Chaotic Modeling and Simulation (CMSIM) : 281–298, 2012

Geodesics Revisited

Pavel Pokorny

Prague Institute of Chemical Technology, Prague, Czech Republic
(E-mail: pavel.pokorny@vscht.cz)

Abstract. Metric tensor and Christoffel symbols are revised and the equation of geodesic is derived from two possible definitions: based on zero tangent acceleration and on minimal length. Geodesics on a torus are shown to split into two distinct classes. Dynamical systems approach is used to investigate these two classes. Application of geodesics in optics and in mechanics are given.
Keywords: dynamical system, geodetic, torus.

1 Introduction

In Euclidean space a segment of a line is the shortest connection of two given points. The segment has also the property that a point moving along the segment with velocity of constant magnitude has zero acceleration. A geodesic (a geodesic curve) is a generalization of the term segment for spaces that are not Euclidean. An example of such a space is a two dimensional surface in a three dimensional Euclidean space.

Historically perhaps the oldest example of such a surface is the sphere (the surface of a ball) because our space for living was limited to the surface of the Earth for a long time. Here comes the origin of the word geodesic. Geo- is the first part of compound words meaning the Earth.

These curved surfaces can be studied in two ways. Either as subsets of an Euclidean space of higher dimension, or as independent curved spaces without any reference to a higher dimensional Euclidean space. The intrinsic geometry of such a curved space can be described by certain matrix depending on the point in the space. This matrix function is called the metric tensor. A space with a constant metric tensor is called a flat (Euclidean) space, while a space with a non-constant metric tensor is called a curved space.

In chapter 2 a metric tensor is introduced and its examples for a sphere and for a torus are given.

In chapter 3 the geodesic is defined as the curve such that a point moving along the curve with the velocity of constant magnitude (i.e. the velocity can change its direction but not its magnitude) has the acceleration vector perpendicular to the given surface, i.e. the acceleration component tangent to the given surface is zero. Such a motion can be expressed by a non-scientific

Received: 17 July 2011 / Accepted: 27 December 2011

ISSN 2241-0503

expression "follow your nose". Given this condition the equation of geodesic is derived. It is a second order differential equation for the functions that parametrically describe the curve.

In chapter 4 alternatively the geodesic is defined as the shortest curve between two given points. Given this condition the same equation of geodesic is derived.

In chapter 5 we show that the magnitude of velocity remains constant for the solution of the equation of geodesic.

In chapter 6 the first integral (i.e. a constant function of state variables) is derived for certain simplified cases.

In chapter 7 the equation of geodesic is applied for a sphere. We show that geodesics on a sphere are the great circles i.e. the circles with the center in the center of the sphere.

In chapter 8 geodesics on a torus are investigated. The geodesics on a torus fall into two classes. Roughly speaking, one class contains geodesics that remain mainly on the outer part of the torus (see Fig. 6) while the other class contains geodesics that wind around the tube of the torus along a spiral (see Fig. 7). Another difference between these two classes is that a geodesic in the first class is either closed or it has self-intersections, while a geodesic in the other class is either closed or it has no self-intersections.

In chapter 9 a physical application of geodesics is given, namely the propagation of light in optically non-homogeneous medium, i.e. where the index of refraction depends on the point in the space. We find the metric tensor appropriate for investigation of the shape of the light ray and the Snell law of refraction is derived from the equation of geodesic found in chapter 3 and 4. This example is interesting in that it is convenient to replace the three dimensional Euclidean space by a curved space described by a non-constant metric tensor for the study of the propagation of light (or in general any wave with varying speed).

In chapter 10 the results of chapter 9 are applied for the study of the shape of the path that brings a mass point from a given initial point to another given point in the shortest possible time (assuming a homogeneous gravitational field). This path is called a brachistochrone and we show that it can be found as a geodesic with appropriate metric tensor.

There are many more examples of geodesics. Besides being an interesting mathematical question of its own, they have many physical and technological application. Spanning from general relativity to cases that seem to have nothing in common with mathematics or physics such as winding a ribbon round handlebars of a bicycle or dressing an injured knee.

Geodesics are sometimes illustrated as the equilibrium position of a spring on a slippery surface. This is a good example for convex parts of the surface; near concave parts of the surface a real spring would go through the air while the geodesic must stay in the given surface. To see this, imagine a thin rubber around an apple. There is a little pit near the stem of the apple. The rubber crosses this pit through the air which the geodesic is not allowed to do.

2 Metric tensor

Consider a M-dimensional manifold embedded into a N-dimensional Euclidean space with parametric equations

$$y = r(x),$$

where $r : R^M \to R^N$.

E.g. a sphere with unit radius can be given by

$$\begin{aligned} y_1 &= r_1(x_1, x_2) = r_1(\vartheta, \varphi) = \sin\vartheta \cos\varphi \\ y_2 &= r_2(x_1, x_2) = r_2(\vartheta, \varphi) = \sin\vartheta \sin\varphi \\ y_3 &= r_3(x_1, x_2) = r_3(\vartheta, \varphi) = \cos\vartheta \end{aligned} \quad (1)$$

and a torus by

$$\begin{aligned} y_1 &= r_1(x_1, x_2) = r_1(u, v) = (a + \cos u)\cos v \\ y_2 &= r_2(x_1, x_2) = r_2(u, v) = (a + \cos u)\sin v \\ y_3 &= r_3(x_1, x_2) = r_3(u, v) = \sin u, \end{aligned} \quad (2)$$

where $a > 1$ is the radius of the axis of the tube; the radius of the tube being 1.

