Renormalization schemes and the double expansion in the field theory of forced turbulence

Juha Honkonen

Department of Military Technology, National Defence University, PO Box 7, 00861 Helsinki, Finland
(E-mail: `juha.honkonen@mil.fi`)

Abstract In the field theory of forced turbulence in arbitrary space dimension the correlation and response functions of the velocity field contain divergences at two dimensions in addition to those brought about by the power-law correlation function of the random force at the critical value of its exponent. Renormalization of the model with an account of both sets of divergences gives rise to expansion of critical exponents and amplitudes in regulators. The structure of renormalization-group equations as well as numerical results heavily depend on the renormalization scheme adopted. Consequences of this ambiguity are analyzed on the basis of results of calculations available in several different renormalization schemes
Keywords: Renormalization group, turbulence, double expansion.

1 Introduction

The basic equation of the theory of hydrodynamic fluctuations is the stochastic Navier-Stokes equation for the velocity field $\mathbf{v}$ of incompressible fluid

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \nu_0 \nabla^2 \mathbf{v} - \nabla p + \mathbf{f}, \qquad \nabla \cdot \mathbf{v} = 0, \tag{1}$$

where $\mathbf{v}(t, \mathbf{x})$ is the transverse velocity field, ν_0 the kinematic viscosity, p the pressure and $\mathbf{f}$ the random force. In view of the subsequent renormalization of parameters of the model we distinguish between unrenormalized (with the subscript "0") quantities and renormalized terms (without the subscript "0"). In the description of thermal fluctuations the correlation function of the zero-mean Gaussian random force is described in terms of transport coefficients ad temperature of the fluid. In the stochastic theory of developed turbulence the probability density function of the random force in equation (1) is chosen to maintain the steady state of the turbulent flow with energy injection at large spatial scales. To this end, it is customary to use the power-function δ sequence (DeDominicis and Martin [1], Adzhemyan *et al.* [2], Yachot and Oszag [3]), in

which the kernel function of the generic random-force correlation function

$$\langle f_i(t,\mathbf{x})f_j(t',\mathbf{x}')\rangle \equiv D_{ij}(t,\mathbf{x};t'\mathbf{x}')$$
$$= \frac{\delta(t-t')}{(2\pi)^d}\int d^d\mathbf{k}\, P_{ij}(\mathbf{k})d_f(k)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \quad (2)$$

in the wave-vector space is of the form

$$d_f(k) = D_{10}k^{4-d-2\varepsilon}, \tag{3}$$

and the transverse projection operator

$$P_{ij} = \delta_{ij} - k_ik_j/k^2.$$

The connection between D_{10} and the average energy injection rate $\overline{\mathcal{E}}$ is determined by the exact relation expressing $\overline{\mathcal{E}}$ in terms of the function $d_f(k)$ in the correlation function (2)

$$\overline{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d}\int d^d\mathbf{k}\, d_f(k). \tag{4}$$

Substituting here the kernel function (3) and introducing the UV cutoff $k \leq \Lambda = (\overline{\mathcal{E}}/\nu_0^3)^{1/4}$ (the inverse dissipation length), we obtain the following connection between the parameters $\overline{\mathcal{E}}$ and D_{10}

$$D_{10} = \frac{4(2-\varepsilon)\Lambda^{2\varepsilon-4}}{\overline{S}_d(d-1)}\,\overline{\mathcal{E}}. \tag{5}$$

According to (4) an idealized injection by infinitely large eddies corresponds to the kernel function

$$d_f(k) = \frac{2(2\pi)^d\,\mathcal{E}\,\delta(\mathbf{k})}{d-1}. \tag{6}$$

In view of the relation

$$\delta(\mathbf{k}) = \lim_{\varepsilon\to 2}(2\pi)^{-d}\int d^d\mathbf{x}(\Lambda x)^{2\varepsilon-4}\exp(i\mathbf{k}\cdot\mathbf{x})$$
$$= S_d^{-1}k^{-d}\lim_{\varepsilon\to 2}\left[(4-2\varepsilon)(k/\Lambda)^{4-2\varepsilon}\right],$$

the powerlike injection with $d_f = D_{10}k^{4-d-2\varepsilon}$ and the amplitude D_{10} from Eq. (5) in the limit $\varepsilon \to 2$ from the the region $0 < \varepsilon < 2$ is a δ-sequence giving rise to (6).

Coefficient functions in the iterative solution of the stochastic problem (1), (2), (3) exhibit singular behaviour in the limit $d \to 2$. To account for their effect to the solution it is customary to start with the modified kernel function (Honkonen and Nalimov [4], Adzhemyan *et al.*[5,6])

$$d_f(k) = D_{10}k^{2-2\Delta-2\varepsilon} + D_{20}k^2 = g_{10}\nu_0^3\,k^{2-2\Delta-2\varepsilon} + g_{20}\nu_0^3\,k^2\,, \tag{7}$$

where $\Delta = (d-2)/2$ measures the deviation of the space dimension d from its critical value two. The effective expansion parameters g_{10} and g_{20} have been introduced in (7) to streamline further notation.

The stochastic problem (1) and (7) may be cast (DeDominicis and Martin [1], Adzhemyan *et al.* [2]) to a field theory with the generating functional

$$G(A) = \int \mathcal{D}v \int \mathcal{D}v'\ \mathrm{e}^{S+Av+\tilde{A}v'}\ ,$$

where v' is a divergenceless auxiliary field, $A, \tilde{A}$ are the source fields, and the De Dominicis-Janssen action

$$\mathcal{S}[\mathbf{v}, \mathbf{v}'] = \frac{1}{2} \int \mathrm{d}t \int \mathrm{d}^d\mathbf{x} \int \mathrm{d}t' \int \mathrm{d}^d\mathbf{x}'\ v'_i(t,\mathbf{x})D_{ij}(t,\mathbf{x};t',\mathbf{x}')v_j(t',\mathbf{x}')$$
$$+ \int \mathrm{d}t \int \mathrm{d}^d\mathbf{x}\ v'_i(t,\mathbf{x})\left[-\partial_t v_i(t,\mathbf{x}) + \nu_0\nabla^2 v_i(t,\mathbf{x}) - v_j(t,\mathbf{x})\partial_j v_i(t,\mathbf{x})\right]\ . \quad (8)$$

Model (8) is logarithmic, i.e. $d_{g_{10}} = d_{g_{20}} = 0$, when $\varepsilon = 0$ and $\Delta = 0$. Due to the Galilei invariance of the action (8), the one-particle irreducible (1PI) Green function $\Gamma_{v'vv}$, which is superficially divergent by power counting, is actually convergent (Forster *et al.* [7], DeDominicis and Martin [1], Adzhemyan *et al.* [2]). Therefore, only the graphs of the 1PI Green functions $\Gamma_{v'v}$ and $\Gamma_{v'v'}$ yield divergent contributions to the renormalization of the model, which leads to the renormalization of the parameters ν_0 and D_{20}. Divergences in the Green functions show in the form of singularities in two (complex-valued) parameters ε and Δ. These two parameters are the regulators of the combined analytic-dimensional regularization (Zavyalov [8]) customarily used in the analysis of this and other double-expansion problems.

Relations between the renormalized and bare parameters are expressed by relations of multiplicative renormalization

$$D_{10} = g_{10}\nu_0^3 = g_1\mu^{2\varepsilon}\nu^3, \qquad D_{20} = g_{20}\nu_0^3 = g_2\mu^{-2\Delta}\nu^3 Z_{D_2},$$
$$g_{10} = g_1\mu^{2\varepsilon}Z_{g_1}, \qquad g_{20} = g_2\mu^{-2\Delta}Z_{g_2}, \qquad (9)$$
$$\nu_0 = \nu Z_\nu, \qquad Z_{g_1}Z_\nu^3 = 1, \qquad Z_{g_2}Z_\nu^3 = Z_{D_2}$$

with two independent renormalization constants for the coefficient of viscosity ν_0 and for the amplitude D_{20}. The amplitude D_{10} of the non-local term of the correlator of the random force is not renormalized. The independent renormalization constants Z_ν and Z_{D_2} are found from the condition that the one-irreducible functions $\Gamma_{v'v}$ and $\Gamma_{v'v'}$ are UV finite.

