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# Bifurcation scenario of a network of two coupled rings of cells

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**Abstract.** We study the bifurcation scenario appearing in systems of two coupled rings of cells with $\mathbf{Z}_3 \times \mathbf{Z}_5$ exact symmetry, and $\mathbf{Z}_3$ interior symmetry. This study was motivated by previous work by Antoneli, Dias and Pinto, on two rings of cells coupled through a 'buffer' cell, with $\mathbf{Z}_3 \times \mathbf{Z}_5$ and $\mathbf{D}_3 \times \mathbf{D}_5$ exact and interior symmetry groups. There, quasi-periodic behavior was found through a sequence of Hopf bifurcations. We questioned if an analogous mechanism could explain the appearance of quasi-periodic motion in the examples considered here. Surprisingly, we observe periodic and quasi-periodic states appearing also through Hopf bifurcations. We compute the relevant states numerically.
**Keywords:** Hopf bifurcation, exact symmetry, interior symmetry, coupled cells systems.

## 1 Introduction

Stewart, Golubitsky and Pivato [24] and Golubitsky, Stewart and Török [17] have developed a new theory for networks of coupled cells systems. They focused in patterns of synchrony and associated bifurcations.

Networks of coupled cells may be represented schematically by a directed graph, where the nodes correspond to the individual cells and the edges to the couplings between them. The term 'cells' means nonlinear dynamical systems of ordinary differential equations.

There has been considerable development on the study of synchrony, phase-relations, quasi-periodic motion, synchronized chaos, amongst others, in networks of coupled cells [5,6,12,20,18]. Graphs architecture appear to be an important part in the explanation of these phenomena.

Networks of coupled cells may arise as models of animal and robot locomotion, speciation, visual perception, electric power grids, internet communication [8,9,21,11,22,23,10,7], and many others.

There are special networks of coupled cells that possess some degree of symmetry. We divide these networks in two groups: (i) networks with exact

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symmetry group; and (ii) networks with interior symmetry group. A symmetry of a network is a permutation on the nodes that preserves the network architecture (including cell-types and arrow-types). An interior symmetry generalizes the notion of symmetry. It has been introduced by Golubitsky *et al* [13]. It is a permutation in a subset of the cells that partially preserves the network architecture. In this case, 'forgetting' about some arrows leads to a subnetwork whose symmetry group is the interior symmetry group of the entire network.

In this paper we study interesting dynamical features occurring in two coupled systems of two unidirectional rings, with $\mathbf{Z}_3 \times \mathbf{Z}_5$ exact symmetry and $\mathbf{Z}_3$ interior symmetry, see Fig. 1. We were motivated by previous work in the study of quasi-periodic motion in four examples of networks of two rings coupled through a 'buffer' cell, with $\mathbf{Z}_3 \times \mathbf{Z}_5$ and $\mathbf{D}_3 \times \mathbf{D}_5$ exact and interior symmetry [2–4]. We questioned if the bifurcation scenario observed in those cases was seen in the networks considered here. Surprisingly, here too, we find quasi-periodic states appearing through a sequence of Hopf bifurcations, analogously to what was found in [2–4]. We also obtain a curious feature appearing further away of the third Hopf bifurcation point, similarly to what was found in [2–4] and [12].

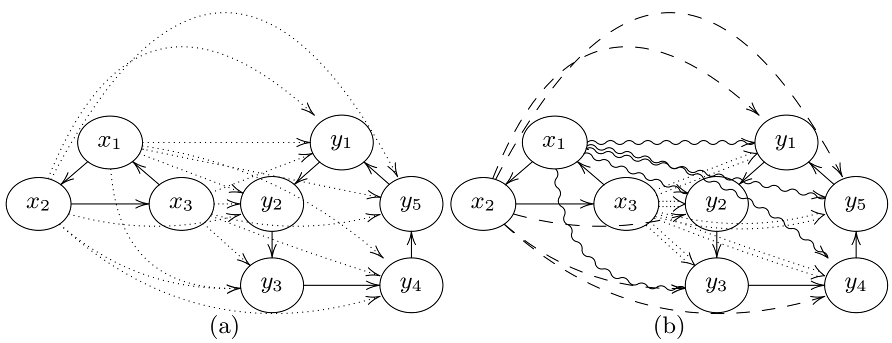


**Fig. 1.** Networks of two coupled unidirectional rings, one with three cells and the other with five.The network on the left (a) has exact $\mathbf{Z}_3 \times \mathbf{Z}_5$-symmetry, the network on the right (b) has interior $\mathbf{Z}_3$-symmetry.

