# Chaos Synchronization and Chaos Control Based on Kannan Mappings 

S. Sh. Alaviani ${ }^{1}$<br>${ }^{1}$ Department of Electrical Engineering, Amirkabir University of Technology Tehran, Iran<br>(E-mail: shaho@aut.ac.ir)


#### Abstract

In this paper, a new method for constructing chaotically synchronizing systems is proposed. Furthermore, a new control method for stabilizing a periodic orbit embedded in a chaotic attractor is proposed. The validity of these methods is shown by a property of Kannan mappings. It is shown that in some cases in which method of contraction mappings, proposed by Ushio (T. Ushio. Chaotic Synchronization and Controlling Chaos Based on Contraction Mappings, Physics Letters A, vol. 198, 14-22, 1995.), cannot be applied to synchronize or control of chaotic systems, the method may be applied. Ultimately, a numerical example is given in order to present the results established.


Keywords: chaos synchronization, chaos control, Kannan mappings.

## 1 Introduction

Chaos, as a very interesting nonlinear phenomenon, has been intensively studied over the past decades. Dynamic chaos has aroused considerable interest in many areas of science and technology due to its powerful applications in chemical reactions, power converters, biological systems, information processing, secure communication, neural networks etc. In the study of chaotic systems, chaos synchronization and chaos control play a very important role and have great significance in the application of chaos.

Chaos synchronization seems to be difficult to observe in physical systems because chaotic behavior is very sensitive to both the initial conditions and noise. However, Pecora and Carroll [1] have successfully proposed a method to synchronize two identical chaotic systems with different initial conditions. Since then, a variety of approaches have been proposed for the synchronization of chaotic systems which include contraction mappings [2], variable structure control $[3,4]$, parameters adaptive control $[5,6]$, observer based control $[7,8]$, nonlinear control [9-11], nonlinear replacement control [12], variable strength linear coupling control [13], active control $[14,15]$ and so on.

On the other hand, chaos control is a very attractive subject in the study of chaotic systems. Since the method for controlling of chaos was first proposed by Ott et al [16], many chaos control methods have been developed extensively over the past decades such as contraction mappings [2], chaotic targeting
method [17,18], delayed feedback control [19] etc. Yu et al [20] used the contraction mapping method, proposed by Ushio [2], to stabilize chaotic discrete neural networks.

Neural networks have been widely used as models of real neural structures from small networks of neurons to large scale neurosystem. In recent years, investigation of chaotic dynamics in neural networks becomes an active field in the study of neural networks dynamics. Numerous chaotic neural network models have been proposed for investigation [20-22]. Among the spectrum of applications of chaos control, neural system is a particularly interesting research object of complex structures that it can be applied [23,24].

In this paper, a new method for synthesis of chaotically synchronizing systems based on Kannan mappings is proposed. Also, a new method based on these mappings to stabilize chaotic discrete systems is proposed. These methods are applied to synchronize and control chaotic discrete neural networks. A similar advantage of the methods proposed in this paper and the methods proposed by Ushio [2] is that the linearization of the system near the stabilized orbit is not required. However, in some cases in which the proposed methods of Ushio [2] are not applicable to synchronize or control chaotic systems, the methods may be applied.

This paper is organized as follows. In section 2, problem of chaos synchronization is studied. In section 3, problem of controlling chaos is discussed. Eventually, a numerical example is given in order to present the result investigated.

## 2 Chaos Synchronization

First, the following theorem which Kannan proved in 1969 is introduced.
Theorem [25] Let $(X, d)$ be a complete metric space. Let $T$ be a Kannan mapping on $X$, that is, there exists $\alpha \in\left[0, \frac{1}{2}\right)$ such that

$$
d(T x, T y) \leq \alpha(d(T x, x)+d(T y, y))
$$

for all $x, y \in X$. Then, there exists a unique fixed point $x_{0} \in X$ of $T$.
We now consider chaotic discrete-time systems described by

$$
x(k+1)=f(x(k)),(1)
$$

where $x(k) \in \mathfrak{R}^{n}$ is the state of the system at time $k$, and $f$ is a mapping from $\mathfrak{R}^{n}$ to itself. We assume that $f$ is rewritten as follows

$$
f:=g+h,(2)
$$

where both $g$ and $h$ are mappings from $\mathfrak{R}^{n}$ to itself and $g$ is a Kannan mapping on a closed set $\Omega \in \mathfrak{R}^{n}$. It is assumed that a chaotic attractor A of Eq. (1) is in $\Omega$. Many methods for constructing synchronized chaotic systems are based upon the decomposition of states of chaotic systems, and it is proved by using conditional Lyapunov exponents whether the constructed systems are
synchronized. Ushio proposes a method based on the partition of the nonlinear mapping, and synchronization of the constructed systems is guaranteed by a property of contraction mappings.