The comma before an index will denote the partial derivative with respect to the variable given by the index after the comma. Thus e.g. for r_k (the k-th component of the vector r) its partial derivative is

$$r_{k,i} = \frac{\partial r_k}{\partial x_i}.$$

Then the differential of y is

$$dy_k = r_{k,i} dx_i$$

(we sum over each index appearing twice in a product) and the square of its norm is

$$||dy||^2 = dy_k dy_k = r_{k,i} r_{k,j} dx_i dx_j = g_{ij} dx_i dx_j,$$

where

$$g_{ij} = r_{k,i} r_{k,j} \quad (3)$$

are the components of the metric tensor.

E.g. for a sphere putting (1) into (3) gives

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\vartheta \end{pmatrix}$$

and for a torus putting (2) into (3) gives

$$g = \begin{pmatrix} 1 & 0 \\ 0 & (a + \cos u)^2 \end{pmatrix}.$$

Later we will need another relation between g and r. We can differentiate

$$g_{ij}(x) = r_{m,i}(x) r_{m,j}(x)$$

with respect to x_k to yield

$$g_{ij,k} = r_{m,ik} r_{m,j} + r_{m,i} r_{m,jk}.$$

We add and subtract to this equation its two cyclic permutations

$$g_{jk,i} = r_{m,ij} r_{m,k} + r_{m,j} r_{m,ik}.$$

$$g_{ki,j} = r_{m,kj} r_{m,i} + r_{m,k} r_{m,ij}$$

and we get

$$g_{ij,k} + g_{jk,i} - g_{ki,j} = 2 r_{m,ik} r_{m,j}.$$

3 Geodesic as the curve with zero tangent acceleration

Consider a curve

$$\alpha = \alpha(t) = r(x(t)),$$

where $\alpha : I \to R^N$ is a sufficiently smooth function, I is the interval $I = [t_1, t_2]$. When we call t the time, we can call

$$\dot{\alpha}_k(t) = r_{k,i} \dot{x}_i(t)$$

the velocity and

$$\ddot{\alpha}_k(t) = r_{k,ij} \dot{x}_i \dot{x}_j + r_{k,i} \ddot{x}_i.$$

the acceleration. We want to find the shape of the curve, so that the acceleration has zero projection to the plane tangent to the given surface

$$\ddot{\alpha}_k r_{k,n} = 0 \qquad \text{for } n = 1, \ldots, M$$

$$(r_{k,ij} \dot{x}_i \dot{x}_j + r_{k,i} \ddot{x}_i) r_{k,n} = 0$$

$$r_{k,i} r_{k,n} \ddot{x}_i + r_{k,ij} r_{k,n} \dot{x}_i \dot{x}_j = 0$$

$$g_{in} \ddot{x}_i + \frac{1}{2}(g_{in,j} + g_{nj,i} - g_{ji,n}) \dot{x}_i \dot{x}_j = 0.$$

It is convenient to denote g^{nm} the element of the inverse matrix to the matrix with elements g_{in} (i.e. $g_{in} g^{nm} = \delta_{im}$ is the element of the unit matrix). Then

$$\ddot{x}_m + \frac{1}{2} g^{nm} (g_{in,j} + g_{nj,i} - g_{ji,n}) \dot{x}_i \dot{x}_j = 0$$

and finally

$$\ddot{x}_m + \Gamma^m_{ij} \dot{x}_i \dot{x}_j = 0, \tag{4}$$

where

$$\Gamma^m_{ij} = \frac{1}{2} g^{nm} (g_{in,j} + g_{nj,i} - g_{ji,n}) \tag{5}$$

is called the Christoffel symbol.

From (3) it follows that the metric tensor g is symmetric, i.e.

$$g_{ij} = g_{ji}$$

and as a result the Christoffel symbol is also symmetric

$$\Gamma_{ij}^m = \Gamma_{ji}^m.$$

We call (4) the equation of geodesic. In this equation the properties of the surface appear only through the metric tensor g and its derivatives (via the Christoffel symbol Γ_{ij}^m). This allows us to work in the M-dimensional space with the metric g without any reference to the N-dimensional Euclidean space.

If the metric tensor g as a function of the point in the space is constant, its derivatives vanish and so do all the Christoffel symbols. The equation of geodesic is then

$$\ddot{x}_m = 0$$

and the geodesic is the straight line in this special case.

4 Geodesic as the shortest curve

Consider a curve

$$x = \alpha(t)$$

where $\alpha : I \to R^n$ is a sufficiently smooth function and the interval I is $I = [t_1, t_2]$.

If g is the metric tensor, then the magnitude of the velocity of a point traveling along the curve α is

$$v_\alpha(t) = \sqrt{g_{ij}(\alpha(t))\ \dot{\alpha}_i(t)\ \dot{\alpha}_j(t)}.$$

Let us denote

$$V(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n) = \sqrt{g_{ij}(x_1, \dots, x_n)\ \dot{x}_i\ \dot{x}_j}$$

in short

$$V(x, \dot{x}) = \sqrt{g_{ij}(x)\ \dot{x}_i\ \dot{x}_j}. \tag{6}$$

Similarly we will write α instead of $\alpha_1, \dots, \alpha_n$ and $\dot{\alpha}$ instead of $\dot{\alpha}_1, \dots, \dot{\alpha}_n$.