The condition of UV finiteness has been formulated in several different ways. In principle the two regulators are independent of each other and in this case the consistent analytic renormalization approach would require removal of divergences in such a way that the renormalized Green functions would be analytic functions of the parameters ε and Δ at the origin (Hnatič *et al.* [9], Adzhemyan *et al.* [5,6]). This approach leads – as will be demonstrated below – to significantly more complex calculations than in the usual minimal subtraction (MS) scheme in similar models. Therefore, it is common to use an approach (to be called the "ray scheme" in the following), in which the regulators are assumed to be proportional to each other, so that only one analytic regulator is left and the calculationally convenient MS scheme may be used (Honkonen and Nalimov [4]).

In any case, the coefficient functions of the renormalization-group equations are defined by relations

$$\gamma_1 = (\beta_1 \partial_{u_1} + \beta_2 \partial_{u_2}) \ln Z_{u_1} = -3\gamma_\nu, \tag{10}$$

$$\gamma_2 = (\beta_1 \partial_{u_1} + \beta_2 \partial_{u_2}) \ln Z_{u_2} = -3\gamma_\nu + \gamma_{D_2}, \tag{11}$$

$$\beta_1 = -u_1(2\varepsilon + \gamma_1), \qquad \beta_2 = -u_2(-2\Delta + \gamma_2), \tag{12}$$

where instead of g_1 and g_2 more convenient charges u_1 and u_2 are used:

$$u_1 \equiv \bar{S}_d g_1, \quad u_2 \equiv \bar{S}_d g_2, \quad \bar{S}_d \equiv \frac{S_d}{(2\pi)^d}, \quad S_d \equiv \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{13}$$

Here, S_d is the surface area of the unit sphere in d-dimensional space and Γ is Euler's Gamma function.

In the present approach any response or correlation function W is calculated in terms of two different sets of parameters. The set of bare parameters $e_0 = (\nu_0, g_{10}, g_{20})$ gives rise to the unrenormalized function W_0, whereas calculation with the set of renormalized parameters $e = (\nu, g_1, g_2)$ yields the renormalized function W. No field renormalization is introduced, therefore

$$W(g_1, g_2, \nu, \mu, \dots) = W_0(g_{10}, g_{20}, \nu_0, \dots),$$

where the ellipsis stands for the arguments not affected by renormalization like the coordinates, times etc. The unrenormalized functions W_0 do not depend on μ, while the renormalized functions W do because of the introduction of μ in renormalization relations (9). The independence of μ of the functions W_0 is expressed by the equation $\mu \partial_\mu|_0 W_0 = 0$, where the subscript reminds that the partial derivative is taken with fixed bare parameters e_0. Written in terms of the renormalized functions and renormalized parameters this is the basic RG equation

$$(\mu \partial_\mu + \beta_1 \partial_{g_1} + \beta_2 \partial_{g_2} - \gamma_\nu \nu \partial_\nu) W = 0. \tag{14}$$

At a fixed point of the RG, defined by vanishing of the coefficient functions $\beta_1(g_{1*}, g_{2*}) = 0$ and $\beta_2(g_{1*}, g_{2*}) = 0$ of (14) the third coefficient function becomes a constant $\gamma_{\nu*} = \gamma_\nu(g_{1*}, g_{2*})$ and the basic RG equation (14) assumes the form of the Euler equation for generalized homogeneity, expressing the scaling behaviour of W (and at the same time $W_0 = W$) governed by the fixed point of the RG equations.

The most important quantity in the asymptotic analysis is the equal-time velocity-velocity correlation function $G_{ij}(\mathbf{x} - \mathbf{x}') = \langle v_i(t, \mathbf{x}) v_j(t, \mathbf{x}') \rangle$. It is convenient to express the Fourier transform of the correlation function

$$\langle v_i(t, \mathbf{x}) v_j(t, \mathbf{x}') \rangle \equiv G_{ij}(\mathbf{r}), \qquad \mathbf{r} \equiv \mathbf{x} - \mathbf{x}'$$

in the form

$$G_{ij}(\mathbf{p}) = P_{ij}(\mathbf{p}) G(p),$$

where $P_{ij}(\mathbf{p})$ is the transverse projection operator and $p \equiv |\mathbf{p}|$. By dimensional arguments the scalar function $G(p)$ can be expressed as

$$G(p) = \nu^2 p^{-d+2} R(s, g_1, g_2), \quad s \equiv p/\mu,$$

where R is a scaling function of dimensionless arguments. Introduce a set of invariant parameters $\bar{e}(s) = (\bar{\nu}(s), \bar{g}_1(s), \bar{g}_2(s))$ corresponding to the set of renormalized parameters $e = (\nu, g_1, g_2)$ as solutions fixed bare parameters e_0. In terms of invariant parameters the correlation function assumes the form

$$G(p) = \nu^2 p^{2-d} R(s, g_1, g_2) = \bar{\nu}^2 p^{2-d} R(1, \bar{g}_1, \bar{g}_2). \tag{15}$$

Equation (15) is valid because both sides of it satisfy the RG equation and coincide at $s = 1$ owing to the normalization of the invariant parameters. The right-hand side of (15) depends on s through the invariant parameters $\bar{e}(s, e)$. They have simple asymptotic behavior as $s \to 0$, which is governed by the infrared-stable fixed point: the invariant charges $\bar{g}$ tend to the fixed-point values $g_* = \mathcal{O}(\varepsilon)$ and the invariant coefficient of viscosity $\bar{\nu}$ exhibits simple power-law behavior. To determine the latter it is convenient to express the invariant parameters $\bar{e} = (\bar{\nu}, \bar{g}_1, \bar{g}_2)$ in terms of the bare variables $e_0 = (\nu_0, g_{10}, g_{20})$ and the wave number p. Due to definition the bare variables e_0 also satisfy the RG equation $\mu \partial_\mu |_0 e_0 = 0$. From this it follows that the sets of these variables are connected by relations

$$\nu_0 = \bar{\nu} Z_\nu(\bar{g}), \qquad g_{10} = \bar{g}_1 p^{2\varepsilon} Z_{g_1}(\bar{g}), \quad g_{20} = \bar{g}_2 p^{-2\Delta} Z_{g_2}(\bar{g}). \tag{16}$$

Relations (16) are valid because both sides of them satisfy the RG equation, and because relations (16) at $s \equiv p/\mu = 1$ coincide with their counterparts in (9) owing to the normalization conditions. Using the connection $Z_g Z_\nu^3 = 1$ between the renormalization constants defined in (9), and eliminating these constants from the first two expressions in (16) we find $g_{10} \nu_0^3 = D_{10} = \bar{g}_1 p^{2\varepsilon} \bar{\nu}^3$, from which it follows that

$$\bar{\nu} = (D_{10} p^{-2\varepsilon} / \bar{g}_1)^{1/3}.$$

In the limit $\bar{g}_1 \to g_{1*}$ the sought asymptotic behavior of the invariant coefficient of viscosity as $s \to \infty$ thus assumes the form

$$\bar{\nu} \to \bar{\nu}_* = (D_{10}/g_{1*})^{1/3} p^{-2\varepsilon/3}, \qquad s \to \infty.$$

Substituting this result into (15) we obtain the relation

$$G(p) \simeq (D_{10}/g_{1*})^{2/3} p^{2-d-4\varepsilon/3} R(1, g_{1*}, g_{2*}), \qquad s \to \infty \tag{17}$$

describing the large-scale asymptotic behaviour of the pair correlation function.

