t+}) = P_{X_t}(\Gamma) \quad a.s.\ (P_a)$$

for $\Gamma \in K$.

Proof. Let us come back to the equality (12) before. Now it will be proved for $D \in K_{t+}$. To this end the following equality will be shown:

$$E_a(f_1(X_{s_1}(\theta_t \omega)) \cdots f_n(X_{s_n}(\theta_t \omega)), D) =$$
$$= E_a(E_{X_t}(f_1(X_{s_1}) \cdots f_n(X_{s_n})), D) \tag{13}$$

for $f_i \in C(S), D \in K_{t+}$ and $0 \le s_1 < s_2 < \cdots < s_n$.

But $D \in K_{t+h}$ for $h > 0$, so that by Corollary 1 it results

$$E_a(f_1(X_{s_1}(\theta_{t+h} \omega)) \cdots f_n(X_{s_n}(\theta_{t+h} \omega)), D) =$$
$$= E_a(E_{X_{t+h}}(f_1(X_{s_1}) \cdots f_n(X_{s_n})), D). \tag{14}$$

Now one can observe that

$$E_a(f_1(X_{s_1})\cdots f_n(X_{s_n}))$$

is continuous in a, if it is considered that

$$\begin{aligned}&E_a(f_1(X_{s_1})\cdots f_n(X_{s_n})) =\\ &= H_{s_1}(f_1\cdots(H_{s_{n-1}-s_{n-2}}(f_{n-1}\cdot H_{s_n-s_{n-1}}f_n))\cdots)\end{aligned}$$

and $H_s : C \longrightarrow C$.

But $X_t(\omega)$ being right continuous in t, one gets

$$f_i(X_{s_i}(\theta_{t+h}\omega)) = f_i(X_{s_i+t+h}(\omega)) \longrightarrow f_i(X_{s_i+t}(\omega)) = f_i(X_{s_i}(\theta_t\omega))$$

as $h \downarrow 0$.

Now, the equality (13) will result by taking the limit in (14) as $h \downarrow 0$.

In this way, for G_i open in S, the following equality will result from (13)

$$\begin{aligned}&E_a(X_{s_i}(\theta_t\omega)\in G_1,\cdots,X_{s_n}(\theta_t\omega)\in G_n,\, D) =\\ &\quad = E_a(P_{X_t}(X_{s_1}\in G_1,\cdots,X_{s_n}\in G_n),D),\end{aligned} \tag{15}$$

and now the Dynkin's theorem can be used.

Remark 5. Theorem (Dynkin's formula). Let us suppose that σ is a stopping time with $E_a(\sigma) < \infty$. Then, for $u \in \mathcal{D}(A)$ it follows:

$$E_a\left(\int_0^\infty Au(X_t)dt\right) = E_a(u(X_\sigma)) - u(a).$$

5 A problem of financial risk

This section is referred, shortly, to a study of Hu Yaozhong[30] involving the so-called *Onsager-Machlup functional*. This operator is computed for the generalized geometric Brownian motion and also the general equation which the most probable path must satisfy is found. We shall consider only some aspects according to our review G. V. Orman[17].

The most probable path is obtained in a form which permit to conclude about the risk when someone want to invest money into several stocks.

Definition 3 *The solution of the following stochastic differential equation*

$$dx_t = x_t\{a(t)dw_t + b(t)dt\}, \quad 0 < t < \infty, \tag{16}$$

is called the ***geometric Brownian motion****, where $a(t), b(t)$ are deterministic functions of t; w_t is a Brownian motion, and dw_t is the Itô integral.*

Now let us given the following stochastic differential equation

$$\begin{aligned}&dx_t = A(t)\,x_t\,dw(t) + B(t)\,x_t\,dt\\ &x_0 = \xi\end{aligned} \tag{17}$$

where

$$A(t) = \text{diag}\,(a_1(t), \cdots, a_d(t)) = \begin{pmatrix} a_1(t) & 0 & \cdots & 0 \\ 0 & a_2(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_d(t) \end{pmatrix},$$

$B(t) = (b_{ij}(t))$ satisfying $b_{ij}(t) \geq 0$ for all $1 \leq i, j \leq d$, $i \neq j$ and $w(t) = (w_1(t), \cdots, w_d(t))$ are standard Brownian motions which are not necessarily independent.