Then the length of the curve α is

$$L(\alpha) = \int_{t_1}^{t_2} v_\alpha(t)\ dt = \int_{t_1}^{t_2} V(\alpha(t), \dot{\alpha}(t))\ dt.$$

For $\epsilon \in R$ and $\beta : I \to R^n$ such that $\beta(t_1) = \beta(t_2) = 0$ we denote

$$\tilde{L}(\epsilon) = L(\alpha + \epsilon\beta) = \int_{t_1}^{t_2} V(\alpha(t) + \epsilon\beta(t), \dot{\alpha}(t) + \epsilon\dot{\beta}(t))\ dt$$

and

$$\tilde{L}'(\epsilon) = \frac{d\tilde{L}}{d\epsilon}.$$

We want

$$\tilde{L}'(0) = 0,$$

meaning that a small change in the shape of the curve does not make it shorter. Thus

$$0 = \int_{t_1}^{t_2} (V_{x_i}\beta_i + V_{\dot{x}_i}\dot{\beta}_i)\ dt.$$

We integrate by parts and we use the assumption $\beta(t_1) = \beta(t_2) = 0$ (meaning that the start point and the end point of the curve are fixed) to get

$$0 = \int_{t_1}^{t_2} (V_{x_i}\beta_i - \dot{V}_{\dot{x}_i}\beta_i)\ dt = \int_{t_1}^{t_2} (V_{x_i} - \dot{V}_{\dot{x}_i})\beta_i\ dt.$$

This must hold for arbitrary functions β_i, thus the bracket must vanish

$$V_{x_i} - \dot{V}_{\dot{x}_i} = 0 \qquad \text{for } i = 1, \ldots, n,$$

thus

$$V_{x_i} - (V_{\dot{x}_i x_k}\dot{x}_k + V_{\dot{x}_i\dot{x}_k}\ddot{x}_k) = 0. \tag{7}$$

To get a unique solution we add the assumption of constant magnitude of the velocity

$$\dot{V} = 0 \tag{8}$$

thus

$$V_{x_k}\dot{x}_k + V_{\dot{x}_k}\ddot{x}_k = 0. \tag{9}$$

When putting (6) into (7) and using (9) the same equation of geodesic (4) is derived. To do it by hand it is convenient to introduce W by

$$W(x, \dot{x}) = g_{ij}(x)\ \dot{x}_i\ \dot{x}_j \tag{10}$$

i.e.

$$V = \sqrt{W}. \tag{11}$$

Putting (11) into (7) yields

$$2WW_{\dot{x}_m x_s}\dot{x}_s + 2WW_{\dot{x}_m\dot{x}_r}\ddot{x}_r - W_{\dot{x}_m}(W_{x_s}\dot{x}_s + W_{\dot{x}_r}\ddot{x}_r) = 2WW_{x_m} \tag{12}$$

where the bracket vanishes because of (9) and (11). When we substitute W from (10) into (12) we get again the equation of geodesic

$$\ddot{x}_m + \Gamma^m_{ij}\dot{x}_i\dot{x}_j = 0,$$

where

$$\Gamma^m_{ij} = \frac{1}{2}g^{nm}(g_{in,j} + g_{nj,i} - g_{ji,n}).$$

5 Constant magnitude of velocity

We have used the assumption of constant magnitude of velocity (8) to simplify (12). It is not clear, however, whether the solution of the resulting equation of geodesic (4) still satisfies the condition (8). We show it does. Let us write the equation of geodesic (4) as a system of first order ODE's

$$\begin{aligned} \dot{x}_m &= X_m \\ \dot{X}_m &= -\Gamma_{ij}^m X_i X_j . \end{aligned} \tag{13}$$

The condition of constant square of the magnitude of velocity

$$W = g_{ij}(x)\ \dot{x}_i\ \dot{x}_j = g_{ij}(x)\ X_i\ X_j = \text{const.}$$

is the equation of a hyper-surface in the state space (of twice the dimension). It is easy to show that the vector field $\boldsymbol{f}$ of (13) is orthogonal to the gradient of W

$$\begin{aligned} \boldsymbol{f} \cdot \nabla W = \begin{pmatrix} X_m \\ -\Gamma_{ij}^m X_i X_j \end{pmatrix} \cdot \begin{pmatrix} g_{ij,m} X_i X_j \\ 2 g_{mk} X_k \end{pmatrix} = \\ = g_{ij,m} X_i X_j X_m - 2 g_{mk} \Gamma_{ij}^m X_i X_j X_k = \\ = g_{ij,m} X_i X_j X_m - g_{mk} g^{nm} (g_{in,j} + g_{nj,i} - g_{ji,n}) X_i X_j X_k = \\ = g_{ij,m} X_i X_j X_m - (g_{in,j} + g_{nj,i} - g_{ji,n}) X_i X_j X_n = 0. \end{aligned}$$

Thus the square of the magnitude of velocity is constant and so is the magnitude itself.

6 First integral

In this chapter we rewrite the equation of geodesic (4) for special cases and then we find its first integral.