For the physical values of the parameters $\Delta = 1/2, \varepsilon = 2$, chosen from the condition that the dimensional parameters of the model are viscosity and energy injection rate, the scaling behavior of the equal-time correlation function G in the three-dimensional space corresponds to the Kolmogorov scaling $G(p) \sim p^{-11/3}$ (DeDominicis and Martin [1], Adzhemyan et al. [2]). The scaling form (17) yields the large-scale asymptotic behavior of the original correlation function, if the fixed point is infrared stable, i.e. if $\bar{g}_1 \to g_{1*}, \bar{g}_2 \to g_{2*}$, when $p \to 0$.

2   Double expansion and the ray scheme

RG calculations with two (or even more) small parameters which may serve as regulators in dimensional or analytic renormalization have been widely used in the analysis of static critical phenomena (Weinrib and Halperin [10], Honkonen and Nalimov [11], Blavatska *et al.* [12]), dynamic critical phenomena (Antonov *et al.* [13,14,15,16,17]), diffusion in random environment (Gevorkian and Lozovik [18], Honkonen and Karjalainen [19], Honkonen [20], Goncharenko and Gopinathan [21], interface growth (Antonov and Kakin [22]) and in stochastic hydrodynamics (Fournier *et al.* [24], Adzhemyan *et al.* [25], Ronis [26], Bollini [27], Hnatič [28], Honkonen and Nalimov [4], Antonov [29], Hnatič *et al.* [30,9], Gladyshev *et al.* [31]). Critical exponents and other relevant quantities may be expressed in a double expansion in these parameters. The two parameters may both be regulators of analytic renormalization or one of them is the regulator of dimensional renormalization. In the following, this pair of parameters will be denoted ε and Δ.

Analytic renormalization would be a natural renormalization scheme to use to construct a double expansion in the two regulators, since it yields the RG functions as analytic functions of the two parameters at the origin. The genuine analytic renormalization involves rather tedious calculations (Zavyalov [8]). Moreover, in analytic renormalization there is no analog of the MS scheme to simplify practical calculations. Therefore, it is invariably assumed (implicitly or explicitly) that both parameters are of the same order of magnitude. This is made explicit by putting them proportional to each other in the ray scheme (Adzhemyan *et al.* [32,5]): $\Delta = \zeta\varepsilon$, where ζ is fixed and finite. This assumption effectively restores the dimensional renormalization with a single small parameter and the MS scheme may be used – at least formally.

Typically there are at least two charges in these models and therefore a rather generic case of two charges and a single anomalous dimension γ (corresponding to a field renormalization) will be analyzed here. It should be emphasized that we are considering coupling constants which serve as expansion parameters of the perturbation theory. When there are several coupling constants, it is customary to classify the order of perturbation theory by the number of loops. In multi-charge problems there are coupling constants, which should be calculated in closed form at each such order of perturbation theory (e.g. ratios of coefficients of viscosity, diffusion and thermal conductivity). We do not discuss such coupling constants here.

Two different structures of β functions are met. In stochastic hydrodynamics two (or more) random sources with different powerlike falloff of correlation functions are often introduced ((Fournier *et al.* [24], Adzhemyan *et al.* [25], Hnatič *et al.* [28,30], Antonov [29]): always random force for the stochastic momentum equation (Navier-Stokes equation) and the random source for either the stochastic diffusion or heat conduction equation (the passive scalar problem) or for Faraday's law (magnetohydrodynamics). Similar constructions have been used in critical dynamics (Antonov *et al.* [33,14,15,16,17]) and the interface growth problem (Antonov *et al.* [22]). Thus, two analytic regulators are used: deviations of exponents of these powerlike correlation functions from

their critical values. The regulators are invariably put explicitly proportional to each other and renormalization is treated in the framework of the usual dimensional renormalization. Nevertheless, a double expansion in the regulators is implied, if not always worked out explicitly. In models of this type the structure of the β functions is similar to the single-charge case, i.e. the renormalized coupling constant is a common factor in the expression for the corresponding β function (for brevity, parameters ε and Δ are omitted in the list of arguments):

$$\beta_1(g_1, g_2) = \mu \frac{\partial}{\partial \mu}\bigg|_0 g_1 = g_1 \left[-\varepsilon - \gamma_1(g_1, g_2)\right], \tag{18}$$

$$\beta_2(g_1, g_2) = \mu \frac{\partial}{\partial \mu}\bigg|_0 g_2 = g_2 \left[-\Delta - \gamma_2(g_1, g_2)\right], \tag{19}$$

$$\gamma_\varphi(g_1, g_2) = \mu \frac{\partial}{\partial \mu}\bigg|_0 \ln Z_\varphi(g_1, g_2) \tag{20}$$

and the coefficient functions γ_1, γ_2 and γ_φ are regular expansions in powers of g_1 and g_2, whose coefficients depend on the regulators ε and Δ. We shall refer to this situation as the regular multi-charge case.

Connections between renormalization constants and the corresponding RG functions in different schemes in this case are

$$Z_i'(g_1', g_2') = F_i(g_1, g_2) Z_i(g_1, g_2), \qquad i = 1, 2, \varphi,$$

$$\gamma_i'(g_1', g_2') = \gamma_i(g_1, g_2) + \sum_{j=1}^{2} \beta_j(g_1, g_2) \frac{\partial}{\partial g_j} \ln F_i(g_1, g_2). \tag{21}$$

Here and henceforth only fixed points with both non-vanishing charges ($g_1^* \neq 0$, $g_2^* \neq 0$) will be considered, if not stated otherwise. At a fixed point g_{1*}, g_{2*} of the RG $\beta_1(g_{1*}, g_{2*}) = 0$ and $\beta_2(g_{1*}, g_{2*}) = 0$. Therefore, the second term on the right side of (21) vanishes rendering the anomalous dimensions equal in the two renormalization schemes.

This is a global argument assuming that all functions in relation (21) are known completely. This is not the case, however, in perturbation theory. Renormalization constants and the RG functions are calculated order by order as power series in the charges g_1, g_2 . Typically, expansions of the coefficient functions start with a linear in charge term. In that case the linear term on the right side of (21) is produced by the function γ_i and the term $g\varepsilon$ multiplied by the coefficient of the linear term of $\partial_{g_j} \ln F_i$. The second contribution to the, say, β_1 function (18) is $\mathcal{O}(g_i^2)$ and it should not be included in the linear contribution to the right side (21) and the vanishing at the fixed point factor is lost! Obviously, the same property holds at every finite order of perturbation theory and we arrive at the conclusion that in the perturbative analytic renormalization the value of the anomalous dimension at a non-trivial ($g_{i*} \neq 0$) fixed point heavily depends on the renormalization scheme.

When the problem is treated in the framework of analytic renormalization, coefficients of the perturbation expansion of the RG functions are regular functions of the two parameters ε and Δ at the origin by construction of the renormalization scheme. Due to the analytic properties of the RG functions

the perturbative non-trivial fixed point may be found in the form of a double expansion in ε and Δ.