### 1.1 Outline of the paper

In section 2, we give a brief summary of the coupled cells networks formalism. In section 2.2, we simulate the coupled cells systems associated to the networks of two coupled rings of cells in Fig. 1. We consider the cases of exact and interior symmetry. In section 3, we state the main conclusions and unravel future research directions.

## 2 Coupled cells network

A coupled cells network consists of a (i) finite set of nodes (or cells) $\mathcal{C}$; (ii) an equivalence relation on cells in $\mathcal{C}$, where the equivalence class of $c$ is the type of cell $c$; (iii) an input set of cells $\mathcal{I}(c)$, that consists of cells whose edges have cell $c$ as head; (iv) an equivalence relation on the edges (or arrows), where the equivalence class of $e$ is the type of edge $e$; (v) and satisfies the condition that 'equivalent edges have equivalent tails and edges'.

We define, for each cell $c$ an internal phase space $P_c$, the total phase space of the network being $P = \prod_{i=1}^{n} P_c$. Coordinates on $P_c$ are denoted by $x_c$, and thus coordinates on $P$ are $(x_1, x_2, \ldots, x_n)$. At time $t$, the system is at state $(x_1(t), x_2(t), \ldots, x_n(t))$.

A vector field $f$ on $P$ that is compatible with the network architecture is said to be *admissible* for that network, and satisfies two conditions: (1) the domain and (2) the pull-back condition. Moreover, condition (1) states that each component $f_i$ corresponding to cell $c_i$ is a function of the cells in $\mathcal{I}(c_i)$. Condition (2) says that if cells $c_i$ and $c_j$ have isomorphic input cells then their corresponding components $f_i$ and $f_j$ are identical up to a suitable permutation of the relevant variables [14].

### 2.1 Symmetry groups

A symmetry of a coupled cells system is the group of permutations of the cells (and arrows) that preserves the network structure (including cell-types and arrow-types) and its action on $P$ is by permutation of cell coordinates. Formally, we have a coupled system given by

$$\dot{x} = f(x) \tag{1}$$

where $f(x)$ is an admissible vector field for the a given network. If $f$ is $\Gamma$ symmetric, then $f(\gamma x) = \gamma f(x)$, $\gamma \in \Gamma$ (equivariance condition). It follows from the "pull-back condition" that this equivariance condition is satisfied for all $\gamma \in \Gamma$, with respect to the action of the symmetry group $\Gamma$ on the phase space $P$, by commuting cells coordinates. A symmetry is thus a transformation of the phase space that sends solutions to solutions.

The network in Figure 1(a) is an example of a network with exact $\mathbf{Z}_3 \times \mathbf{Z}_5$ symmetry.

An interior symmetry generalizes the concept of exact symmetry and it was introduced by Golubitsky *et al* [13]. It is a group of permutations that acts in a subset of cells (but not on the entire set of cells) and partially preserves the network structure (cell-types and edges-types).

The network in Figure 1(b) is an example of a coupled cells system with $\mathbf{Z}_3$ 'interior symmetry'. Moreover, if we ignore the couplings from cells $x_1,\ x_2,\ x_3$ to cells $y_1,\ y_2,\ y_3,\ y_4,\ y_5$, then the resulting network is $\mathbf{Z}_3$-exactly symmetric. Moreover, the network has interior $\mathbf{Z}_3$-symmetry on the set of cells $\{x_1, x_2, x_3\}$.

### 2.2 Numerical results

In this section we simulate the coupled cells systems associated with the two networks depicted in Fig. 1. We use the following function for the internal dynamics of each of the eight cells [2,12]:

$$f(x) = \mu x - \frac{1}{10}x^2 - x^3$$

where $\mu$ is a real parameter.

The coupled cells system of equations associated to the network ($a$) in Fig. 1 is given by:

$$\begin{aligned} \dot{x_j} &= f(x_j) + c_1\,(x_j - x_{j+1}),\ j = 1, \ldots, 3 \\ \dot{y_j} &= f(y_j) + c_2\,(y_j - y_{j+1}) + d\,(y_j - x_1) + d\,(y_j - x_2) + d\,(y_j - x_3), \\ &\quad j = 1, \ldots, 5 \end{aligned} \tag{2}$$

where $c_1 = 0.75,\ c_2 = 0.60,\ d = 0.2$, and the indexing assumes $x_4 \equiv x_1$ and $y_6 \equiv y_1$.