This paper proposes another method based on partitioning of the nonlinear mapping, and synchronization of the constructed systems is guaranteed by a property of Kannan mappings. In the following subsections, we study synthesis methods for in-phase and anti-phase synchronization of chaotic systems.

### 2.1 In-phase synchronization

That the difference of the states of two systems converges to zero is called in-phase synchronization or synchronization. We construct a system described by

$$
w(k+1)=g(w(k))+h(x(k)),
$$

where $w(k) \in \mathfrak{R}^{n}$ is the state of the system, and $x(k) \in \mathfrak{R}^{n}$ is the state of Eq. (1). Suppose that initial state $x(0)$ of Eq. (1) is in the basin of the attractor A, and both states $x(k)$ and $w(k)$ of Eq. (1) and (3) are in $\Omega$ for each $k \in \mathrm{~N}$, where N denotes the set of all natural numbers. We assume that there exist a closed set $\Omega \in \mathfrak{R}^{n}$ and a nonnegative constant $0 \leq \alpha<\frac{1}{2}$ such that for any $x, y \in \Omega$ the mapping $g$ satisfies

$$
\|g(x)-g(y)\| \leq \alpha(\|x-g(x)\|+\|y-g(y)\|)
$$

We show that Eq. (1) and (3) are in-phase synchronized, so

$$
\begin{aligned}
& \|x(k+1)-w(k+1)\|=\|g(x(k))-g(w(k))\| \\
& \quad \leq \alpha(\|x(k)-g(x(k))\|+\|w(k)-g(w(k))\|) .
\end{aligned}
$$

According to Theorem, we obtain

$$
\lim _{k \rightarrow \infty}\|x(k)-w(k)\|=0
$$

Thus, in-phase chaotic synchronization of Eqs. (1) and (3) is achieved. Note that $w(0)$ is not necessarily in the basin of A .

Let us consider the following fully connected network composed of mneurons, as given in [20]:

$$
x_{k+1}^{i}=\varphi_{\mu}\left(\sum_{j=1}^{m} W_{i j} x_{k}^{j}\right), i=1,2, \ldots, m
$$

where $\varphi_{\mu}(z)=\left(1+e^{-\mu z}\right)^{-1}$ is assumed to be the sigmoid function. Let $m=2$, i.e., consider the case where we have a 2 D fully connected neural network defined as

$$
\begin{aligned}
& x_{k+1}=\varphi_{\mu}\left(W_{11} x_{k}+W_{12} y_{k}\right),(4-a) \\
& y_{k+1}=\varphi_{\mu}\left(W_{21} x_{k}+W_{22} y_{k}\right),(4-b)
\end{aligned}
$$

Altering the matrix $W=\left(w_{i j}\right)$ of connecting, this map can generate various complex dynamical patterns, including deterministic chaos [23]. We start our study with a 2D neural network with matrix

$$
W=\left(\begin{array}{ll}
-a & a \\
-b & b
\end{array}\right)
$$

This simplified neural network is dynamically equivalent to a one-parameter family of s-unimodal maps; it is well known that this map will generate chaotic via the Feigenbaum scenario.

We partition the neural network as follows

$$
\begin{gathered}
h\left(x_{k}, y_{k}\right)=\left(\begin{array}{cc}
\varphi_{\mu}\left(w_{11} x_{k}+w_{12} y_{k}\right)-\sqrt{\left|x_{k}\right|} & 0 \\
0 & \left.\varphi_{\mu}\left(w_{21} x_{k}+w_{22} y_{k}\right)-\sqrt{\left|y_{k}\right|}\right)
\end{array}\right) \\
g\left(x_{k}, y_{k}\right)=\left(\begin{array}{cc}
\sqrt{\left|x_{k}\right|} & 0 \\
0 & \sqrt{\left|y_{k}\right|}
\end{array}\right) .
\end{gathered}
$$

The mapping $g$ satisfies Kannan mapping for any $x, y \in \mathfrak{R}$. Then, we have the following new system

$$
\begin{aligned}
& w_{1}(k+1)=\varphi_{\mu}\left(w_{11} x_{k}+w_{12} y_{k}\right)-\sqrt{\left|x_{k}\right|}+\sqrt{\left|w_{1}(k)\right|},(5-a) \\
& w_{2}(k+1)=\varphi_{\mu}\left(w_{21} x_{k}+w_{22} y_{k}\right)-\sqrt{\left|y_{k}\right|}+\sqrt{\left|w_{2}(k)\right|} \cdot(5-b)
\end{aligned}
$$

So in-phase synchronization of System (4) and System (5) is achieved.
Remark 1 Because $\sqrt{|x|}, x \in \mathfrak{R}$ is not contraction mapping, the results given in [2] are not applicable to show the synchronization of System (4) and System (5).

