Chaotic Modeling and Simulation (CMSIM) 2: 257–264, 2013

Analysis of 1-D Linear Piecewise-smooth Discontinuous Map

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Abstract. In this paper we analyze the stable periodic orbits existing in the 1-D linear piecewise-smooth discontinuous map with respect to variations in the parameters of the map. We analytically show how to calculate the range of parameter μ such that the orbits of specific periodicity can exist. Moreover, for a given period, the relation between the probability of occurrence of orbits of that period and the corresponding length of range of μ is established. Further, we show that this probability can be maximized by varying the parameter of the map. We prove that there exist a unique value of this parameter such that this probability is maximum. We provide diagrams generated by numerical simulations to illustrate these results and to depict the effects of variations in the parameters of the map on the ranges of existence of orbits.
Keywords: Border collision bifurcation, piecewise-smooth, discontinuous map, periodic orbit.

1 Introduction

Piecewise-smooth dynamical systems are being extensively studied over the last decade because of their applications in various fields like electrical engineering, physics, economics etc. Examples are DC-DC converters in discontinuous mode [1,2], impact oscillators [3], economic models [4] etc. One of the major reasons for interest in piecewise-smooth systems is the existence of a phenomenon, unique to such systems, called *border collision bifurcation.* Though this term was coined by Nusse [5] in 1992, the phenomenon was earlier reported by Feigin [6] in 70's.

The 1-D linear piecewise-smooth discontinuous map is defined as [7]:

$$x_{n+1} = f(x_n, a, b, \mu, l) = \begin{cases} ax_n + \mu & \text{for } x_n \leqslant 0 \\ bx_n + \mu + l & \text{for } x_n > 0 \end{cases} \tag{1}$$

Over the last decade, several authors have published the analytical as well as numerical work which analyzes the 1-D piecewise-smooth discontinuous map in detail [8–11]. Recently in [12] it was shown that exactly $\phi(n)$ stable periodic

Received: 30 March 2012 / Accepted: 15 October 2012

ISSN 2241-0503

orbits exist in the map given by Equation (1) when $a, b \in (0,1)$, $l = -1$ and $\mu \in (0,1)$; where n is the period and ϕ is Euler's number. In this paper we extend this analysis to investigate the effect of variation in parameters a, b and n on the range of existence of periodic orbits.

1.1 Notation

Let $\mathcal{L} := (-\infty, 0]$ (the closed left half plane) and $\mathcal{R} := (0, \infty)$ (the open right half plane). Given a particular sequence of points $\{x_n\}_{n\geqslant 0}$ through which the system evolves, one can convert this sequence into a sequence of $\mathcal{L}s$ and $\mathcal{R}s$ by indicating which of the two sets ($\mathcal{L}$ or $\mathcal{R}$) the corresponding point belongs to. Since a periodic orbit has a string of $\mathcal{L}s$ and $\mathcal{R}$ that keeps repeating, we call this repeating string, a *pattern* and denote it by σ. The length of the string σ is denoted by $|\sigma|$ and gives the number of symbols in the pattern i.e., the period of the orbit. The range of existence of this pattern σ is denoted by $P_\sigma = (p_1, p_02]$ where p_2 and p_1 are the upper and the lower limits respectively. The sum of geometric series $1 + k + k^2 + \cdots + k^n$ is denoted by S_n^k.

1.2 Preliminaries

Definition 1. A pattern σ is termed *admissible* if $P_\sigma \neq \varnothing$.

Definition 2. If a pattern consists of a single chain of consecutive $\mathcal{L}s$ followed by a singleton $\mathcal{R}$ then it called an $\mathcal{L}$-prime pattern. Similarly, if a pattern consists of a single chain of consecutive $\mathcal{R}s$ followed by a singleton $\mathcal{L}$ then it called an $\mathcal{R}$-prime pattern. Together, we call them *prime patterns.*

Example 1. $\mathcal{L}^n\mathcal{R}$ is a $\mathcal{L}$-prime pattern and $\mathcal{L}\mathcal{R}^n$ is a $\mathcal{R}$-prime pattern. $\mathcal{L}\mathcal{R}$ is both $\mathcal{L}$-prime as well as $\mathcal{R}$-prime.

Definition 3. A pattern made up of two or more prime patterns is called a composite pattern.