Regularity of the fixed points and RG functions imply that anomalous dimensions are obtained in the form of regular expansions in ε and Δ. Little reflection shows that the second term in relation (21) at a fixed point gives rise to a contribution which is of higher order by $\mathcal{O}(\varepsilon)$ or $\mathcal{O}(\Delta)$ in comparison with the double expansion of the anomalous dimension in the two renormalization schemes. However, in practical calculations instead of the analytic renormalization the ray scheme is used, in which the regulators are proportional to each other and the renormalization is carried out as in dimensional renormalization with an additional finite and fixed parameter $\zeta = \Delta/\varepsilon$. At one-loop order the γ's are linear functions of the charges vanishing at the origin with coefficients which are regular functions of ε and Δ in the ray scheme as well. At higher orders, divergent subgraphs produce denominators of the structure $(m\varepsilon + n\Delta)$, where integers m and n are determined by the interaction in the model and the structure of the subgraph. The contribution of a given graph to a renormalization constant is a product of such denominators from divergent subgraphs (including the graph itself, if it is superficially divergent) multiplied by a function analytic in regulators at the origin. In the analytic renormalization there is no trace of these denominators left in the analytic coefficient functions of the RG. In the ray scheme, however, a common factor ε is extracted giving rise to denominators of the structure $(m + n\zeta)$. As a consequence, coefficients of the emerging Laurent series in ε are meromorphic functions of ζ. In particular, in the subsequent minimal subtraction scheme with respect to ε this is true for the coefficient functions of the RG (for a detailed example, see Adzhemyan *et al* [5]). These meromorphic functions produce contributions to finite renormalization the MS calculation with respect to ε. It should borne in mind, however, that the ray scheme has not been proven to be a consistent renormalization procedure. On the contrary, the fact that the coefficient functions of the RG produced within it contain traces of explicit UV singularities in the analytic renormalization – which has been proven to be a consistent renormalization procedure (see, e.g. Zavyalov [8]) – casts serious doubts about the validity of the results obtained in the ray scheme beyond the one-loop order (nonwithstanding this, physical results may coincide, but a separate check is needed, see Adzhemyan *et al* [5]).

Apart from the case of several powerlike correlation functions leading to analytic renormalization with several regulators, the other case widely met is dimensional regularization amended by analytic regularization (only one analytic regulator will be considered here, although several have been introduced). In this case either in propagators or interactions the wave-number dependence contains the combination $a + bk^{-2\alpha}$, in which $\alpha > 0$ (in propagators this combination is usually multiplied by the factor k^2). For small α a non-trivial problem of renormalization of field operators with this structure arises (Weinrib and Halperin [10], Honkonen and Nalimov [11,4], Blavatska *et al.* [12], Antonov *et al.* [33,14,15,16,17,22], Goncharenko and Gopinathan [21]), since in the limit $\alpha \to 0$ the terms in $a + bk^{-2\alpha}$ become indistinguishable and it is not clear, which of them should be renormalized. The problem is solved by the

prescription of the counter terms to renormalization of the local (analytic in k^2) contribution (Honkonen and Nalimov [4], Adzhemyan *et al.*[5]). The basic idea is that renormalization produces only local counterterms. Construction of renormalization constants is carried out in the regularized model, in which the local and non-local term are clearly distinguishable ($\alpha > 0$ although small) and the counterterms have the structure of the local term and thus contribute to the renormalization of that term only. If the original model did not contain the local term at the outset (which is often the case when models with long-range effects are constructed), then it is usually brought about by the renormalization procedure as in Weinrib and Halperin [10]. In the field-theoretic approach such "generation terms" are to the original model to make it multiplicatively renormalizable, which is very convenient from the technical point of view.

In many cases the analytic properties of RG functions in problems with combined dimensional and analytic regulators are analogous to those of the case with two analytic regulators. A different situation takes place, for instance, in critical systems with quenched disorder (Weinrib and Halperin [10], Honkonen and Nalimov [11], Blavatska *et al.* [12], Goncharenko and Gopinathan [21]) and in stochastic hydrodynamics with competing long-range and short-range correlations (Honkonen and Nalimov [4]). The interplay of long-range and short-range correlations is accompanied by the appearance of generation terms. Generation terms are contributions to renormalization of a charge produced by other charges only. Generation terms produce contributions to renormalization constant of the corresponding charges in which the charge corresponding to the generation term stands in the denominator of a polynomial functions of other charges. This introduces significant changes to conclusions obtained from connections between renormalization constants and charges in different schemes.

Contrary to the regular multi-charge case the fixed-point values of charges in the analytic renormalization are not regular functions of the regulators (although the RG functions are). Therefore, critical exponents may not be regular functions of regulators either. Another feature of this class of models is that the very number of the fixed points becomes scheme dependent. This may be seen in the example of stochastic hydrodynamics near two dimensions, in which one-loop calculations in four different schemes are available (Honkonen and Nalimov [4], Hnatič *et al.* [9], Adzhemyan *et al.*[5,6]). In the MS scheme in the ray approach the one-loop solution for the two charges $g_1^* \neq 0$ and $g_2^* \neq 0$ is obtained from a system of equations which is essentially linear and the solution is unique (Honkonen and Nalimov [4]), whereas in the other schemes the one-loop equation for charges is quadratic (Hnatič *et al.* [9], Adzhemyan *et al.* [5]) with two different solutions corresponding to different choices of the sign of the quadratic root in the solution. In most cases only the stable fixed point with a regular expansion in regulators has been discussed, however, with modifications taking into account the additional solution (Hnatič *et al.* [9], Antonov [33]).

3   Stochastic Navier-Stokes problem near two dimensions

Problems inherent in the MS ray scheme are illustrated here by an analysis of the problem of randomly stirred fluid near two dimensions (1), (2) with the kernel function containing both the powerlike term and the local generation term (7). The model is logarithmic in two space dimensions ($\Delta = 0$) and $\varepsilon = 0$.

Standard power counting (see, e.g., Vasil'ev [23]) shows that in this problem near two dimensions real degrees of divergence of 1PI Green functions $\Gamma_{v'v}$ and $\Gamma_{v'v'}$ are

$$\delta'_{\Gamma_{v'v}} = n\Delta - m\varepsilon\,, \qquad \delta'_{\Gamma_{v'v'}} = (n-1)\Delta - m\varepsilon\,,$$

where m and n are powers of g_1 and g_2, respectively, in the graph of the 1PI Green function. Contribution of a divergent subgraph to renormalization constant is the product of factors of the structure

$$\frac{1}{n\Delta - m\varepsilon}\,, \qquad \frac{1}{(n-1)\Delta - m\varepsilon} \qquad (22)$$

multiplied by function analytic in ε, Δ at the origin. Terms of the usual ε expansion of the stochastic Navier-Stokes problem are obtained by expanding expressions (22) and the analytic coefficients in ε at fixed (say, $\Delta = \frac{1}{2}$ for $d = 3$) with the subsequent extraction of divergences in the MS scheme. Coefficients of this ε expansion are singular in the limit $d \to 2$ and turn out to be numerically large even at the physical dimension $d = 3$. These singularities have been summed up with the use of the double expansion (Adzhemyan *et al.* [5]), but results of calculations carried out in the ray scheme need additional checking, because traces of UV-divergences remain in high-order graphs.

All singularities are removed in the analytic renormalization: therefore, to be sure that renormalization is carried out consistently, the starting point should be the analytically regularized model logarithmic with respect to dimensional regulator as well. This is sometimes called the principle of maximum divergences. As in all renormalization schemes, there is a lot of freedom in the choice of any concrete version of the analytic renormalization. In particular, if the model is analytically regularized, renormalization may be carried out with the use of a scheme based on the subtraction of necessary coefficients of the Taylor expansion at a given value of external momenta of a divergent graph. Due to the renormalization theorem the result is free from UV divergences and thus an analytic function of regulators at the origin. Such "normalization point" schemes have been used in the stochastic Navier-Stokes problem as well (Hnatič *et al.* [9], Adzhemyan *et al.*[5,6]).