### 2.2 Anti-phase synchronization

That the states of synchronized systems have the same absolute values but opposite signs is called anti-phase synchronization. We can say that anti-phase synchronization holds if

$$
\lim _{k \rightarrow \infty}\left\|x_{1}(k)+x_{2}(k)\right\|=0
$$

where $x_{i}, i=1,2$, is the state of the system. Suppose that the state $x(k)$ is both in the basin of the chaotic attractor A and in $\Omega$, and $w(k)$ is in $\Omega$. Then,

$$
\begin{aligned}
& \|x(k+1)+w(k+1)\|=\|g(x(k))+g(w(k))\| \\
& \quad \leq \alpha(\|x(k)+g(x(k))\|+\|w(k)+g(w(k))\|)
\end{aligned}
$$

According to Theorem, we obtain

$$
\lim _{k \rightarrow \infty}\|x(k)+w(k)\|=0
$$

Thus, anti-phase chaotic synchronization of $x(k)$ and $w(k)$ is achieved.

## 3 Chaos Control

Consider the following chaotic discrete-time systems with an external input

$$
Z_{k+1}=f\left(Z_{k}\right)+B u_{k}
$$

where $Z_{k} \in \mathfrak{R}^{n}$ and $u_{k} \in \mathfrak{R}^{l}$ are the state and input of the system, and $B$ is an $n \times l$ constant matrix. Eq. (6) without input has a chaotic attractor $A$. Let $Z^{*}=f\left(Z^{*}\right)$ be a periodic orbit embedded in $A$. We consider the following input

$$
u_{k}=\left\{\begin{array}{lr}
D\left(z_{k}\right)-D\left(z^{*}\right) \text { if }\left\|z_{k}-z^{*}\right\|<\varepsilon  \tag{7}\\
0 & \text { otherwise }
\end{array}\right.
$$

where $D$ is a mapping from $\mathfrak{R}^{n}$ to $\mathfrak{R}^{l}$, and $\varepsilon$ is a sufficiently small positive constant. Assume that the mapping $f+B D$ is a Kannan mapping on a closed set $\Omega \in \mathfrak{R}^{n}$, and the chaotic attractor $A$ is within $\Omega$. Suppose that the initial state $z_{0}$ of Eq.(6) is within $\Omega$; then, the following behavior $z_{k}$ controlled by Eq.(7) is expected

$$
\begin{aligned}
& \left\|z_{k+1}-z^{*}\right\|=\left\|(f+B D) z_{k}-(f+B D) z^{*}\right\| \\
& \leq \alpha\left(\left\|z_{k}-(f+B D) z_{k}\right\|+\left\|z^{*}-(f+B D) z^{*}\right\|\right)
\end{aligned}
$$

Since $0 \leq \alpha<\frac{1}{2}$, according to Theorem, we get $\lim _{k \rightarrow \infty}\left\|z_{k}-z^{*}\right\|=0$, and the periodic orbit $z^{*}$ can be stabilized in $\Omega$.

As in [20], we consider the neural network defined as follows:

$$
\begin{aligned}
& x_{k+1}=\varphi_{\mu}\left(w_{11} x_{k}+w_{12} y_{k}\right)+u_{1 k},(8-a) \\
& y_{k+1}=\varphi_{\mu}\left(w_{21} x_{k}+w_{22} y_{k}\right)+u_{2 k},(8-b)
\end{aligned}
$$

where $u_{1 k}, u_{2 k} \in \mathfrak{R}$ are control inputs. Then, we have

$$
z_{k}=\binom{x_{k}}{y_{k}}, f\left(x_{k}, y_{k}\right)=\binom{\varphi_{\mu}\left(w_{11} x_{k}+w_{12} y_{k}\right)}{\varphi_{\mu}\left(w_{21} x_{k}+w_{22} y_{k}\right)}
$$

and $B=\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$.
Now, let us consider the following mapping

$$
D\left(x_{k}, y_{k}\right)=\binom{-\varphi_{\mu}\left(w_{11} x_{k}+w_{12} y_{k}\right)+\sqrt{\left|x_{k}\right|}}{-\varphi_{\mu}\left(w_{21} x_{k}+w_{22} y_{k}\right)+\sqrt{\left|y_{k}\right|}}
$$