Assuming dimension 2, denoting $x_1 = x,\ x_2 = y$ and assuming a diagonal metric tensor g, i.e. $g_{12}(x,y) = g_{21}(x,y) = 0$ we arrive at

$$\ddot{x} + \frac{g_{11,1}}{2g_{11}} \dot{x}\dot{x} + \frac{g_{11,2}}{g_{11}} \dot{x}\dot{y} - \frac{g_{22,1}}{2g_{11}} \dot{y}\dot{y} = 0 \tag{14}$$

$$\ddot{y} - \frac{g_{11,2}}{2g_{22}} \dot{x}\dot{x} + \frac{g_{22,1}}{g_{22}} \dot{x}\dot{y} + \frac{g_{22,2}}{2g_{22}} \dot{y}\dot{y} = 0. \tag{15}$$

Further, assuming that g depends on x only and not on y, formally written $_{,2} = 0$ meaning $\frac{\partial}{\partial y} = 0$ or $g_{11,2} = g_{22,2} = 0$ we get even more simple equations

$$\ddot{x} + \frac{g_{11,1}}{2g_{11}} \dot{x}\dot{x} - \frac{g_{22,1}}{2g_{11}} \dot{y}\dot{y} = 0 \tag{16}$$

$$\ddot{y} + \frac{g_{22,1}}{g_{22}} \dot{x}\dot{y} = 0. \tag{17}$$

We will use this result (assuming further $g_{11} = g_{22}$) in chapter 9.

In chapters 7 and 8 we will work with the metric tensor where one of its diagonal elements is constant. Assuming $g_{11}(x, y) = 1$ allows us to simplify the equation of geodetic even more

$$\ddot{x} - \frac{g_{22,1}}{2}\dot{y}\dot{y} = 0 \tag{18}$$

$$\ddot{y} + \frac{g_{22,1}}{g_{22}}\dot{x}\dot{y} = 0. \tag{19}$$

Now we can find the first integral of this system of ODE's. Multiplying (18) by $\dot{y}$ and multiplying (19) by $\dot{x}$ and subtracting the second equation from the first one we get

$$\ddot{x}\dot{y} - \ddot{y}\dot{x} - \frac{g_{22,1}}{2}\dot{y}^3 - \frac{g_{22,1}}{g_{22}}\dot{x}^2\dot{y} = 0.$$

When multiplying this equation by $\frac{2\dot{x}}{g_{22}^2\dot{y}^3}$ we get (after simple manipulation)

$$\frac{d}{dt}\left(\frac{1}{g_{22}(x(t))^2}(\frac{\dot{x}}{\dot{y}})^2 + \frac{1}{g_{22}(x(t))}\right) = 0$$

which is equivalent to

$$\frac{1}{g_{22}^2}(\frac{dx}{dy})^2 + \frac{1}{g_{22}} = \text{const.} \tag{20}$$

We will use this first integral of the equation of geodesic in chapters dealing with the sphere and with the torus.

7 Geodesics on a sphere

Putting $x_1 = \vartheta$, $x_2 = \varphi$ and

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\vartheta \end{pmatrix}$$

into the equation of geodesic (4) yields

$$\ddot{\vartheta} - \sin\vartheta\cos\vartheta\ \dot{\varphi}^2 = 0$$

$$\ddot{\varphi} + 2\cot\vartheta\ \dot{\varphi}\ \dot{\vartheta} = 0.$$

Its first integral (20) is

$$\frac{1}{\sin^4\vartheta}\left(\frac{d\vartheta}{d\varphi}\right)^2 + \frac{1}{\sin^2\vartheta} = \frac{1}{\sin^2\vartheta_0}, \tag{21}$$

where $\vartheta_0 = \min(\vartheta)$ and φ_0 are the coordinates of "the north most" point of the curve.

This is the equation of a circle lying in the plane going through the origin. Such a plane has the equation

$$x \cdot x_P = 0,$$

where

$$x = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$$

and

$$x_P = \begin{pmatrix} \sin\vartheta_P\cos\varphi_P \\ \sin\vartheta_P\sin\varphi_P \\ \cos\vartheta_P \end{pmatrix}$$

where

$$\varphi_P = \varphi_0 + \pi, \qquad \vartheta_P = \frac{\pi}{2} - \vartheta_0$$

are coordinates of the normal vector to the plane. After some manipulation

$$\varphi = \varphi_0 + \arccos\left(\tan\vartheta_0\cot\vartheta\right).$$

Differentiating gives

$$\frac{d\varphi}{d\vartheta} = \frac{1}{\sin^2\vartheta}\ \frac{\tan\vartheta_0}{\sqrt{1-(\tan\vartheta_0\cot\vartheta)^2}}$$

and

$$\frac{1}{\sin^4\vartheta}(\frac{d\vartheta}{d\varphi})^2 = \frac{1}{\sin^2\vartheta_0} - \frac{1}{\sin^2\vartheta}$$

which agrees with(21).