Example 2. $\mathcal{LLLRLLR}$ is a composite pattern as it is made of two prime patterns namely $\mathcal{LLLR}$ and $\mathcal{LLR}$.

Remark 1. Some authors use the term *maximal* or *principal* to describe prime pattern [13].

Recall that the range of existence of an orbit is denoted by P_σ. We illustrate with an example how to calculate P_σ.

Example 3. Consider a pattern $\mathcal{LLR}$ which means: $x_0, x_1 \leqslant 0, x_1 > 0$ and $x_3 = x_0$. Using Equation (1) these inequalities can be rewritten as:

$$
\begin{aligned}
&x_0 \leqslant 0,\\
&x_1 = ax_0 + \mu \leqslant 0,\\
&x_2 = a^2x_0 + (a+1)\mu > 0,\\
&x_3 = x_0 = a^2bx_0 + (ab+b+1)\mu - 1 \Rightarrow x_0 = \frac{(ab+b+1)\mu - 1}{1 - a^2b}.
\end{aligned}
$$

Substituting the value of x_0 in x_1 and x_2 we get:

$$x_1 = a\Big(\frac{(ab+b+1)\mu - 1}{1-a^2b}\Big) + \mu \leqslant 0,$$
$$x_2 = a^2\Big(\frac{(ab+b+1)\mu - 1}{1-a^2b}\Big) + (a+1)\mu > 0.$$

After simplification we get:

$$\mu > \frac{a^2}{a^2+a+1},$$
$$\mu \leqslant \frac{a}{ab+a+1}.$$

Hence, $\mathcal{P}_{\mathcal{LLR}} = \Big(\frac{a^2}{a^2+a+1}, \frac{a}{ab+a+1}\Big]$.

In a similar way we can find the range of existence (P_σ) for the prime patterns $\mathcal{L}^n\mathcal{R}$ and $\mathcal{L}\mathcal{R}^n$ for any $n \geqslant 2$. The method is explained in detail in [12]. We directly use the formulas from [12] here:

$$\mathcal{P}_{\mathcal{L}^n\mathcal{R}} = \left(\frac{a^n}{S_n^a}, \quad \frac{a^{n-1}}{a^{n-1}b + S_{n-1}^a}\right] \tag{2}$$

and

$$\mathcal{P}_{\mathcal{L}\mathcal{R}^n} = \left(\frac{ab^{n-1} + S_{n-2}^b}{ab^{n-1} + S_{n-1}^b}, \quad \frac{S_{n-1}^b}{S_n^b}\right]. \tag{3}$$

1.3 Characterization of Patterns

We have seen earlier that the prime patterns are admissible and the range of existence of prime patterns is given by Equations (2) and (3). The immediate question is other than prime patterns, which type of patterns are admissible? It is shown in [12] that only specific type of patterns are admissible. For example, it is shown that admissible patterns can not contain consecutive chain of $\mathcal{L}s$ and $\mathcal{R}s$ simultaneously. Moreover, admissible composite patterns are always made up of exactly two prime patterns of successive lengths. Further, it is shown that these results lead to the final conclusion that exactly $\phi(n)$ number of distinct patterns are admissible for a given n.

For a given n, the algorithm to generate the $\phi(n)$ patterns and to calculate the range of existence of these patterns is discussed in detail in [12]. We now extend this analysis to find out the effects of variations in parameters on the range of existence of patterns.

2 Effects of Variations in Parameters on The Range of Existence of Patterns

In this section we analyze the effects of variations in parameters a, b and n on the range of existence of patterns. Recall that the range of existence of pattern

σ is expressed as $\mathcal{P}_\sigma = (p_1, p_2]$. Let the length occupied on the parameter line μ corresponding to the i^{th} pattern of length n is denoted by Γ_i^n. That is, $\Gamma_i^n = p_2 - p_1$. Let the total length occupied corresponding to all the patterns of length n is denoted by Γ^n. That is, $\Gamma^n = \sum_{i=1}^{\phi(n)} \Gamma_i^n$. We now find out the expression for Γ^n. In this paper we consider the case of $a = b$.