To illustrate the root of the problem consider calculation of RG functions in the ray scheme. In the MS scheme the coefficient functions are calculated by acting on simple pole terms in ε of a renormalization constant by the operator

$$\left(-\varepsilon g_1 \partial_{g_1} + \Delta g_2 \partial_{g_2}\right) g_1^m g_2^n = \left(-n\varepsilon + m\Delta\right) g_1^m g_2^n = \varepsilon\left(-n + m\zeta\right) g_1^m g_2^n\,.$$

The coefficient $(-n + m\zeta)$ cancels only the denominator corresponding to the superficial degree of divergence of the graph, all contributions of divergent subgraphs remain in the form of a meromorphic function of ζ. To infer physival

information of the model, results of the expansion in regulators are extrapolated to values of regulators corresponding to the physical case. Physical values of the regulators in the stochastic Navier-Stokes problem are $\varepsilon = 2$ (corresponding to energy injection at the origin in the wave-vector space) and $\Delta = \frac{1}{2}$ (corresponding to $d = 3$). From expressions (22) we see that for $m = 1$, $n = 4$ the UV denominators vanish at the physical values of the regulators! It is readily seen that this takes place for five-loop graphs of the model. Therefore, starting from six-loop contributions to RG functions in the ray scheme terms will appear which simply are meaningless (denominators vanish) at the physical values of the parameters $\varepsilon = 2$, $\Delta = \frac{1}{2}$. Although it may be that calculations to such high order will never be carried out, the very existence of this phenomenon invalidates the MS ray scheme for the stochastic Navier-Stokes problem.

4   Improved ε expansion in the RG analysis of turbulence

A specific feature of the renormalization-group approach in the theory of developed turbulence is that the formal small parameter ε is not connected with the space dimension and it is determined only by the noise correlator of random forcing in the stochastic Navier-Stokes equation (DeDominicis and Martin [1], Adzhemyan *et al.* [2]). Its physical value $\varepsilon = 2$ is not small Adzhemyan *et al.* [36,37], hence reasonable doubts arise about the effectiveness of such an expansion. For some paramount physical quantities like the critical dimensions of the velocity field and the coefficient of viscosity the ε expansion terminates at the first term due to the Galilei invariance of the theory (DeDominicis and Martin [1], Adzhemyan *et al.* [2]). Therefore, exact values are predicted for these quantities. However, there are other physically important quantities, viz. the skewness factor, the Kolmogorov constant and critical dimensions of various composite operators, for which the ε series do not terminate (Adzhemyan *et al.* [38,32,5]), therefore the question about the effectiveness of the expansion remains open.

Consider a quantity A calculated at the fixed point of the RG in the renormalized field theory of developed turbulence. In d dimensions it is a function of the parameters ε and d: $A = A(\varepsilon, d)$. In practice, calculations are often carried out in the ε expansion, whose coefficients for the quantity $A(\varepsilon, d)$ depend on the space dimension d

$$A(\varepsilon, d) = \sum_{k=0}^{\infty} A_k(d)\varepsilon^k. \tag{23}$$

Analysis shows that these coefficients $A_k(d)$ have singularities at small dimension $d \leq 2$. The singularity at $d = 2$ – the nearest to the physical value $d = 3$ – gives rise to new divergences as $d \to 2$ not eliminated by the renormalization of the d-dimensional theory (Adzhemyan *et al.* [5], Ronis [26], Honkonen and Nalimov[4]). These divergences manifest themselves in the form of poles in the parameter $(d - 2) \equiv 2\Delta$ in the coefficients of the ε expansion $A_k(d)$, which

therefore may be expressed as Laurent series of the form

$$A_k(d) = \sum_{l=0}^{\infty} a_{kl}\, \Delta^{l-k}.$$

A two-loop calculation of the Kolmogorov constant and skewness factor at various values of space dimension d carried out in [38] has shown that at $d = 3$ the relative part of the two-loop contribution is comparable with the one-loop contribution. The two-loop contribution, however, rapidly decreases as d increases, and at $d = 5$ it gives only 30 %, and at $d \to \infty$ decreases to 10 %. On the contrary, when the space dimension decreases from $d = 3$ to $d = 2$ rapid growth of the two-loop correction term is observed. This growth is due to diagrams which contain singularities at $d = 2$. Analysis has shown that it is just these diagrams which form the main part of two-loop contribution at $d = 3$. Therefore, the nearest singularity strongly manifests itself at the realistic value $d = 3$ and allows to improve the ε expansion by means of summation of singular contributions in all orders of this expansion (Adzhemyan *et al.* [32,5]).

Divergences in Δ may be absorbed to suitable additional counterterms, which gives rise to a different renormalized field theory (the physical unrenormalized field theory is, of course, the same in both cases) with two formal small parameters ε and Δ. In the MS ray scheme – used by Adzhemyan *et al.* [32,5] – these parameters satisfy the relation $\zeta \equiv \varepsilon/\Delta = const$. In this case new ε expansion proposed by Honkonen and Nalimov [4], an alternative to the expansion (23), may be constructed for A:

$$A(\varepsilon, \zeta) = \sum_{k=0}^{\infty} b_k(\zeta)\varepsilon^k, \qquad \zeta \equiv \varepsilon/\Delta. \tag{24}$$

Account of the additional information in expansion (24) has led to much better of agreement between the calculated and experimental values of the Kolmogorov constant and skewness factor in the "improved ε expansion" of Adzhemyan *et al.* [32,5] in comparison with the results of the usual ε expansion (Adzhemyan *et al.* [38]).

However, due to problems in the MS ray scheme the analytic renormalization should be preferred. In fact, the "normalization point" scheme used in Adzhemyan *et al.* [5] to corroborate the results of the improved ε expansion is an analytic-renormalization scheme with a specific choice of normalization. In particular, all RG functions in this approach calculated to two-loop order by Adzhemyan *et al.* [5] are analytic functions of regulators, contrary to the RG functions in the MS ray scheme.

Here, a slightly different version of the normalization point scheme – proposed in Adzhemyan *et al.* [6] – will be used. The only difference between the two cases is in the normalization condition, the rest of the analysis is completely analogous in both cases.

Let us first remind the results of the original one-loop calculation (Honkonen and Nalimov [4]) with the use of the MS ray scheme (we recall that at one-loop

level the MS ray scheme is fully applicable)

$$\gamma_1 = -\frac{3}{16}(u_1 + u_2), \quad \gamma_2 = \frac{(u_1 + u_2)^2}{16\,u_2} - \frac{3}{16}(u_1 + u_2), \tag{25}$$

where the normalization (13) is used.

The anomalous asymptotic behavior of the long-range model above two dimensions is governed by the fixed point

$$u_{1*} = \frac{32\varepsilon(3\Delta + 2\varepsilon)}{9(\Delta + \varepsilon)}, \qquad u_{2*} = \frac{32\varepsilon^2}{9(\Delta + \varepsilon)}. \tag{26}$$

It should be noted that the fixed point $u_{1*} \neq 0$, $u_{2*} \neq 0$ is unique due to the degeneracy of the β functions (25) in the ray scheme. We recall that in general there is a quadratic equation for the fixed-point values of the charges.

In the normalization-point scheme, introduce normalized scalar one-irreducible functions as

$$\overline{\Gamma}_{v'v} = \frac{\langle v_i' v_i \rangle_{1-\mathrm{ir}}|_{\omega=0}}{\nu p^2 (1 - d)}, \quad \overline{\Gamma}_{v'v'} = \frac{\langle v_i' v_i' \rangle_{1-\mathrm{ir}}|_{\omega=0}}{\nu^3 p^2 (d - 1)} - g_1 (\mu/p)^{2\Delta + 2\varepsilon} - g_2$$

and determine the renormalization constants from the conditions

$$\overline{\Gamma}_{v'v}\bigg|_{\substack{p=0, \\ \mu=m}} = 1, \qquad \overline{\Gamma}_{v'v'}\bigg|_{\substack{p=0, \\ \mu=m}} = 0. \tag{27}$$

These normalization conditions are different from those used in Adzhemyan *et al.* [5].