8 Geodesics on a torus

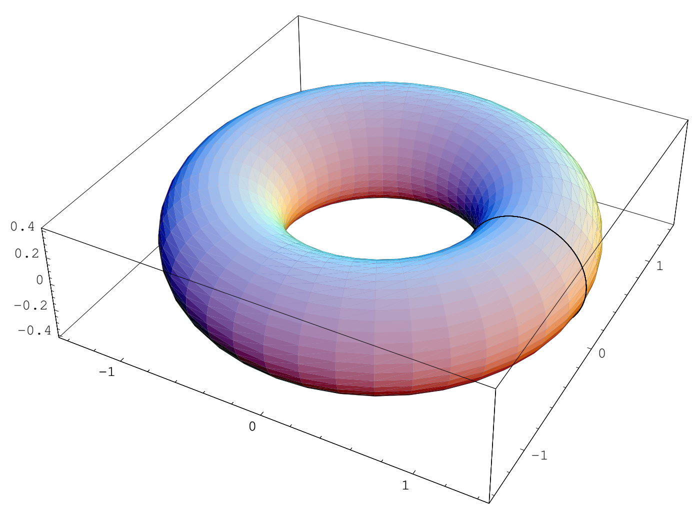

Fig. 1. Geodesic (24).

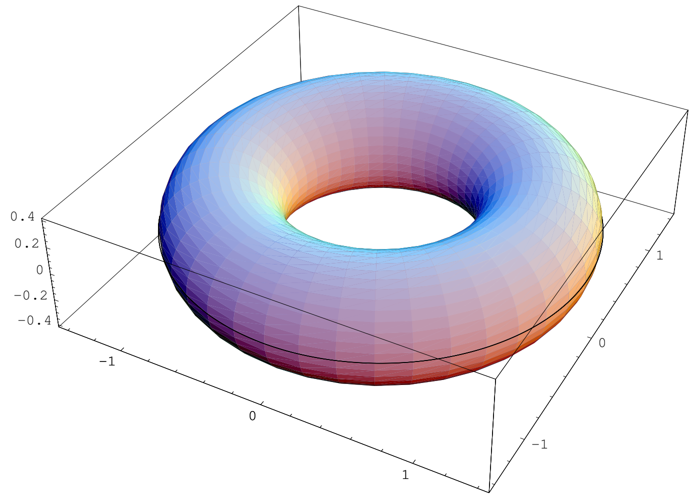

Fig. 2. Geodesic (25).

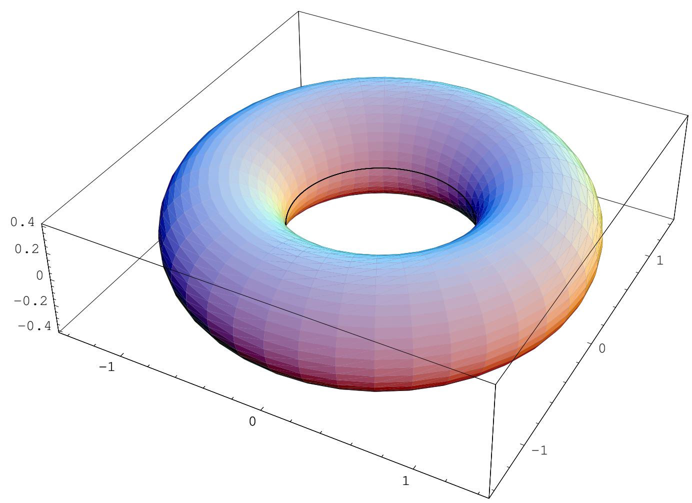

Fig. 3. Geodesic (26).

Substituting $x_1 = u$, $x_2 = v$, and the metric tensor

$$g = \begin{pmatrix} 1 & 0 \\ 0 & (a + \cos u)^2 \end{pmatrix} \tag{22}$$

into the equation of geodesic (4) gives

$$\ddot{u} + (a + \cos u) \sin u \; \dot{v}^2 = 0$$
$$\ddot{v} - 2 \frac{\sin u}{a + \cos u} \; \dot{u} \; \dot{v} = 0.$$

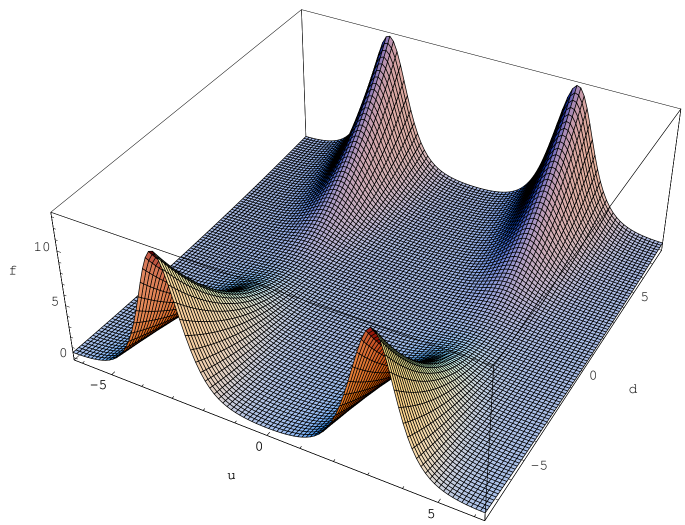

Fig. 4. The graph of the function (29) shows the minimum for $(u, d) = (0, 0)$ and a saddle for $(u, d) = (\pi, 0)$.

This system of two differential equations of the second order can be written as a system of four equations of the first order

$$\begin{aligned} \dot{u} &= U \\ \dot{U} &= -(a + \cos u)\ V^2\ \sin u \\ \dot{v} &= V \\ \dot{V} &= 2\frac{\sin u}{a + \cos u}\ U\ V. \end{aligned} \tag{23}$$

The 4 dimensional state space (u, U, v, V) of this system is divided by the hyper-plane $V = 0$ into two half-spaces. The hyper-plane $V = 0$ contains the solution

$$\begin{aligned} u &= k_1 t + k_2 \\ U &= k_1 \\ v &= k_3 \\ V &= 0, \end{aligned} \tag{24}$$

where $k_1,\ k_2,\ k_3 \in R$.