If $a_1(t), \cdots, a_d(t)$ are continuous functions with bounded derivative one considers

$$B(t) = (b_{ij}(t))_{1 \leq i,j \leq d}$$

where $b_{ij}(t) \geq 0$, $\forall\, i \neq j$.

Denote $A(t) = \text{diag}\,(a_1(t), \cdots, a_d(t))^T$ and consider the stochastic differential equation

$$\begin{aligned} &dx_i(t) = a_i(t)\, x_i(t)\, dw_i(t) + \sum_{j=1}^{d} b_{ij}(t)\, x_j(t)\, dt \\ &x_i(0) = \xi_i \qquad\qquad i = 1, \cdots, d\,. \end{aligned} \tag{18}$$

Or its integral form

$$x_i(t) = \xi_i + \int_0^t a_i(s)\, x_i(s)\, dw_i(s) + \sum_{j=1}^{d} \int_0^t b_{ij}(s)\, x_j(s)\, ds, \quad i = 1, 2, \cdots, d\,. \tag{19}$$

The problem is to perform asymptotic evaluation of the probability

$$P\{ \sup_{0 \leq t \leq T} |\, x(t) - \Phi(t)\,| < \varepsilon \} \quad \text{as} \quad \varepsilon \to \infty\,,$$

where $|\cdot|$ denotes the Euclidian norm in d-dimensional space, and $\Phi : [0, T] \to R$ is a function with continuous and bounded first and second derivatives. [To develop such aspects see, for example, L. Onsager and S. Machlup[15], Y. Takahashi and S. Watanabe[28], O. Zeitouni[31]].

Now we comeback to the geometric Brownian motion. The equation (16) has been successfully applied to the financial problems such as modeling the prices of stocks. For $i = 1, 2, \cdots, d$ we have the stochastic differential equation

$$\begin{aligned} &dx_i(t) = x_i(t)\, [a_i(t)\, dw_i(t) + b_i(t)\, dt] \\ &x_i(0) = \xi_i\,. \end{aligned} \tag{20}$$

On the other hand, the most probable path Ψ_i associated to the equation (20) is proved that satisfies the following conditions

$$\begin{aligned} &\Psi_i^{'}(t) = b_i(t)\, \Psi_i(t) - \frac{1}{2}\, a_i^2(t)\, \Psi_i(t), \\ &\Psi_i(0) = \xi_i, \qquad\qquad i = 1, 2, \cdots, d \end{aligned} \tag{21}$$

or, equivalently

$$\Psi_i^{'}(t) = \Psi_i(t)\,[b_i(t) - \frac{1}{2}\,a_i^2(t)],$$
$$\Psi_i(0) = \xi_i, \qquad i = 1, 2, \cdots, d\,. \tag{22}$$

From this equation we come to the conclusion that if an investment is made in a stock with the mean return $b(t)$ and the volatility $a(t)$, then the real return rate is most likely be given by the equality

$$c(t) = b(t) - \frac{1}{2}\,a^2(t) \tag{23}$$

instead of $b(t)$. That is to say the interest rate is most likely to be $b(t) - \frac{a^2(t)}{2}$ instead of $b(t)$. The quantity $c(t)$ in (23) is referred to as *the most probable interest rate*.

In conclusion, when an investment is made into several stocks with the mean return $b_i(t)$ and the volatility $a_i(t)$ it is recommended to compare the most probable interest rate

$$c_i(t) = b_i(t) - \frac{1}{2}\,a_i^2(t)$$

instead of the mean interest rate $b_i(t)$.

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