The coupled cells system of equations associated to the network ($b$) in Fig. 1 is given by:

$$\begin{aligned} \dot{x_j} &= f(x_j) + c_1\,(x_j - x_{j+1}),\ j = 1, \ldots, 3 \\ \dot{y_j} &= f(y_j) + c_2\,(y_j - y_{j+1}) + d_1\,(y_j - x_1) + d_2\,(y_j - x_2) + d_3\,(y_j - x_3), \\ &\quad j = 1, \ldots, 5 \end{aligned} \tag{3}$$

where $d_1 = 0.1,\ d_2 = 0.2,\ d_3 = 0.3$ and all other parameters and indexes are as above.

Note that if $d_1 = d_2 = d_3$ then the structure of the coupled cell system (3) is consistent with the network of Figure 1(a) and thus has $\mathbf{Z}_3 \times \mathbf{Z}_5$ exact symmetry.

We vary parameter $\mu \in [-1.0, 2.0]$, going from positive values to negative values. We obtain a branching pattern similar to the schematic bifurcation diagram presented in Fig. 2.

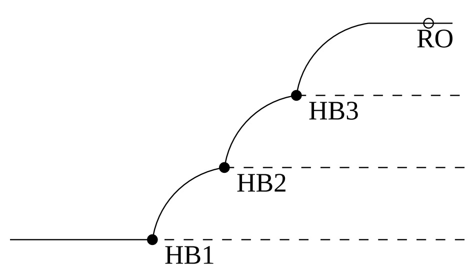


**Fig. 2.** Schematic (partial) bifurcation diagram for the coupled cell systems in Fig. 1, near the equilibrium point. Solid lines represent stable solutions, dashed lines correspond to unstable solutions [2].

In Table 1, we give a summary of the values of the Hopf bifurcation points and the corresponding solutions in the two rings for the networks in Fig. 1.

| branch | $\mu$ | 3-ring | 5-ring | network | Figure |
|---|---|---|---|---|---|
| trivial | 2.0 | equilibrium | equilibrium | equilibrium | |
| | 2.0 | equilibrium | equilibrium | equilibrium | |
| $1^{\text{st}}$ (HB1) | 1.66 | equilibrium | rotating wave | periodic | Fig 3 |
| | 1.66 | equilibrium | rotating wave | periodic | Fig 3 |
| $2^{\text{nd}}$ (HB2) | 1.04 | rotating wave | rotating wave | quasi-periodic | Fig. 4 |
| | 1.04 | rotating wave | rotating wave | quasi-periodic | Fig. 5 |
| $3^{\text{rd}}$ (HB3) | 1.015 | rotating wave | rotating wave | quasi-periodic | Fig. 6 |
| | 1.015 | rotating wave | rotating wave | quasi-periodic | Fig. 7 |
| $3^{\text{rd}}$ (RO) | $-0.5$ | relax. osc. | relax. osc. | quasi-periodic | Fig. 8 |
| | $-0.5$ | relax. osc. | relax. osc. | quasi-periodic | Fig. 9 |

**Table 1.** Summary of the dynamical behavior of coupled cell systems associated to the networks in Fig. 1. In the first column we indicate some branches of solutions with the respective bifurcation points. The second, third and fourth columns show the type of asymptotic stable solutions in the rings and the full systems in the corresponding branch. See text for more details.

The first branch of Hopf bifurcation, $1^{\text{st}}$ (HB1), comes from a trivial branch of equilibria. The solutions corresponding to the primary branch can be explained using the Equivariant Hopf Theorem [16] for coupled cells systems in the symmetric case, and the Interior Symmetry Breaking Hopf Theorem [1] for coupled cells systems with interior symmetry.

Fig. 3 shows the time series after (HB1) in the coupled cells systems (2)-(3). On the panel on the left we plot the time series for the network with $\mathbf{Z}_3 \times \mathbf{Z}_5$ exact symmetry and on the right panel we plot the time series for the network with $\mathbf{Z}_3$ interior symmetry. In both cases, we observe a rotating wave on the 5-ring (periodic solution in which the cells in the 5-ring have the same wave form but they are 1/5 out of phase) and the cells in the 3-ring stay in equilibrium.

By varying further the parameter $\mu$, there is a secondary Hopf bifurcation point (HB2) where the time series of the cells in the 3-ring appear to show a rotating wave (periodic solution in which the cells in the 3-ring have the same wave form but they are 1/3 out of phase). Figures 4-5 (left) show the time series after the secondary Hopf bifurcation (HB2) in the coupled cell systems (2)-(3). The Hopf bifurcation "occurs" in the 3-ring, leading to a rotating wave on the 3-ring. Cells in both rings appear to be at a rotating wave state. The full solution is quasi-periodic (solution fills in the visible region), see Figures 4-5 (right).

