Universal fractal time of biological growth

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Abstract: The West-Brown-Enquist universal growth curve has been mapped on the power law function with the time-dependent scaling factor and exponent representing the temporal fractal dimension of the growth of species like mammals, birds, fish, crustaceans, regardless of taxon, cellular metabolic rate and body size. The results obtained permit formulation of three important rules governing the biological growth: (i) growing biological systems possess its own, internal, universal fractal time, which differs from the linear scalar time of the external observer, (ii) fractal structure of the universal time is lost during growth, (iii) the universal growth belongs to the class of macroscopic non-local quasi-quantum phenomena.

Keywords: Biological growth, Fractal time, Mapping method, Nonlocality, Quasi-quantum phenomena

1. Introduction

Recently, an idea has been developed that the growth curves describing neuronal differentiation [1] or malignant tumour progression [2] can be successfully fitted by the fractal function

$$y(t) = a_t(t)t^{b_t(t)} \tag{1}$$

with the scaling factor $a_t(t)$ and exponent $b_t(t)$ as the functions of time. The formula (1) is derived by the mapping the sigmoidal Gompertz function [3]

$$G(t) = G_0 e^{\frac{b}{a}\left(1-e^{-at}\right)} \tag{2}$$

widely used to describe the time-evolution of the biological system (organism, organ, tissue, bacterial colony, tumour etc.) on the power one. In this approach we employ the generalized spline interpolation method [4], which permits interpolating the Gompertz function (2) by a family of power law curves

$$\left\{ y_i(t) = a_t(t_i)t^{b_t(t_i)} + y_0 \quad i = 1, 2, \ldots N \right\} \tag{3}$$

determined at the points $\{t_i, y_i(t_i)\}$. Defining the sets of parameters $a_t=\{a_t(t_i), i=1,2\ldots\}$, $b_t=\{b_t(t_i), i=1,2\ldots\}$, one may derive the fractal function (1) assuming

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that functions (2) and (3) are isovalued and isosloped for the each moment of time. In such circumstances the equality of the functions (2) and (3) as well as their first derivatives provides the set of equations [1,2]

$$b_t(t) = bte^{-at}\frac{e^{\frac{b}{a}\left(1-e^{-at}\right)}}{e^{\frac{b}{a}\left(1-e^{-at}\right)}-1} \qquad a_t(t) = t^{-b_t}G_0\left[e^{\frac{b}{a}\left(1-e^{-at}\right)}-1\right] \qquad (4)$$

Here, G_0 stands for the initial mass, volume, diameter or number of proliferating cells, a - is the retardation constant whereas b denotes the initial growth or regression rate constant.

The main objective of the present study is extension of the research area on the ontogenic growth described by the West-Brown-Enquist function [5]

$$m(t) = M\left[1 - c_0\exp(-c_1 t)\right]^{\frac{1}{c_2}} \qquad (5)$$

derived from the first principles: the conservation of metabolic energy, the allometric scaling of metabolic rate, energetic costs of producing and maintaining biomass. In the above formulae

$$c_0 = 1 - \left(\frac{m_0}{M}\right)^{1/4} \quad c_1 = \frac{a}{4M^{1/4}} \quad c_2 = \frac{1}{4} \quad M = m(t=\infty) \quad m_0 = m(t=0) \quad (6)$$

m_0 denotes the initial mass of the system, M is the maximum body size reached whereas a is the metabolic parameter. The function (5) can be expressed in dimensionless form

$$r(\tau) = \left[\frac{m(t)}{M}\right]^{1/4} = 1 - \exp(-\tau) \qquad \tau = c_1 t - \ln(c_0) \qquad (7)$$

which has been named the *universal growth function* [5]. It almost perfectly describes the growth of all known species like mammals, birds, fish, crustaceans, regardless of taxon, cellular metabolic rate and body size [5].

Recently, the West-Brown-Enquist model has been successfully applied to fit the data for the different types of the tumours [6]. In particular, it has been demonstrated [7] that instead of the function (7) its generalized form

$$R(\tau) = \left[\frac{m(t)}{M}\right]^{1-p} = 1 - \exp(-\tau) \qquad \tau = (1-p)c_1 t - \ln(c_0) \qquad (8)$$

$$c_0 = 1 - \left(\frac{m_0}{M}\right)^{1-p} \quad c_1 = aM^{p-1} \quad c_2 = 1 - p \qquad (9)$$

better reproduces the tumor growth than the original function (5). Here, parameter p takes the value in the range<2/3, 1> [7]. Hence, the both universal growth functions (7) and (8) can be applied as input for the mapping procedure to obtain the fractal function (1) characterizing the growth of the species like mammals, birds, fish, crustaceans and different kinds of tumors.

The main purpose of the present work is mapping the functions (7) and (8) on the power law one using the spline interpolation method outlined above. As

the result one gets the time-dependent universal fractal dimension $b_\tau(\tau)$ and the scaling factor $a_\tau(\tau)$ appearing in generalized τ-dependent function (1), which includes universal time τ instead of t. The analytical formulae $b_\tau(\tau)$ and $a_\tau(\tau)$ will be employed to calculate their values at an arbitrary moment of time and to interpret the biological growth in the terms of the quasi-quantum model of life [8].