Adzhemyan *et al.* [6] have proposed to use the renormalization scheme (27) *without* the ε, Δ expansion to take advantage in numerical results of the scheme-dependence of renormalized quantities and have shown that such a scheme reproduces correctly the leading terms of expansion in both regimes $\varepsilon \to 0$, $\Delta = const$ and $\varepsilon \sim \Delta \to 0$ simultaneously.

In fact, a far stronger conjecture can be put forward upon the analysis of the structure of the renormalized field theory at the fixed point of the RG: with the use of maximum divergences the analytically renormalized stochastic Navier-Stokes model gives rise to an ε expansion, whose coefficients are calculated exactly for arbitrary space dimension $d \geq 2$ (i.e. not only as expansions in Δ as in the double expansion).

One-loop calculation of renormalization constants with the normalization condition (27) yields

$$Z_\nu = 1 + \frac{d - 1}{4(d + 2)}\left(-\frac{u_1}{2\varepsilon} + \frac{u_2}{2\Delta}\right), \tag{28}$$

$$Z_{D_2} = 1 + \frac{d^2 - 2}{4d(d + 2)}\left[-\frac{u_1^2}{2(2\varepsilon + \Delta)u_2} - \frac{u_1}{\varepsilon} + \frac{u_2}{2\Delta}\right]. \tag{29}$$

From (9), (28) and (29) the renormalization constants of the charges u_1 and u_2 are determined as

$$Z_{u_1} = 1 + \frac{3(d-1)}{4(d+2)}\left(\frac{u_1}{2\varepsilon} - \frac{u_2}{2\Delta}\right),$$

$$Z_{u_2} = 1 + \frac{d^2 - 2}{4d(d+2)}\left[-\frac{u_1^2}{2(2\varepsilon + \Delta)\,u_2} - \frac{u_1}{\varepsilon} + \frac{u_2}{2\Delta}\right] + \frac{3(d-1)}{4(d+2)}\left(\frac{u_1}{2\varepsilon} - \frac{u_2}{2\Delta}\right).$$

The corresponding one-loop RG functions are

$$\gamma_1 = -\frac{3(d-1)}{4(d+2)}(u_1 + u_2)\,, \tag{30}$$

$$\gamma_2 = \frac{(d^2 - 2)(u_1 + u_2)^2}{4d(d+2)\,u_2} - \frac{3(d-1)}{4(d+2)}(u_1 + u_2). \tag{31}$$

With the use of (12), (30) and (31), the coordinates of the nontrivial fixed point $u_{1*} > 0$, $u_{2*} > 0$ are found as the solution of the equations $\beta_1(u_*) = 0$, $\beta_2(u_*) = 0$ in the form

$$u_{1*} + u_{2*} = \frac{8\varepsilon(d+2)}{3(d-1)}\,, \tag{32}$$

$$u_{2*} = \frac{\varepsilon^2}{2\varepsilon + d - 2}\,\frac{16(d^2 - 2)(d+2)}{9d(d-1)^2}\,. \tag{33}$$

We are analyzing asymptotic behaviour of correlation functions and other relevant quantities calculated in the form of perturbation expansion in the charges u_1 and u_2. Physical expressions for the results are obtained at the fixed point of the RG. Inspection of expressions for the fixed point values in both presented cases (26) and (32), (33) reveals that fixed-point values of both charges are small, when ε is small irrespective of the value of the space dimension. In particular, for the physical value of space dimension $d = 3$ the parameter Δ is not small and both (26) and (32), (33) indicate that $u_{1*} = O(\varepsilon)$ and $u_{2*} = O(\varepsilon^2)$ giving rise to an improved ε expansion, which takes into account additional divergences near $d = 2$ to all orders in the deviation $\Delta = (d-2)/2$. Coefficients of the improved expansion are calculated as functions of the space dimension in a model, which is UV renormalized at two dimensions. These results are then extrapolated to values with a finite deviation from $d = 2$, since Δ is not needed as an expansion parameter in the renormalized model at the fixed point. This seems similar to the extrapolation of the results of the $\varepsilon = 4 - d$ expansion in the theory of critical phenomena to the physical value $\varepsilon = 4 - 3 = 1$. There is an important difference, however. In the theory of critical phenomena the extrapolation is made to values of ε at which there is no regular way to directly take into account the IR divergences (below $d_c = 4$). In the Navier-Stokes problem the extrapolation is made to values of Δ, for which there are less and less severe IR divergences than in the critical theory.

5 Calculation of the Kolmogorov constant through the skewness factor

In the field-theoretic RG approach several ways have been proposed (Adzhemyan *et al.* [39,38,5,6], Honkonen [34], Hnatič *et al.* [30,9]) to calculate the (non-universal) amplitude factor – the Kolmogorov constant – in Kolmogorov's 5/3 law for the turbulent energy spectrum

$$E(k) \sim C_K' \overline{\mathcal{E}}^{2/3} k^{-5/3} , \tag{34}$$

where $\overline{\mathcal{E}}$ is the average energy injection rate per unit mass. Different approximations for the connection of model parameters and the average energy injection rate lead have resulted in different values for the Kolmogorov constant. At present the most reliable approach appears to be that based on the connection between the Kolmogorov constant and the skewness factor (Adzhemyan *et al.* [38]), whose value at the fixed point is a function of regulators and independent of the model parameter D_{10} of energy injection. The renormalization scheme and the improved ε expansion of Adzhemyan *et al.* [6] allows to obtain best match with experimental data up to date.

The Kolmogorov constant is not determined uniquely in the ε expansion in the model with power-law injection (3) (for details, see Adzhemyan *et al.* [5]). On the other hand physical quantities independent of the amplitude D_{10} (3) (universal quantities) are determined unambiguously in the framework of the ε expansion. The skewness factor

$$\mathcal{S} \equiv S_3 / S_2^{3/2}, \tag{35}$$

is an example of such a quantity. In (35) S_n are structure functions defined by relations

$$S_n(r) \equiv \big\langle [v_r(t, \mathbf{x} + \mathbf{r}) - v_r(t, \mathbf{x})]^n \big\rangle, \qquad v_r \equiv (\mathbf{v} \cdot \mathbf{r})/r, \quad r \equiv |\mathbf{r}|.$$

According to the Kolmogorov theory, the second-order structure function $S_2(r)$ in the inertial range is of the form

$$S_2(r) = C_K \mathcal{E}^{2/3} r^{2/3}, \tag{36}$$

where $\mathcal{E}$ is the average energy dissipation rate per unit mass (in the steady state it coincides with the mean energy injection rate $\overline{\mathcal{E}}$, see Eq. (34)) and C_K is the Kolmogorov constant, the value of which is not determined in the framework of the phenomenological approach. Although there is strong experimental evidence that the Kolmogorov scaling $S_n(r) \sim r^{n/3}$ does not hold in the inertial range for the structure functions of order $n \geq 4$, for the second-order structure function $S_2(r)$ the experimental situation about anomalous scaling [i.e., deviation of the power of r from the Kolmogorov value 2/3 in (17)] in the inertial range is still controversial and in any case this deviation is small (Barenblatt *et al.* [40], Benzi *et al.* [41]). Therefore, we shall use the Kolmogorov asymptotic expression (36) for the second-order structure function $S_2(r)$ in the following analysis.