Figures 6-7 show the time series after the tertiary Hopf bifurcation (HB3) in the coupled cells systems (2)-(3). Cells in the 3- and 5- rings appear to be at a rotating wave state. The full solution is quasi-periodic.

Figures 8-9 show the time series further away from the tertiary Hopf bifurcation (HB3) in the coupled cell systems (2)-(3). In Figures 8-9, we plot, on

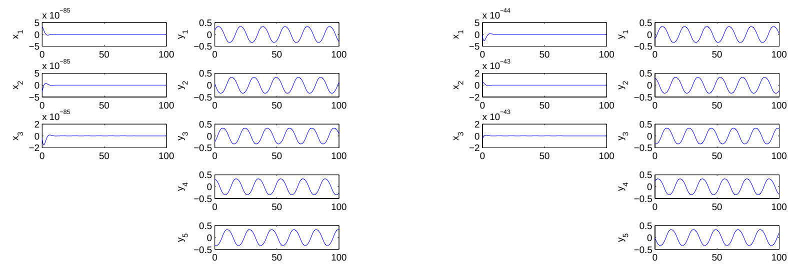

**Fig. 3.** Simulation of the coupled systems (2) and (3). Time series from the eight cells after the first Hopf bifurcation point (HB1). (Left) Exact symmetry $\mathbf{Z}_3 \times \mathbf{Z}_5$. Cells in the 3-ring are at equilibrium and cells in the 5-ring display a rotating wave. (Right) Interior symmetry $\mathbf{Z}_3$. Cells in the 3-ring are at equilibrium and cells in the 5-ring display a rotating wave.

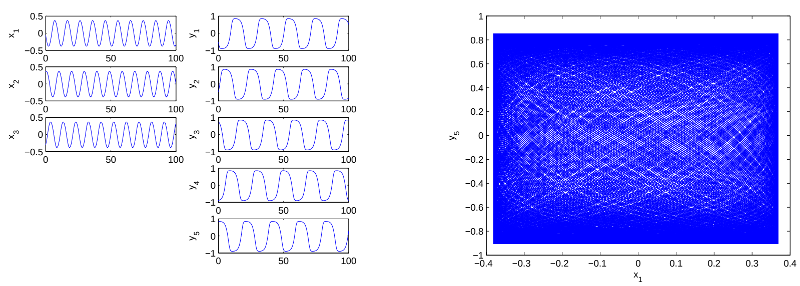

**Fig. 4.** Simulation of the coupled system (2) with $\mathbf{Z}_3 \times \mathbf{Z}_5$ exact symmetry, after the second Hopf bifurcation point (HB2). (Left) Time series from the eight cells. (Right) Cell $x_1$ vs cell $y_5$.

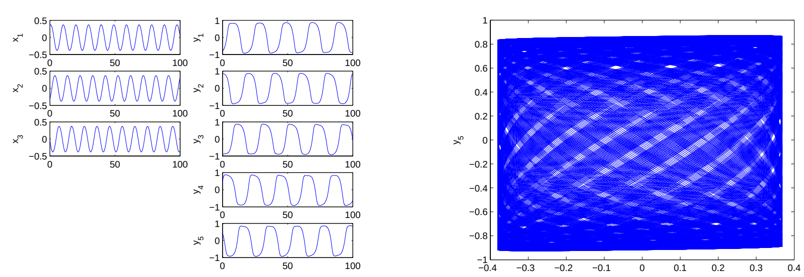

**Fig. 5.** Simulation of the coupled system (3) with $\mathbf{Z}_3$ interior symmetry, after the second Hopf bifurcation point (HB2). (Left) Time series from the eight cells. (Right) Cell $x_1$ vs cell $y_5$.

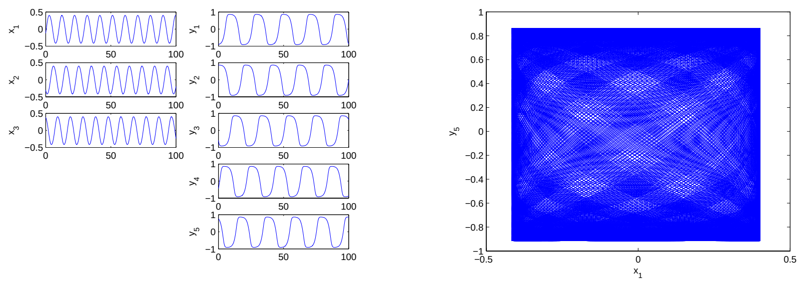

**Fig. 6.** Simulation of the coupled system (2) with $\mathbf{Z}_3 \times \mathbf{Z}_5$ exact symmetry, after the third Hopf bifurcation point (HB3). (Left) Time series from the eight cells. (Right) Cell $x_1$ vs cell $y_5$.