Then, the mapping $f+B D$ is a Kannan mapping. Thus, the following control input can stabilize any periodic orbit embedded in a chaotic attractor of (6)

where $z^{*}=\left(x^{*}, y^{*}\right)$ denotes a stabilized periodic state with period 1 . To obtain the necessary information of an approximate location of the desired periodic orbit, the strategy described in Ref. [26] is utilized. We collect a long data string of observed $z_{1}, z_{2}=f\left(z_{1}\right)$ and so on. If two successive $z_{3}$ are closed to each other, say $z_{100}$ and $z_{101}$, then there will typically be a period-1 orbit $z^{*}$ nearby. Having observed a first such close return pair, we then search the succeeding data for other close return pairs $\left(z_{k}, z_{k+1}\right)$ restricted to the small region of the original close return. Because orbits on a strange attractor are ergodic, we will get many such pairs if the data string is long enough. When the first close return pair is detected, the first point of the pair is taken as a reference point. There are a number of close return pairs detected, which are close to reference point, where $z_{j, 1}$ and $z_{j, 2}$ are respectively used to denote the first point and its successive point of the $j$ th collected return pair, $j=1,2, \ldots, M$, where $M$ is the maximum number of collected return pairs. The mean value

$$
z^{*}=\frac{1}{2 M} \sum_{j=1}^{M}\left(z_{j, 1}+z_{j, 2}\right),
$$

is regarded as an approximate fixed point $z^{*}$. This fixed point can be used to define a neighborhood $\left|z_{i}-z^{*}\right| \leq \varepsilon$ in which control input is activated.

Remark 2 In comparison with the results given in [20], it can be seen that using controller $u_{k}$, proposed in this section, the results of [20] cannot show the control of the chaotic discrete neural network.

## 3 Numerical Example

Consider the following chaotic neural network

$$
\begin{array}{ll}
x_{k+1}=\varphi_{\mu}\left(-5 x_{k}+5 y_{k}\right)+u_{1 k}, & (10-a) \\
y_{k+1}=\varphi_{\mu}\left(-25 x_{k}+25 y_{k}\right)+u_{2 k}, & (10-b)
\end{array}
$$

where $\varphi_{\mu}(z)=\left(1+e^{-\mu z}\right)^{-1}$ is assumed to be the sigmoid function. The system has chaotic behavior for $\mu=5.5$, and the approximate period- 3 orbit is estimated at $(0.999496,1.00000)^{T},(0.593963,0.870103)^{T}$ and $(0.503459,0.517291)^{T}$, when the condition $\left|z_{i}-z_{i+2}\right| \leq 0.005$ is satisfied [20].

We first show the simulation results of chaotically synchronizing System (10) and System (5) without control input. So System (5) becomes as follows

$$
\begin{aligned}
& w_{1}(k+1)=\varphi_{\mu}(-5 x(k)+5 y(k))-\sqrt{|x(k)|}+\sqrt{\left|w_{1}(k)\right|}, \\
& w_{2}(k+1)=\varphi_{\mu}(-25 x(k)+25 y(k))-\sqrt{|y(k)|}+\sqrt{\left|w_{2}(k)\right|},(11-b)
\end{aligned}
$$



Fig. 1. The error $x(k)-w_{1}(k)$


Fig. 2. The error $y(k)-w_{2}(k)$
The system is simulated with initial conditions $x(0)=0.5, y(0)=0.6, w_{1}(0)=0.9, w_{2}(0)=0.7$, and the differences are showed in Figs. (1) and (2). These figures show that system (10) is synchronized with system (11).

Now, we show the simulation results of chaos control of System (10) using controller $u_{k}$ proposed in previous section.

Behaviors of the state variables $x$ and $y$ and the input controls $u_{1}$ and $u_{2}$ are shown in Figs. 3-6, when a periodic orbit with perio $\mathrm{d}=3$ is stabilized with $\varepsilon=0.002$.


Fig. 3. Behavior of $x$.


Fig. 4. Behavior of $y$.
Figs. 3 and 4 show behaviors of the state variables $x$ and $y$, respectively, with initial condition $\left(\begin{array}{ll}1.9 & 2.1\end{array}\right)^{T}$.