Corresponding to each solution in one half-space there is one solution in the other half-space. These two solutions are symmetrical with respect to the hyper-plane $V = 0$. The theorem of the uniqueness of solution implies that $V(t)$ is either always positive or always zero or always negative. Meaning the solutions neither cross nor touch the hyper-plane $V = 0$. Thus we can limit our attention to solutions satisfying $V(t) = \dot{v}(t) > 0$.

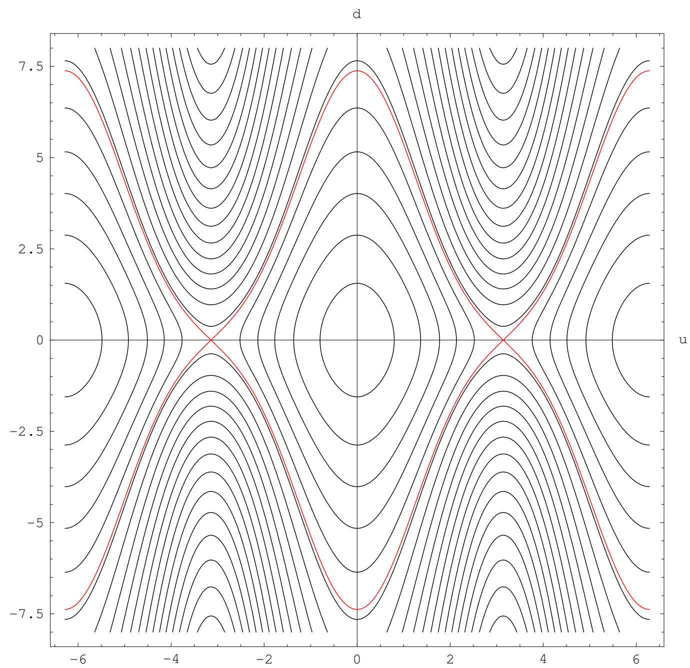

Fig. 5. The contour-lines of the function (29) are closed curves near a minimum. A separatrix (shown in red) going from the saddle separates a region with closed contour-lines from the region with non-closed contour-lines. Closed contour-lines correspond to geodesics that remain mainly in the outer part of the torus (see Fig. 6). Non-closed contour-lines correspond to geodesics that wind around the tube of the torus (see Fig. 7).

Among these solutions there are two special solutions satisfying $\dot{u}(t) = 0$, namely

$$\begin{aligned} u &= 0 \\ U &= 0 \\ v &= k_1 t + k_2 \\ V &= k_1 , \end{aligned} \tag{25}$$

and

$$\begin{aligned} u &= \pi \\ U &= 0 \\ v &= k_1 t + k_2 \\ V &= k_1 . \end{aligned} \tag{26}$$

The behavior of nearby trajectories can be studied by linear expansion. The Jacobi matrix of partial derivatives of the system (23) evaluated on the trajectory (25) has two zero eigenvalues and two purely imaginary complex conjugate eigenvalues

$$\lambda_{3,4} = \pm i k_1 \sqrt{a+1}. \tag{27}$$

This means it acts like a center; nearby trajectories rotate around it in the u-U plane spanned by the corresponding eigenvectors.

The Jacobi matrix of partial derivatives of the system (23) evaluated on the trajectory (26) has two zero eigenvalues and two real eigenvalues with opposite signs

$$\lambda_{3,4} = \pm k_1 \sqrt{a-1}.$$

Thus the trajectory (26) is a saddle with one stable and one unstable directions in the u-U plane. The saddle itself is not a stationary point but rather a closed trajectory. In fact, there are no stationary points, the velocity has a constant magnitude.

The first integral (20) for torus is

$$\frac{1}{(a+\cos u)^4}(\frac{du}{dv})^2 + \frac{1}{(a+\cos u)^2} = \text{const}. \tag{28}$$

Thus geodesics on a torus can be described by contour-lines of the function

$$f(u,d) = \frac{1}{(a+\cos u)^4}(d)^2 + \frac{1}{(a+\cos u)^2}. \tag{29}$$

The graph of the function (29) is shown in Fig. 4 and its contour-lines are shown in Fig. 5. For fixed u it is a quadratic function of d with positive coefficients, thus having a minimum. For $d = 0$ it is a periodic function of u with the period 2π having a minimum for $u = 0$ and a maximum for $u = \pi$. As a function of two variables f has a minimum in $u = 0, d = 0$ and a saddle in $u = \pi, d = 0$. The contour-lines near a minimum are closed curves, the contour-line leaving a saddle is a separatrix separating a region with closed contour-lines near a minimum (with bounded values of u) and a region with contour-lines which are neither closed not bounded (here $u(t)$ is a monotone function). This is shown in Fig. 4 depicting the graph of the function (29) and in Fig. 5 with its contour-lines.