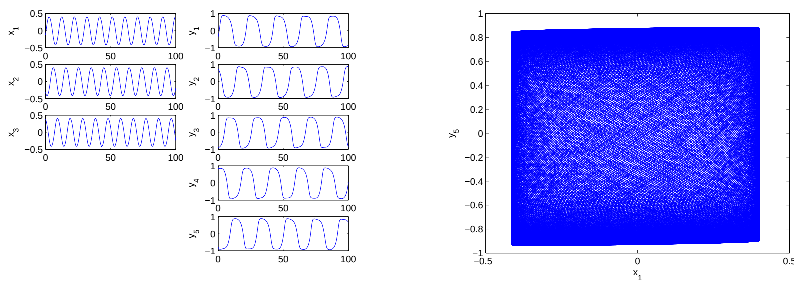

**Fig. 7.** Simulation of the coupled system (3) with $\mathbf{Z}_3$ interior symmetry, after the third Hopf bifurcation point (HB3). (Left) Time series from the eight cells. (Right) Cell $x_1$ vs cell $y_5$.

the left panel, the time series for the eight cells and on the right panel cell $x_1$ vs cell $y_5$, for the cases with exact symmetry and interior symmetry, respectively.

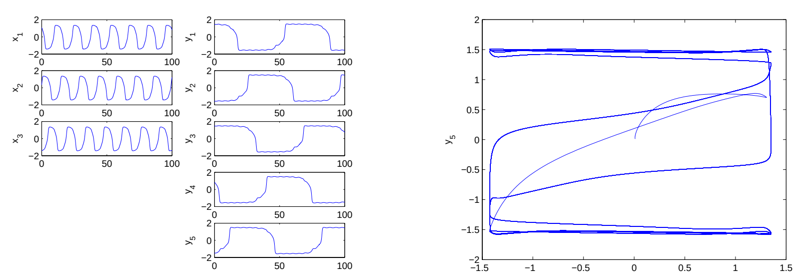

**Fig. 8.** Simulation of the coupled system (2) with $\mathbf{Z}_3 \times \mathbf{Z}_5$ exact symmetry, further away of the third Hopf bifurcation point (HB3). (Left) Time series from the eight cells. (Right) Cell $x_1$ vs cell $y_5$.

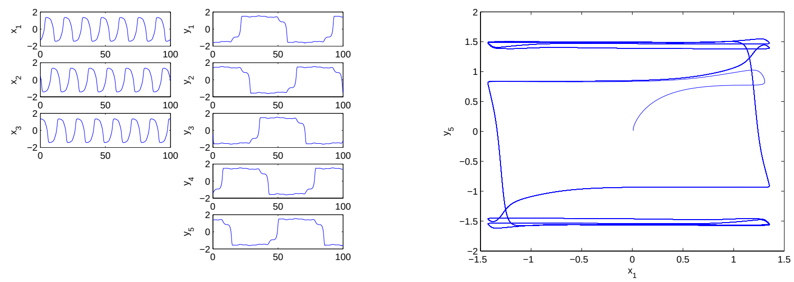

**Fig. 9.** Simulation of the coupled system (3) with $\mathbf{Z}_3$ interior symmetry, further away of the third Hopf bifurcation point (HB3). (Left) Time series from the eight cells. (Right) Cell $x_1$ vs cell $y_5$.

The full solution is quasi-periodic that is, the time series on the 3-ring looks like a (approximate) rotating wave and the time series on the 5-ring a (approximate) rotating wave.

## 3 Conclusion

In this paper we study the dynamical behavior of two networks consisting of two coupled rings of cells that admit $\mathbf{Z}_3 \times \mathbf{Z}_5$ exact and $\mathbf{Z}_3$ interior symmetry groups.

We find equilibria, rotating waves, quasi-periodic motion, and relaxation oscillations. The bifurcation diagram that explains the occurrence of these phenomena is similar to the one found in Antoneli *et al* [2–4]. There, authors study two rings coupled through a 'buffer' cell with $\mathbf{Z}_3 \times \mathbf{Z}_5$ and $\mathbf{D}_3 \times \mathbf{D}_5$ exact and interior symmetry groups. Analogously of what was found in [2–4], here too, the exotic behavior found further away of the third Hopf bifurcation point, reveals itself when a relaxation oscillation occurs. Relaxation oscillations are solutions that appear through canard explosions [19,25].

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