Approximately Conserved Quantities in The Matinyan Yang Mills Higgs System

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Abstract. The Matinyan Yang Mills Higgs System (abbreviated MYMH), a generalization of the truncations of the Toda Lattice, is a classical Hamiltonian system given below. It is well known that the Toda lattice is, while its truncations are not integrable [1]. Approximate integrals for the Toda system have been constructed. The present Hamiltonian has similar algebraic terms with somewhat different coefficients than the Toda truncations.

The MYMH system has been used for modelling the suppression of chaotic behavior in the classical Yang Mills system. [2] This has become necessary in light of our recent understanding on the stability of the universe and the mechanism for the onset of instability. In recent work, this analysis was done by numerical simulation. In this study, we present results for analyzing the possible candidates for approximately conserved quantities (approximate invariants) in light of our work on the Toda truncations. One possible approach is to start from the basic second order invariants, the energy and the angular momentum component which cease to be invariant in the higher orders. [3] The other approach would be to construct higher order invariants by selecting suitable combinations constructed from the truncated expressions which will have a higher degree Poisson bracket than the truncation order.

Keywords: MYMH systems, Toda systems, Yang Mills models, truncations..

1 Introduction

The Matinyan Yang Mills Higgs System (abbreviated MYMH) is a classical Hamiltonian system given by:

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 y^2}{2} + \frac{1}{2}g(x^2 + y^2) - \frac{1}{2}y^2 + \frac{1}{8}x^4 + \frac{1}{4}py^4$$
(1)

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The present Hamiltonian has similar algebraic terms with somewhat different coefficients than the second and fourth order Toda truncations. The Higgs term $py^4/4$ is an important reason for the onset of stability.

$$\dot{x} = p_x \tag{2a}$$

$$\dot{y} = p_y \tag{2b}$$

$$\dot{p_x} = -xy^2 - gx - \frac{1}{2}x^3$$
 (2c)

$$\dot{p_y} = -x^2y - gy + y - py^3 \tag{2d}$$

where $g, p \in \Re$

Let us analyze the characteristic equation of the system in natural basis:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -y^2 - g - \frac{1}{2}x^2 & 0 & 0 & 0 \\ 0 & -x^2 - g + 1 - py^2 & 0 & 0 \end{pmatrix}$$

The corresponding eigenvalues found from characteristic equation are:

$$\lambda_{1,2} = \pm \sqrt{-g - py^2 - x^2 + 1}$$
 and $\lambda_{3,4} = \pm \frac{\sqrt{-2g - x^2 - 2y^2}}{\sqrt{2}}$

It is clear that two of the eigenvalues are purely imaginary, while the other two start out as real \pm values but as x^2 and y^2 increases due to the positive eigenvalue, they also become purely imaginary. The system thus transitions into local stability.

In addition to the argument above, one can derive the Lyapunov function $\frac{1}{2}(x^2 + y^2)$ with the approximate time derivative containing dominant large x and y terms:

$$\frac{1}{2}\frac{d(x^2+y^2)}{dt} = -\frac{1}{2}x^4 - \frac{p}{4}y^4 \tag{3}$$

To derive this relation start with \ddot{x} and \ddot{y} which are :

$$\ddot{x} = -xy^2 - gx - \frac{1}{2}x^3$$
 (4a)

$$\ddot{y} = -x^2y - gy + y - \frac{p}{4}y^3 \tag{4b}$$

Then it is easy to calculate the quantity:

$$x\ddot{x} + y\ddot{y} = -gx^2 + (1-g)y^2 - \frac{1}{2}x^4 - \frac{p}{4}y^4$$
(5)

The time derivative is:

$$\frac{d(x\dot{x}+y\dot{y})}{dt} = -gx^2 + (1-g)y^2 - \frac{1}{2}x^4 - \frac{p}{4}y^4 + (\dot{x}^2 + \dot{y}^2) \tag{6}$$

In the case of $x \gg 1$ and $y \gg 1$:

$$\frac{1}{2}\frac{d(x^2+y^2)}{dt} = x\dot{x} + y\dot{y} = -\frac{1}{2}x^4 - \frac{p}{4}y^4 \tag{7}$$

This shows that in that limit fourth power terms will dominate second power terms. Since before this occurs, a stage where $O(x^2)$, $O(y^2)$ terms will be the comparable to the fourth degree terms, an initial instability which will transition to semi stability followed by asymptotic stability can be seen.