We can find the equation of the separatrix. From

$$\frac{du}{dv} = 0 \qquad \text{for } u = \pi$$

it follows

$$\frac{1}{(a+\cos u)^4}(\frac{du}{dv})^2 + \frac{1}{(a+\cos u)^2} = \frac{1}{(a-1)^2}.$$

Thus for the separatrix for $u = 0$ it is

$$\frac{du}{dv} = 2\frac{a+1}{a-1}\sqrt{a}.$$

The angle α_C, formed by the critical geodesic and the plane $z = 0$ in $u = 0$ (i.e. on the outer edge of the torus) is

$$\tan \alpha_C = \frac{dz}{dy} = \frac{1}{a+1}\frac{du}{dv} = \frac{2\sqrt{a}}{a-1}.$$

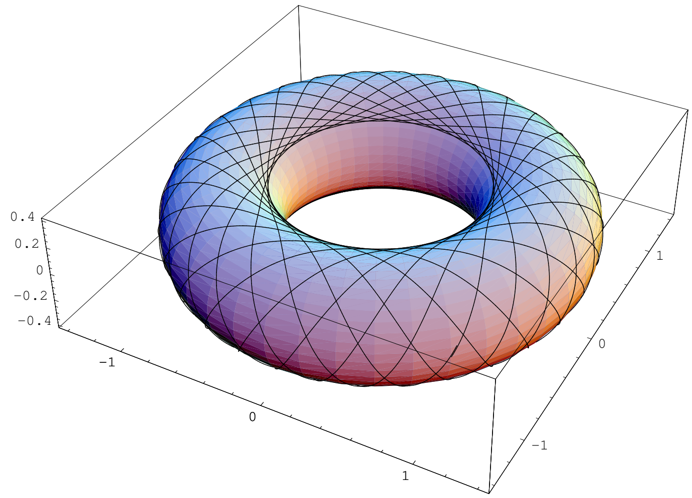

Fig. 6. If the angle α, formed by the geodesic and the plane $z = 0$, is less than a critical value α_C, the geodesic remains mainly on the outer part of the torus. For this graph $\alpha = 60°$, $\alpha_C \doteq 64.6°$ and $a = \frac{5}{2}$. Only a finite part of the geodesic is shown.

E.g. for $a = \frac{5}{2}$ the critical angle is $\alpha_C \doteq 64,6°$. Fig. 6 shows an example of a geodesic for $\alpha < \alpha_C$ and Fig. 7 for $\alpha > \alpha_C$.

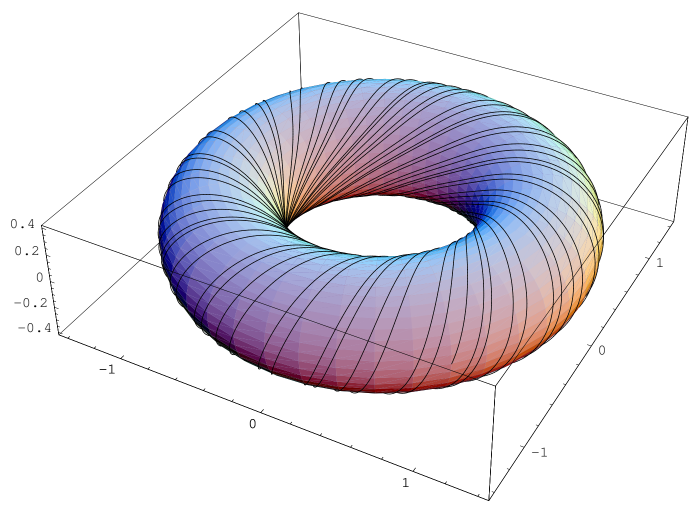

Fig. 7. If the angle α formed by the geodesic and the plane $z = 0$ on the outer part of the torus is greater than the critical angle α_C then the geodesic winds around the tube of the torus. For this graph $\alpha = 69°$, $\alpha_C \doteq 64.6°$ and $a = \frac{5}{2}$. Only a finite part of the geodesic is shown.

Are the geodesics on a torus closed curves? From (28) it follows that u as a function of v is periodic for small u (i.e. for $u_0 < \pi$). Let us denote its period T. If T is a rational multiple of 2π (as the increase of v by 2π corresponds to the same point) then the geodesic is a closed curve. The period T can be evaluated as follows. From (28) we find

$$dv = \frac{a + \cos u_0}{(a + \cos u)\sqrt{(a + \cos u)^2 - (a + \cos u_0)^2}} du$$

and

$$T = 4 \int_0^{u_0} \frac{a + \cos u_0}{(a + \cos u)\sqrt{(a + \cos u)^2 - (a + \cos u_0)^2}} du.$$

It is sufficient to integrate over one quarter of the period because the function (29) is even in both u and d.

The period T is a continuous function of two variables: a (the ratio of the radius of the axis of the tube of the torus and the radius of the tube of the torus) and u_0 (maximum of u on the geodesic) thus

$$T = T(a, u_0).$$

When a or u_0 is varied continuously then the ratio $\frac{T}{2\pi}$ will achieve rational and irrational values and in every neighborhood of a closed geodesic there will be infinitely many non-closed ones and vice versa. Almost every geodesic will be non-closed.

It is possible to compute the period $T(a, u_0)$ for small amplitude u_0

$$\lim_{u_0 \to 0} T(a, u_0) = \frac{2\pi}{\sqrt{a+1}},$$

so that e.g. for $a = 3$ the geodesic for small u_0 will almost close after two turns around the torus. This is in agreement with (27).