The amplitude of the third-order structure function $S_3(r)$ is determined in the Kolmogorov theory exactly (see, e.g., Monin and Yaglom [42] and Frisch [43]):

$$S_3(r) = -\frac{12}{d(d+2)}\,\mathcal{E}\,r. \tag{37}$$

All these expressions together with (35), (36) allow to connect the Kolmogorov constant with the skewness factor:

$$C_K = \left[-\frac{12}{d(d+2)\mathcal{S}}\right]^{2/3}. \tag{38}$$

Among the three quantities $S_2(r)$, $S_3(r)$ and $\mathcal{S}$ only the last one is a unique well-defined function of regulators. Thus, relation (38) (valid only at the physical value $\varepsilon = 2$) may be used to determine C_K by means of the calculated value $\mathcal{S}(\varepsilon = 2)$.

To find the RG representation of the skewness factor (35) the RG representations of the functions $S_2(r)$ and $S_3(r)$ have to be determined. The function $S_2(r)$ is connected with the Fourier transform of the pair correlation function $G(p)$ by relation

$$S_2(r) = 2\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\,G(k)\left[1 - (\mathbf{k}\cdot\mathbf{r})^2/(kr)^2\right]\{1 - \exp\left[\mathrm{i}(\mathbf{k}\cdot\mathbf{r})\right]\}. \tag{39}$$

Therefore, the RG representation of $S_2(r)$ can be specified with the aid of the RG representation (17). A similar RG representation can be written for the function $S_3(r)$. It is, however, more convenient to use the exact result analogous to expression (37)

$$S_3(r) = -\frac{3(d-1)\,\Gamma(2-\varepsilon)\,(r/2)^{2\varepsilon-3}D_{10}}{(4\pi)^{d/2}\,\Gamma(d/2+\varepsilon)}. \tag{40}$$

This relation clearly demonstrates that the amplitude of the structure function expressed through D_{10} has a singularity at $\varepsilon \to 2$. In this case the singularity is in the form $\sim(2-\varepsilon)^{-1}$. After the substitution of the amplitude $D_{10}\sim(2-\varepsilon)$ into (40) the singularity on the right-hand side of (40) is canceled by the node of D_{10}. This leads to a finite expression for $S_3(r)$ at $\varepsilon = 2$ which coincides with (37).

Relations (17), (39) and (40) could be used as the basis for the construction of the ε expansion of the skewness factor (35), but on this way there is an additional complication. The point is that the behavior $S_2(r)\sim r^{2-2\varepsilon/3}$, which is determined by power counting from (39) and (17), is valid only at $\varepsilon > 3/2$, because at $\varepsilon < 3/2$ the integral (39) diverges as $k \to \infty$ [it means that in this case the leading contribution to $S_2(r)$ is given by the term $\langle v_r^2(t,\mathbf{x})\rangle$ independent of r]. The derivative $r\partial_r S_2(r)$, however, is free from this flaw and according to (39) it assumes the form

$$r\partial_r S_2(r) = 2\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\,G(k)\left[1 - (\mathbf{k}\cdot\mathbf{r})^2/(kr)^2\right](\mathbf{k}\cdot\mathbf{r})\sin(\mathbf{k}\cdot\mathbf{r}). \tag{41}$$

The integral in (41) converges at all values $0 < \varepsilon < 2$. On the other hand, at the physical value $\varepsilon = 2$ the amplitudes in $S_2(r)$ and $r\partial_r S_2(r)$ differ from each other only by the factor $2/3$, therefore the dependence on regulators is sought the following analogue of the skewness factor (Adzhemyan *et al.* [38,32,35])

$$Q(\varepsilon) \equiv \frac{r\partial_r S_2(r)}{|S_3(r)|^{2/3}} = \frac{r\partial_r S_2(r)}{(-S_3(r))^{2/3}}. \tag{42}$$

The Kolmogorov constant and the skewness factor are expressed through the value $Q(\varepsilon = 2)$ according to (35), (36) and (37) by relations:

$$C_K = \left[\frac{3Q(2)}{2}\right]\left[\frac{12}{d(d+2)}\right]^{2/3}, \quad \mathcal{S} = -\left[\frac{3Q(2)}{2}\right]^{-3/2}. \tag{43}$$

Substituting expressions (41), (17) and (40) into (42) we obtain

$$Q(\varepsilon) = \left[\frac{4(d-1)}{9u_{1*}^2}\right]^{1/3} A(\varepsilon)\, R(1, u_{1*}, u_{2*}), \tag{44}$$

where

$$A(\varepsilon) = \frac{\Gamma(2 - 2\varepsilon/3)\Gamma^{1/3}(d/2)\Gamma^{2/3}(d/2+\varepsilon)}{\Gamma(d/2 + 2\varepsilon/3)\Gamma^{2/3}(2 - \varepsilon)} = 1 + \mathcal{O}(\varepsilon^2). \tag{45}$$

At the leading order the scaling function $R(1, u_{1*}, u_{2*})$ in (44) is

$$R(1, u_{1*}, u_{2*}) \approx \frac{u_{1*} + u_{2*}}{2}. \tag{46}$$

Putting $A(\varepsilon) \approx 1$ we obtain in one-loop approximation

$$Q(\varepsilon) = \left[\frac{4(d-1)}{9(u_{1*})^2}\right]^{1/3} \cdot \frac{u_{1*} + u_{2*}}{2}. \tag{47}$$

Results of the original one-loop calculation (26) (Honkonen and Nalimov [4]) give rise to values

$$Q(4) = 1,30\,; \qquad C_K = 1,68\,; \qquad \mathcal{S} = -0,37\,.$$

The values $C_K \approx 2.01$ and $\mathcal{S} \approx -0.28$ are considered the most reliable experimental values of these quantities (Sreenivasan [44]). We remind that the usual ε expansion for $d = 3$ at one-loop order gives the values $C_K \approx 1.47$ and $\mathcal{S} \approx -0.45$ (Adzhemyan *et al.* [38]). Substitution of values (32) and (33) into (47) yields

$$Q(4) = 1,46\,; \qquad C_K = 1,89\,; \qquad \mathcal{S} = -0,31\,, \tag{48}$$

which are significantly closer to the recommended experimental values than those of the ε expansion in fixed dimension and the double expansion. The agreement of the one-loop result (48) with the experimental results is quite reasonable.

6   Conclusion

In this report the effect of various renormalization schemes on the asymptotic results for physical quantities has been analyzed. Differences between the regulator expansion and the loop expansion have been pointed out. It is demonstrated that the popular minimal-subtraction scheme in the ray approximation is not a consistent UV renormalization procedure in calculations beyond one-loop order. It is conjectured that a consistent analytic renormalization scheme should be used and that normalization-point schemes are probably the most economic way to implement analytic renormalization. It is shown that in the paradigmatic problem of randomly stirred fluid analytic renormalization starting from maximum divergences gives rise to an expansion in the deviation of the falloff power of force correlations from the logarithmic value such that the coefficient functions of this expansion may be extrapolated to arbitrary values of the space dimension greater than the critical dimension two.