2. The Method

The West et al. [5] universal curve representing the growth of different biological species is employed to formulate the model considered here and to derive the universal power law function describing the biological growth in the space-time with universal temporal fractal dimension. All operations (plotting, integration, differentiation) were performed by making use of a Maple vs.14 processor for symbolic calculations. To avoid errors, the derivation of the mathematical formulae was also carried out by the Maple software. Proceeding along the line of the generalized spline interpolation method outlined above, the universal growth function (7) can be converted into power law using the set of nonlinear equations

$$a_\tau \tau^{b_\tau} = 1 - \exp(-\tau) \qquad b_\tau a_\tau \tau^{b_\tau - 1} = \exp(-\tau) \tag{10}$$

which assume that power law (3) and the universal growth (7) functions are isovalued and isosloped for the each moment of the universal time. The solution of the nonlinear set of equations (8) is the universal fractal dimension and the scaling factor

$$b_\tau(\tau) = \frac{\tau \exp(-\tau)}{1 - \exp(-\tau)} \qquad a_\tau(\tau) = \tau^{-b_\tau} \left[1 - \exp(-\tau) \right] \tag{11}$$

which define the universal fractal function

$$y(\tau) = a_\tau(\tau) \tau^{b_\tau(\tau)} \tag{12}$$

describing the biological growth in the space-time with the universal time τ defined by (7). The equation (10) permits formulating the power law function governing the growth of all species, independently of their biological characteristics, e.g. a, M and m_0. The plots of the universal fractal dimension $b_\tau(\tau)$ and scaling factor $a_\tau(\tau)$ generated from (9) are presented in Fig. 1.
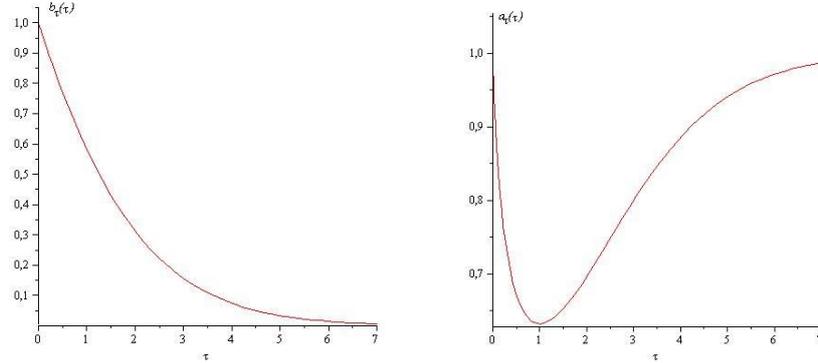
Fig. 1. The universal fractal dimension $b_\tau(\tau)$ and scaling factor $a_\tau(\tau)$ generated from Eqs. (11).

One may prove that the universal growth function (8) satisfies the first- and second-order differential equations

$$\frac{d}{d\tau}R(\tau) - \frac{\exp(-\tau)}{1-\exp(-\tau)}R(\tau) = 0, \quad \frac{d^2}{d\tau^2}R(\tau) + \frac{\exp(-\tau)}{1-\exp(-\tau)}R(\tau) = 0 \quad (13)$$

The second term in the above equations represents the well-known in the quantum physics Hulthèn potential [9] widely used in description of the electrostatic interactions between micro-particles. One may prove that this equation is a special case of the quantal non-local Horodecki-Feinberg equation [10,11] for the time-dependent Hulthèn potential [9]

$$\frac{d^2}{d\tau^2}\Psi_v + \beta^2\frac{\exp[-\tau]}{1-\exp[-\tau]}\Psi_v = \varepsilon_v^2\Psi_v \quad \varepsilon_v = \frac{\beta^2-v^2}{2\beta v} \quad v=1,2... \quad (14)$$

$$\beta^2 = \frac{2mc^2V_0}{\hbar^2c_1^2} \qquad \varepsilon_v^2 = -\frac{2mc^3P_v}{\hbar^2c_1^2}$$

$$\Psi_v = \exp(-\varepsilon\tau)[1-\exp(-\tau)]_2F_1[2\varepsilon_v+1+v,1-v,2\varepsilon_v+1;\exp(-\tau)]$$

For the *critical screening* [12] (β=1) and ground quantum state (v=1) the quantum wave function Ψ_1 reduces to the macroscopic growth function $R(\tau)$

$$\Psi_1 = [1-\exp(-\tau)]\exp(-\varepsilon_1\tau) \xrightarrow{\beta=1,\varepsilon_1=0} R(\tau) = 1-\exp(-\tau) \quad (15)$$

This result indicates that the biological growth according to the universal growth function (8) belongs to the class of macroscopic non-local quasi-quantum phenomena. The notion *quasi-quantum* refers to the possibility of application of the quantum language and formalism in description of macroscopic objects [8] as this equation does not contain the Planck's constant and is a special case of the quantal Horodecki-Feinberg equation.

3. Conclusions

The calculations performed reveal that the fractal dimension $b_\tau(\tau)$ decreases to zero from the maximum value equal to one. On the contrary, the scaling factor at the beginning takes the value one $a_\tau(\tau=0)=1$ and then decreases to 0.6321 for $\tau=1$ (it corresponds to $b_\tau(\tau=1)=0.5820$); then it tends to the asymptotic value equal to one $a_\tau(\tau\to\infty)=1$. The results obtained permit formulation of three interesting rules governing the biological growth in the space-time with universal time τ, from which the first two conform acceptably with the results of our previous investigation on the neuronal differentiation and tumorigenessis [1,2]:

(i) Growth of biological systems according to the universal West et al. function (8) can be described in the space-time with internal universal fractal time, which differs from the linear $b_\tau(\tau=0)=1$ scalar time of the external observer.

(ii) Fractal structure of the universal time is lost during biological growth.

(iii) Universal growth belongs to the class of macroscopic non-local quasi-quantum phenomena.

As far as point (i) is concerned, if we assume that time is a continuous variable, it is clear that at the beginning of growth of a biological system it takes place in the space-time with the temporal fractal dimension $b_\tau(\tau=0)=1$ equal to the extrasystemic (physical) time. During growth, it is continuously transformed into intrasystemic universal fractal time, which diminishes to zero as growth continues.

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