$\mathbf{2}$ Finding Approximate Integrals of Hamiltonian System

The Hamiltonian H provides an integral of motion \mathbf{I} (without explicit time dependence) where the Hamiltonian $H(x, y, p_x, p_y)$ becomes invariant under the flow generated by the Hamilton equations of motions and therefore I and H satisfies the Poisson bracket (henceforth referred to as PB) relation [I, H] = 0.

In this section, we consider the problem of finding approximately conserved quantities of the example Hamiltonian system given in the introduction.

We use the condition that the PB of the partial sums of the Hamiltonian truncated to given order \mathbf{n} and partial sums of the truncations to order \mathbf{n} of invariant should give a PB that vanishes to degree at least n and if possible to a higher degree l > n. If the bracket vanishes at a higher order, the truncation may be looked upon as a better approximate integral. To this end, we evaluate PBs of the partial sums for truncated YMH Hamiltonian to successive homogeneous orders and the analytic isolating integral given by I_{ij} . We generate lists of the first four order truncations for YMH Hamiltonian H_T and the original Toda third integral I_{ij} where j term is considered to define the truncation order given by H_T .

$$H_T(1) = 0 \tag{8a}$$

$$H_T(2) = \frac{p_x^2 + p_y^2}{2} + \frac{g(x^2 + y^2)}{2} - \frac{y^2}{2}$$
(8b)

$$H_T(3) = 0 \tag{8c}$$

$$H_T(4) = \frac{x^2 y^2}{2} + \frac{x^4}{8} + \frac{p y^4}{4}$$
(8d)

The corresponding truncations of I_{ii} can be of interest as approximately conserved quantities.

$$I_{ii}(0) = 0 \tag{9a}$$

$$I_{ii}(2) = (p_x y - x p_y)$$
(9b)
$$I_{ii}(3) = 4(3n x^2 - 6n^2n - 6xun + 2n^3 - n u)$$
(9c)

$$I_{ii}(3) = 4(3p_xx^2 - 6p_y^2p_x - 6xyp_y + 2p_x^3 - p_xy)$$
(9c)

$$I_{ii}(4) = 24(p_x y - x p_y)(x^2 + y^2)$$
(9d)

Notice that $(p_x y - x p_y)$ is related to the rotational symmetry about the z-axis.

It is possible to construct a candidate third integral involving the linear combination these quantities. This proposed integral is composed of only third

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degree, but not fourth degree terms. Furthermore, $I_{ii}(2)$ is not present in the integral.

The terms (x^2+y^2) and (xp_y-yp_x) repeatedly appear in the Toda truncations. [4], [5].We will see that $(xp_y - yp_x)$ will be immaterial in YMH case. Similar results covering the existence of approximate integral in Toda case ,for specific values of parameters can be found in [1]. In order to generalize this to the YMH system, we construct the following candidate third integral

$$\frac{1}{2}(p_x^2 + p_y^2) + \frac{c_1}{2}(x^2 + y^2) + c_3(p_yx - p_xy) + (c_4x + c_5y)(x^2 + y^2)$$
(10)

where, for the following specific values of the parameters

 $2c_2 = c_1, 4c_1 = g^2, g^2 = 1, c_3 = 0$. this third integral ansatz has a **zero third degree** Poisson Bracket with the Hamiltonian. However, the **fourth order degree terms** present in the Poisson Bracket cannot be reduced by a similar approach.

We generate the Poisson Brackets(PBs) of list elements of $H_T(j_1)$ and $I_{ii}(j_2)$. According to the truncations of $H_T(j)$ and $I_{ii}(j)$ given above, the PBs of the partial sums of equal terms up to fifth order truncations are obtained.

Poisson Bracket	$I_{ii}(l)$	$I_{ii}(2)$	$I_{ii}(3)$	$I_{ii}(4)$
$H_{I}(l)$	0	0	0	0
H _T (2)	0	12xy	$\begin{array}{r} 4(-6gp_{y}^{2}x+6p_{y}^{2}x\\ -12gp_{y}p_{x}y\\ +18p_{y}p_{x}y+p_{y}p_{x}\\ +6gp_{x}^{2}x-6gy^{2}x\\ +6y^{2}x-xyg\\ +3gx^{3} \end{array}$	$\begin{array}{c} 24(2p_{y}^{2}xy \\ -2p_{x}p_{y}y^{2} \\ +2p_{y}p_{x}x^{2}-2p_{x}^{2}xy \\ +y^{2x}+x^{2}y) \end{array}$
$H_T(3)$	0	0	0	0
H _T (4)	0	$6xy(-2py^2 + 2y^2 - x^2)$	$\begin{array}{c} 2(-12p_{y}^{2}y^{2}x\\ -6p_{y}^{2}x^{2}\\ -24p_{x}p_{y}y_{x}y^{2}p\\ -24p_{x}p_{y}yx^{2}\\ +12p_{y}^{2}y^{2}x - 6p_{y}^{2}x^{2}\\ -12p_{y}^{4}x - 2y^{2}x\\ -6y^{2}x^{2} - yx^{3}\\ +3x^{5} \end{array}$	$ \begin{array}{r} 12xy(-2y^4p \\ +2y^4 - 2y^2xp \\ +x^2y^2 - x^4 \end{array} $