Our results in this chapter differ from the classical ones based on the intrinsic geometry of the torus $T^2 = R^2/N$ inherited from the Euclidean plane (u, v), where the geodesics are straight lines in the plane (u, v) and thus having the constant slope. When wound on the torus a geodesic is either a closed curve or it fills densely the entire surface of the torus. We, however, assume the metric tensor (22) based on the geometry of the torus as embedded into a three dimensional Euclidean space. This non-constant metric tensor gives rise to two distinct classes of geodesic curves (cf. Fig. 6 and Fig. 7).

9 Geodesic as the light beam

If the optical index of refraction

$$n = \frac{c}{v},$$

where c is the speed of light in vacuum and v is the speed of light in the given medium, is independent of the point in space, then the light propagates along a straight line. If the index of refraction depends of the point in space

$$n = n(x, y, z),$$

refraction of light takes place. This is of fundamental importance for the human eye and for a large range of optical devices.

The light beam propagates along such a curve, to minimize the time necessary to reach a given point from another given point. This is called the Fermat principle. The element of time is

$$dt = \frac{dl}{v},$$

where dl is the element of length. Then

$$c\,dt = \frac{c}{v}\,dl = n\,dl$$

$$(c\,dt)^2 = n^2((dx)^2 + (dy)^2 + (dz)^2),$$

meaning the shape of the beam is a geodesic in the space with metric

$$g = \begin{pmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^2 \end{pmatrix}.$$

Let us consider a special case when the index of refraction depends on a single space variable (say, y)

$$n = n(y).$$

Then it is sufficient to consider the shape of the beam in a plane. Using the equation of geodesic (4) for $x_1 = x,\ x_2 = y$ we get

$$\ddot{x} + 2\frac{n'}{n}\dot{x}\dot{y} = 0 \tag{30}$$

$$\ddot{y} + \frac{n'}{n}(\dot{y}\dot{y} - \dot{x}\dot{x}) = 0. \tag{31}$$

Multiplying (30) by $\dot{y}$ and multiplying (31) by $\dot{x}$ and subtracting the first equation from the second one we get

$$\ddot{y}\dot{x} - \ddot{x}\dot{y} - \frac{n'}{n}\dot{x}(\dot{x}^2 + \dot{y}^2) = 0.$$

When multiplying this equation by $\frac{2\dot{y}n^2}{\dot{x}^3}$ we get (after simple manipulation)

$$\frac{d}{dt}\left(\frac{1}{n(y(t))^2}((\frac{\dot{y}}{\dot{x}})^2 + 1)\right) = 0$$

which is equivalent to

$$\frac{1}{n^2}((\frac{dy}{dx})^2 + 1) = \text{const.}$$

$$\frac{1}{n^2}\frac{(dl)^2}{(dx)^2} = \text{const.}$$

$$\frac{1}{n^2}\frac{1}{\sin^2\alpha} = \text{const.}$$

and finally

$$n\sin\alpha = \text{const} \tag{32}$$

where α is the angle formed by the beam and the normal vector to the plane of constant index of refraction.

A special case

$$n(y) = \frac{1}{y}$$

i.e.

$$g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

for $y > 0$ is called the Poincare metric. Then (32) gives

$$\frac{1}{y}\sin\alpha = K.$$

Comparing with the equation of a circle with radius R

$$\sin\alpha = \frac{y}{R}$$

shows that the geodesics in Poincare metric are semicircles for $K = \frac{1}{R} > 0$ and straight lines

$$\alpha = 0$$

i.e.

$$x = \text{const.}$$

for $K = 0$.

10 Brachistochrone

Brachi- is the first part of compound words meaning short and chronos means time. Brachistochrone is the name for the curve bringing a mass point from a given point to another given point in the shortest possible time (assuming homogeneous gravitational field). To find it we make use of the results from the previous chapter.

The conservation of mechanical energy

$$\frac{1}{2}mv^2 + mgh = mgh_0$$

lets us introduce a quantity playing a similar role as the index of refraction for light

$$n = \frac{c}{v} = \frac{\text{const}}{\sqrt{y_0 - y}}.$$

Using the law of refraction

$$n \sin\alpha = \text{const}$$

we get

$$\frac{\sin\alpha}{\sqrt{y_0 - y}} = \text{const}.$$

When we describe the curve as the graph of a function

$$y = y(x)$$

we get the equation

$$(1 + y'^2)(y_0 - y) = \text{const}.$$

It is easy to show that this is the cycloid. Starting with the parametric equation of cycloid

$$\begin{aligned} x &= R\omega t + R\cos\omega t \\ y &= R\sin\omega t \end{aligned}$$

and differentiating with respect to time

$$\begin{aligned} \dot{x} &= R\omega - R\omega\cos\omega t \\ \dot{y} &= R\omega\cos\omega t \end{aligned}$$

we find

$$y'^2 = (\frac{\dot{y}}{\dot{x}})^2 = \frac{\cos^2\omega t}{(1 - \sin\omega t)^2}$$

and

$$(1 + y'^2)(y_0 - y) = 2R = \text{const}.$$

Meaning that the cycloid is also a geodesic with a suitable metric.

References

1.Arnold V. I.: Ordinary Differential Equations. MIT Press 1998.
2.Horsky J. et al.: Mechanika ve fyzice. Academia Prague 2001.
3.Ljusternik L. A.: Kratcajsije linii. Gostechizdat Moscow 1955.