Consider the pattern of length $N = n+1$. We substitute $a = b$ in Equations (2) and (3) to get:

$$P_{\mathcal{L}^n\mathcal{R}} = \left(\frac{a^n}{S_n^a}, \frac{a^{n-1}}{S_n^a}\right] \text{ and } P_{\mathcal{L}\mathcal{R}^n} = \left(\frac{a^n + S_{n-2}^a}{S_n^a}, \frac{S_{n-1}^a}{S_n^a}\right].$$

Note that $\Gamma^N_{P_{\mathcal{L}^n\mathcal{R}}} = \Gamma^N_{P_{\mathcal{L}\mathcal{R}^n}} = \frac{a^{n-1}(1-a)}{S_n^a}$. We denote it by γ^N. Since, for $a = b$ the map becomes symmetric, all the patterns of length N have $\Gamma_i^N = \gamma^N$. This gives $\Gamma^N = \sum_{i=1}^{\phi(N)} \Gamma_i^N = \phi(N)\gamma^N$. Substituting for γ^N and N we get $\Gamma^{n+1} = \phi(n+1)\frac{a^{n-1}(1-a)}{S_n^a} = \phi(n+1)\frac{a^{n-1}(1-a)^2}{1-a^{n+1}}$. For consistency, we use the formula for n which is:

$$\Gamma^n = \phi(n)\gamma^n = \phi(n)\frac{a^{n-2}(1-a)^2}{1-a^n}. \tag{4}$$

From the above equation it is clear that Γ^n depends on the parameters a and n. Recall that Γ^n is the length of range of existence of patterns as defined earlier. Hence, any change in Γ^n due to the variations in a and n can be interpreted as the effect on the range of existence of patterns.

2.1 Probability of Occurrence of a Pattern

We have seen that the total length occupied on the parameter line μ corresponding to all the patterns of length n is expressed by Γ^n. We know $\mu \in (0, 1)$. This leads us to the question: for a randomly selected μ from the set $(0, 1)$, what is the probability that it corresponds to a pattern of length n? Since $\mu \in (0, 1)$, the total length of the parameter line is unity and Γ^n is the total length occupied on parameter line μ corresponding to all the patterns of length n. Hence, the probability of occurrence of a pattern of length n is Γ^n. The Equation (4) gives the formula for this probability in terms of a and n.

2.2 Maximizing the Probability of Occurrence of a Pattern

For $n = 2$, $\Gamma^2 = \frac{1-a}{1+a}$ and $a \in (0, 1)$. Clearly, it is a monotonically decreasing function. Hence, the suprimum is achieved at $a = 0$. For all $n > 2$, Γ_n is not monotonic. With bit more analysis we can show that Γ^n attains maxima for a particular value of $a \in (0, 1)$. This can be calculated by differentiating Γ^n with respect to a.

$$\frac{d}{da}(\Gamma^n) = \frac{d}{da}(\phi(n)\gamma^n) = \phi(n)\left(a^n - \frac{n}{2}a + \frac{n}{2} - 1\right). \tag{5}$$

We check that the expression $a^n - \frac{n}{2}a + \frac{n}{2} - 1$ has only one real root in $(0, 1)$. At that root, $\frac{d^2}{da^2}(\Gamma^n) = a^{n-1} - \frac{1}{2}$ is negative. Hence, for a given n, there is an unique value of a such that Γ_n is maximum.

Example 4. We plot Γ^n versus n for different values of a. In these plots, n is varied from 2 to 14. These graphs (see Figure 1a to Figure 1e) show that as n increases, the position of maxima for Γ_n increases too. This means, higher the value of a, greater is the probability of occurrence of high period orbits. For the same values of a, figures 1b to 1f shows the bifurcation diagrams. We note that above results are validated by the bifurcation diagrams.

The graphs of Γ^n versus a, for different values of n, are plotted in figures from 1g to 1i. In these plots, a is varied from 0.01 to 0.99. From these plots we can see that Γ^2 is indeed a monotonically decreasing function. For vary small values of a, Γ_2 almost completely occupies the parameter line. For example, when $a = 0.1$, $\Gamma^2 = 0.818$. For all $n > 2$ is clear from the graphs that Γ^n is not monotonic and the maxima attained varies as n changes.