Then, we introduce definition of partial sums of $H_T(j)$ and $I_{ii}(j)$ given by:

$$I_{iss}(n) = \sum_{j=1}^{n} I_{ii}(j)$$
 (11a)

$$H_{Ts}(n) = \sum_{j=1}^{n} H_{Ts}(n)$$
(11b)

We use these above definitions to calculate the PB of the partial sums of the list elements of truncated Toda Hamiltonian H_T and the isolating integrals I_{ii} by defining:

$$\{H_{Ts}(n); I_{iis}(n)\}\tag{12}$$

where $n = 1, \ldots, \infty$.

According to the truncations of $H_T(j)$ and $I_{ii}(j)$ given above, the PBs of the partial sums of equal terms up to fifth order truncations are obtained as

$$[H_{Ts}(1), I_{iis}(1)] = 0 (13a)$$

$$[H_{Ts}(2), I_{iis}(2)] = 12xy \tag{13b}$$

$$[H_{Ts}(3), I_{iis}(3)] = 4(-6gp_y^2 x + 6p_y^2 x - 12gp_y p_x y + 18p_y p_x y + p_y p_x + 6gp_x^2 x - 6gy^2 x + 6y^2 x - xyg + 3gx^3$$
(13c)
$$[H_{Ts}(4), I_{iis}(4)] = 2(-12p_y^2 y^2 x + 24p_y^2 x y - 6p_y^2 x^3 - 12gp_y^2 x$$

$$\Gamma_{s}(4), I_{iis}(4) = 2(-12p_{y}^{*}y^{*}x + 24p_{y}^{*}xy - 6p_{y}^{*}x^{*} - 12gp_{y}^{*}x - 24bp_{x}p_{y}y^{3} - 24p_{x}p_{y}y^{2}) + O(5)terms$$
(13d)

In the third order the PB gives third degree terms and fourth order term gives fifth degree terms. If we recall that order n terms are homogeneous polynomials of degree n, the fact that the PB of fourth order terms is fifth degree is an indicator of an approximate conservation rule.

3 Simulation Results for YMH System

In this section, we have simulated the YMH System under the different set of initial conditions. One can see the simulation results for the case where g=p=0 and initial conditions $(x, y, p_x, p_y) = (1.0, 1.0, 1.0, 1.0)$ in Fig. 3.



Fig. 1. :Simulation of YMH System (g=p=0)

In Fig. 2, $p_x(t)$ vs. x(t) is plotted for the same case.

Secondly, we have changed the parameter values p=1.5 and g=1.0 with the same initial conditions and have obtained the following results given in Fig 3:

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Fig. 3. Simulation Results for g=2.0 and p=1.5



In Fig. 3, $p_x(t)$ vs. x(t) is plotted for the second case. In Fig 3, it can be

seen that around the point (0,0) there is limit cycle which creates intuitions that there will be a stable regime followed by a transition to chaos via period doubling bifurcations for these parameter values. We analyze the simulation values for investigating chaotic parameters. Firstly, we try to determine the delay time. We used mutual information analysis to find delay time as 5. Then,



Fig. 5. Mutual Information Analysis

we find the embedding dimension of the system using False Nearest Neighbors analysis. Plot of analysis is given Figure 6 and we found that the embedding dimension is 2.





Fig. 6. False Nearest Neighbors

Then, we applied the largest Lyapunov exponent test to see the chaotic behavior of the system using delay time and embedding dimensions. We have found the largest Lyapunov exponent as 0,0625. It is a positive value which means there is a chaos on the system for the given parameter values



Fig. 7. Largest Lyapunov Exponent Test

4 Conclusion

The Matinyan-Yang Mills Higgs system starts from instability but reaches an asymptotically semi stable state. This has been verified both by the asymptotic behavior of the Lyapunov function and by numerical simulation. The present study hence shows the dual role of the Higgs field by both initiating unstability and transitioning to asymptotic stability.

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