References

1. C. DeDominicis and P. C. Martin. Energy spectra of certain randomly-stirred fluids. *Phys. Rev. A*, 19, 1, 419–422, 1979.
2. L. Ts. Adzhemyan, A. N. Vasil'ev and Yu. M. Pis'mak. Renormalization-group approach in the theory of turbulence: The dimensions of composite operators. *Theor. Math. Phys.*, 57, 2, 1131–1141, 1983.
3. V. Yakhot and S. A. Orszag. Renormalization-group analysis of turbulence. *Phys. Rev. Lett.*, 57, 14, 1722–1724, 1986.
4. J. Honkonen and M. Yu. Nalimov. Two-parameter expansion in the renormalization-group analysis of turbulence. *Z. Phys. B*, 99, 2, 297–303, 1996.
5. L. Ts. Adzhemyan, J. Honkonen, M. V. Kompaniets and A. N. Vasil'ev. Improved ε expansion for three-dimensional turbulence: Two-loop renormalization near two dimensions. *Phys. Rev. E*, 71, 3, 036305 (19), 2005.
6. L. Ts. Adzhemyan, M. Hnatich and J. Honkonen. Improved ε expansion in the theory of turbulence: summation of nearest singularities by inclusion of an infrared irrelevant operator.*Eur. Phys. J. B*, 73, 275–285, 2010.
7. D. Forster, D. R. Nelson and M. J. Stephen. Large-distance and long-time properties of a randomly stirred fluid. *Phys. Rev. A*, 16, 2, 732–749, 1977.
8. O. I. Zavyalov. *Perenormirovannye diagrammy Feinmana (Renormalised Feynman diagrams)*, Nauka, Moskva, 1979.
9. M. Hnatich, J. Honkonen and M. Jurcisin. Stochastic magnetohydrodynamic turbulence in space dimensions $d \geq 2$. *Phys. Rev. E*, 64, 5, 056411 (11), 2001.
10. A. Weinrib and B. I. Halperin. Critical phenomena in systems with long-range-correlated quenched disorder. *Phys. Rev. B*, 27, 1, 413– 427, 1983.
11. J. Honkonen and M. Yu. Nalimov. Crossover between field theories with short-range or long-range exchange or correlations. *J. Phys. A*, 22, 751–763, 1989.
12. V. Blavats'ka, C. von Ferber and Yu. Holovatch. Polymers in long-range-correlated disorder. *Phys. Rev. E*, 64, 4, 041102(10), 2001.
13. N. V. Antonov, M. Hnatich and J. Honkonen. Effects of mixing and stirring on the critical behaviour. *J. Phys. A*, 39, 7867–7887, 2006.
14. N. V. Antonov and A. A. Ignatieva. Critical behaviour of a fluid in a random shear flow: renormalization group analysis of a simplified model. *J. Phys. A*, 39, 13593–13620, 2006.

15. N. V. Antonov, V. I. Iglovikov and A. S. Kapustin. Effects of turbulent mixing on the nonequilibrium critical behaviour. *J. Phys. A*, 42, 135001(19), 2009.

16. N. V. Antonov, A. S. Kapustin and A. V. Malyshev. Effects of turbulent transfer on critical behaviour. *Theor. Math. Phys.*, 169, 1, 1470–1480, 2011.

17. N. V. Antonov and A. S. Kapustin. Critical behaviour of the randomly stirred dynamical Potts model: novel universality class and effects of compressibility. *J. Phys. A*, 45, 505001(20), 2012.

18. Zh. S. Gevorkian and Yu. E. Lozovik. Classical diffusion in random fields with long-range correlations. *J. Phys. A*, 20, L659–L664, 1987.

19. J. Honkonen and E. Karjalainen. Diffusion in a random medium with long-range correlations. *J. Phys. A*, 21, 4217–4234, 1988.

20. J. Honkonen. Stochastic processes with stable distributions in random environments. *Phys. Rev. E*, 53, 1, 327–331, 1996.

21. I. Goncharenko and A. Gopinathan. Vicious walks with long-range interactions. *Phys. Rev. E* 82, 1, 011126(8), 2010.

22. N. V. Antonov and P. I. Kakin. Random interface growth in a random environment: renormalization group analysis of a simple model. *Theor. Math. Phys.*, 185, 1, 1391–1407, 2015.

23. A. N. Vasil'ev". *The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics*, Chapman Hall/CRC, Boca Raton, 2004.

24. J. D. Fournier, P. L. Sulem and A. Pouquet. Infrared properties of forced magnetohydrodynamic turbulence. *J. Phys. A*, 15, 1393–1420, 1982.

25. L. Ts. Adzhemyan, A. N. Vasil'ev and M. Gnatich. Quantum-field renormalization group in the theory of turbulence: Magnetohydrodynamics. *Theor. Math. Phys.*, 64, 2, 777–785, 1985.

26. D. Ronis. Field-theoretic renormalization group and turbulence. *Phys. Rev. A*, 36, 7, 3322–3331, 1987.

27. C. G. Bollini and J. J. Giambiagi. Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter. *Nuovo Cimento*, 12B, 1, 20–26, 1972.

28. M. Hnatich. Quantum-field renormalization group in turbulence theory: Chemically active scalar admixture. *Theor. Math. Phys.*, 83, 3, 601–608, 1990.

29. N. V. Antonov. The renormalization group in the problem of turbulent convection of a passive scalar impurity with nonlinear diffusion. *JETP*, 85, 5, 898–906, 1997.

30. M. Hnatich, J. Honkonen, D. Horváth and R. Semančík. Advection of passive scalar near two dimensions. *Phys. Rev. E*, 59, 4, 4112–4121, 1999.

31. A. V. Gladyshev, E. Jurčišinová, M. Jurčišin, R. Remecký and P. Zalom. Anomalous scaling of a passive scalar field near two dimensions. *Phys. Rev. E* 86, 3, 036302(12), 2012.

32. L. Ts.Adzhemyan, J. Honkonen, M. V. Kompaniets and A. N. Vasil'ev. Improved ε expansion for three-dimensional turbulence: Summation of nearest dimensional singularities. *Phys. Rev. E* 68, 5, 055302(R)(4), 2003.

33. N. V. Antonov. Renormalization group, operator product expansion and anomalous scaling in models of turbulent advection. *J. Phys. A*, 39, 7825–7865, 2006.

34. J. Honkonen. Asymptotic behavior of the solution of the two-dimensional stochastic vorticity equation. *Phys. Rev. E*, 58, 4, 4532–4540, 1998.

35. L. Ts.Adzhemyan, N. V. Antonov, M. V. Kompaniets and A. N. Vasil'ev. Renormalization group in the statistical theory of turbulence: Two-loop approximation. *Acta Physica Slovaca*, 5, 6, 565–571, 2002.

36. L. Ts. Adzhemyan, N. V. Antonov and A. N. Vasil'ev. Quantum field renormalization group in the theory of fully developed turbulence. *Phys. Usp.*, 39, 12, 1193–1219, 1996.

37. L. Ts. Adzhemyan, N. V. Antonov and A. N. Vasil'ev. *The Field Theoretic Renormalization Group in Fully Developed Turbulence.*, Gordon & Breach, London, 1999.

38. L. Ts. Adzhemyan, N. V. Antonov, M. V. Kompaniets and A. N. Vasil'ev. Renormalization-group approach to the stochastic Navier-Stokes equation: Two-loop approximation. *Int. J. Mod. Phys. B*, 17, 10, 2137–2170, 2003.

39. L. Ts. Adzhemyan, N. V. Antonov and A. N. Vasil'ev. Infrared divergences and the renormalization group in the theory of fully developed turbulence. *Sov. Phys. JETP*, 68, 4, 733–742, 1989.

40. G. I. Barenblatt, A. J. Chorin and V. M. Prostokishin. Comment on the paper On the scaling of three-dimensional homogeneous and isotropic turbulence by Benzi et al. *Physica D*, 127, 105–110, 1999.

41. R. Benzi, S. Ciliberto, C. Baudet and G. Ruiz Chavarria. On the scaling of three-dimensional homogeneous and isotropic turbulence. *Physica D*, 80, 385–398, 1995.

42. A. S. Monin and A. M. Yaglom. *Statistical Fluid Mechanics:Vol 2*, MIT Press, Cambridge, 1975.

43. U. Frisch. *Turbulence: The Legacy of A. N. Kolmogorov*, Cambridge University Press, Cambridge, 1995.

44. K. R. Sreenivasan. On the universality of the Kolmogorov constant. *Phys. Fluids* 7, 11, 2778–2784, 1995.