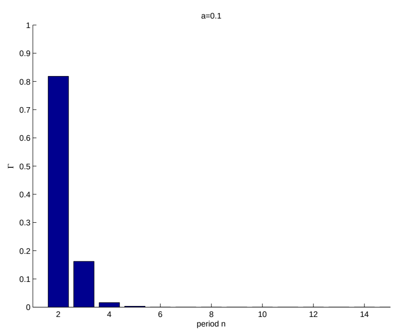

Fig. 1a. Graph showing Γ^n for different values of n. $a = 0.1$

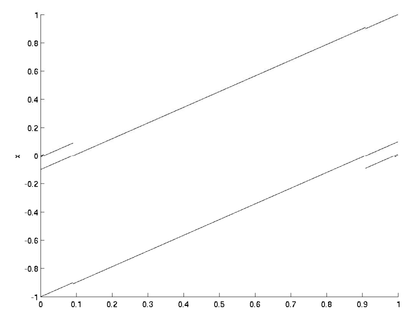

Fig. 1b. Bifurcation Diagram for $a = b = 0.1$

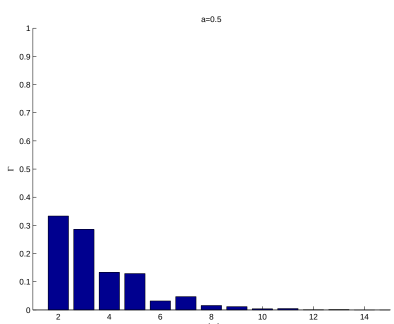

Fig. 1c. Graph showing Γ^n for different values of n. $a = 0.5$

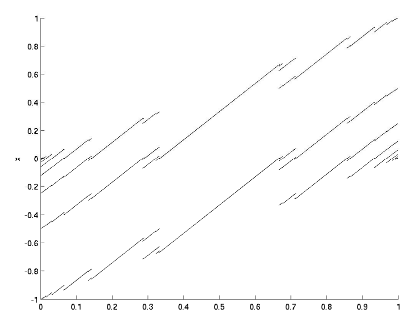

Fig. 1d. Bifurcation Diagram for $a = b = 0.5$

2.3 Patterns Completely Span The Parameter Line μ

We know that Γ^n gives the total length occupied on the parameter line μ corresponding to all the patterns of length n. We have shown that for $a = b$,

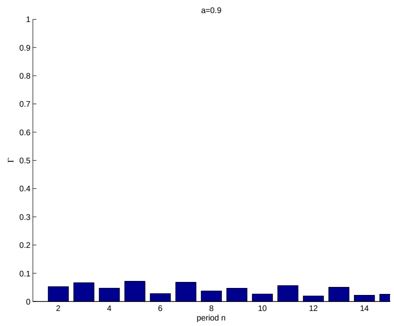

Fig. 1e. Graph showing Γ^n for different values of n. $a = 0.9$

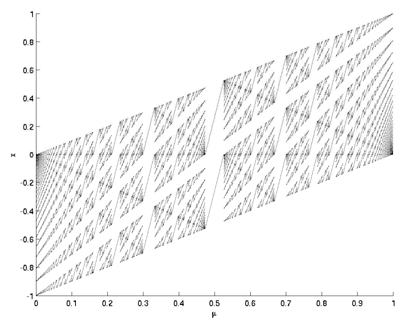

Fig. 1f. Bifurcation Diagram for $a = b = 0.9$

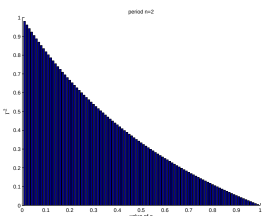

Fig. 1g. Graph showing Γ^n for different values of a. $n = 2$

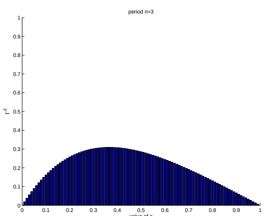

Fig. 1h. Graph showing Γ^n for different values of a. $n = 3$

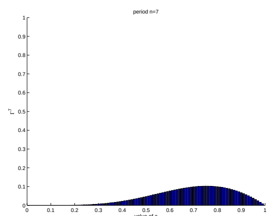

Fig. 1i. Graph showing Γ^n for different values of a. $n = 7$

Γ^n can be maximized by appropriately choosing the value of a. Now let the total length occupied on the parameter line μ corresponding to all the possible patterns be denoted by Γ. That is, $\Gamma = \sum_{n=2}^{\infty} \Gamma^n$. The following lemma proves that $\Gamma = 1$ and it completely spans the parameter line μ.

Lemma 1. *For every $\mu \in (0, 1)$, there exists a pattern.*

Proof. We know that $P_{\mathcal{L}^n\mathcal{R}} = (\sigma_1, \sigma_2] = \left(\frac{a^n}{S_n^a}, \quad \frac{a^{n-1}}{a^{n-1}b+S_{n-1}^a}\right]$ and $P_{\mathcal{L}^{n-1}\mathcal{R}} = (\sigma_1', \sigma_2'] = \left(\frac{a^{n-1}}{S_{n-1}^a}, \quad \frac{a^{n-2}}{a^{n-2}b+S_{n-2}^a}\right]$. Hence, for any arbitrarily given $\mu \in (0, 1)$ we can find an 'n' such that

Step 1: either $\mu \in P_{\mathcal{L}^n\mathcal{R}}$ or $\mu \in P_{\mathcal{L}^{n-1}\mathcal{R}}$ or $\sigma_2 < \mu < \sigma_1'$.

For the first two cases the pattern exists as μ belongs to the range of existence of a pattern. For the last case we proceed further by calculating $\mathcal{P}_{\mathcal{L}^{\mathrm{n}}\mathcal{R}\mathcal{L}^{\mathrm{n}-1}\mathcal{R}} = (\sigma_1'', \sigma_2'']$. Now again we have three cases:

Step 2: either $\mu \in \mathcal{P}_{\mathcal{L}^{\mathrm{n}}\mathcal{R}\mathcal{L}^{\mathrm{n}-1}\mathcal{R}}$ or $\sigma_2 < \mu < \sigma_1''$ or $\sigma_2'' < \mu < \sigma_1'$.

For the first case the pattern exists as μ belongs to the range of existence of a pattern. For the second case we again go to Step 1 but this time with $P_{\mathcal{L}^n\mathcal{R}}$ and $\mathcal{P}_{\mathcal{L}^{\mathrm{n}}\mathcal{R}\mathcal{L}^{\mathrm{n}-1}\mathcal{R}}$. Similarly for the third case we go to Step 1 with $\mathcal{P}_{\mathcal{L}^{\mathrm{n}}\mathcal{R}\mathcal{L}^{\mathrm{n}-1}\mathcal{R}}$ and $P_{\mathcal{L}^{n-1}\mathcal{R}}$. Without the loss of generality we assume the second case to be true i.e. μ always lay in the left side partition or nearer to $P_{\mathcal{L}^n\mathcal{R}}$. Then, before every time we take Step 2, we construct the new pattern of form $(\mathcal{L}^{\mathrm{n}}\mathcal{R})^{\mathrm{k}}\mathcal{L}^{\mathrm{n}-1}\mathcal{R}$

with $k = 2, 3, 4 \ldots$ With the help of generalized map method explained in [12] this pattern can be written as $\mathcal{L}'^{\mathrm{k}}\mathcal{R}'$ where, $\mathcal{L}' = \mathcal{L}^{\mathrm{n}}\mathcal{R}$ and $\mathcal{R}' = \mathcal{L}^{\mathrm{n}-1}\mathcal{R}$.

This process is nothing but constructing a series of intervals $\mathcal{P}_{\mathcal{L}'^{\mathrm{k}}\mathcal{R}'}$. This series of intervals must converge at σ_2. This is because, if it converges at some other point (say $\tilde{\sigma_1}$) then we get a finite length subinterval $(\sigma_2, \tilde{\sigma_1}]$. We arbitrarily select any point from this interval (say $\tilde{\mu}$). Now as we argued for the case of $P_{\mathcal{L}^n\mathcal{R}}$, similar arguments can be made here i.e. we can select a large enough k (since limits of $\mathcal{P}_{\mathcal{L}'^{\mathrm{k}}\mathcal{R}'}$ involve a and b with k in power) such that $\mathcal{P}_{\mathcal{L}'^{\mathrm{k}}\mathcal{R}'}$ lies to the left of $\tilde{\mu}$. This is contradiction to the earlier assumption that series converges to $\tilde{\sigma}$. Hence, the series must converge to σ_2.

3 Conclusions

In this paper we have analyzed the stable periodic orbits of the 1-D linear piecewise-smooth discontinuous map with respect to change in the parameters. We have analytically calculated the range of parameters for which period-n orbits exist. The length of this range is considered as the probability of occurrence of period-n orbit. Further, we have shown that this probability can be maximized by varying the parameter of the map and we prove that there exist an unique value of this parameter such that this probability is maximum.

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