Fig. 5. Behavior of input control $u_{1}$.


Fig. 6. Behaviorf of input control $u_{2}$.

Figs. 5 and 6 show behaviors of the input controls $u_{1}$ and $u_{2}$, respectively.
These figures show that System (10) is stabilized by the controller proposed in this paper.

## 5 Conclusions

In this paper, a new method based on Kannan mappings for chaotic synchronization is proposed. Furthermore, a new method based on the mappings is presented to stabilize chaotic discrete systems. These methods are applied to synchronize and control of chaotic discrete neural networks. Finally, a numerical example is given to validate the methods presented.

## References

1. L. M. Pecora and L. Carroll. Synchronization in Chaotic Systems, Physical Review Letters, vol. 64, 821-824, 1990.
2. T. Ushio. Chaotic Synchronization and Controlling Chaos Based on Contraction Mappings, Physics Letters A, vol. 198, 14-22, 1995.
3. L. Liao and N. S. Huang. Control and Synchronization of Discrete Time Chaotic Systems via Variable Structure Control Technique, Physics Letters A, vol. 234, 262268, 1997.
4. C. Yang, Z. Ge, C. Chang, and S. Li. Chaos Synchronization and Chaos Control of Quantum-CNN Chaotic System by Variable Structure Control and Impulse Control, Nonlinear Analysis, Real World Applications, vol. 11, 1977-1985, 2010.
5. D. Vassiliadis. Parametric Adaptive Control and Parameter Identification of Low Dimensional Chaotic Systems, Physica D: Nonlinear Phenomena, vol. 71, 319-341, 1994.
6. S. Chen and J. Lu. Parameters Identification and Synchronization of Chaotic Systems Based upon Adaptive Control, Physics Letters A, vol. 299, 353-358, 2002.
7. E. Solak, O. Morgul, and U. Ersoy. Observer-Based Control of A Class of Chaotic Systems, Physics Letters A, vol. 279, 47-55, 2001.
8. J. A. Ramirez, H. Puebla, and I. Cervantes. Convergence Rate of Observer-Based Approach for Chaotic Synchronization, Physics Letters A, vol. 289, 193-198, 2001.
9. L. Huang, R. Feng, and W. Wang. Synchronization of Chaotic Systems via Nonlinear Control, Physics Letters A, vol. 320, 271-275, 2004.
10. J. H. Park. Chaos Synchronization of A Chaotic System via Nonlinear Control, Chaos, Solitons, and Fractals, vol. 25, 579-584, 2005.
11. H. K. Chen. Global Chaos Synchronization of New Chaotic Systems via Nonlinear Control, Chaos, Solitons, and Fractals, vol. 23, 1245-1251, 2005.
12. R. Tang and J. Xue. Chaotic Synchronization by Replacing Nonlinear Terms with Signals, Chaos, Solitons, and Fractals, vol. 28, 228-235, 2006.
13. Z. Ge and P. Tsen. Two Theorems of Generalized Unsynchronization for Coupled Chaotic Systems, Nonlinear Analysis, Theory, Method, and Applications, vol. 69, 4230-4240, 2008.
14. H. N. Agiza and M. T. Yassen. Synchronization of Rossler and Chen Chaotic Dynamical Systems Using Active Control, Physics Letters A, vol. 278, 191-197, 2001.
15. M. T. Yassen. Chaos Synchronization Between Two Different Chaotic Systems Using Active Control, Chaos, Solitons, and Fractals, vol. 23, 131-140, 2005.
16. E. Ott, C. Grebogi, and J. A. Yorke. Controlling Chaos, Physical Review Letters, vol. 64, 1190-1199, 1990.
17. M. Kopel. Improving The Performance of An Economic System: Controlling Chaos, Evelutionary Economics, vol. 7, 269-289, 1997.
18. L. Kass. Stabilizing Chaos in A Dynamical Macroeconomic Model, Economic Behavior and Organization, vol. 33, 313-332, 1998.
19. J. A. Holyst and K. Urbanowicz. Chaos Control in Economical Model by Time Delayed Feedback Method, Physica A, vol. 287, 587-598, 2000.
20. H. Yu, Y. Liu, and J. Peng. Control of Chaotic Neural Networks Based on Contraction Mappings, Chaos, Solitons, and Fractals, vol. 22, 787-792, 2004.
21. K. Aihara, T. Takabe, and M. Toyoda. Chaotic Neural Network, Physics Letters A, vol. 144, 333-340, 1990.
22. S. Nara, P. Davis, and H. Totsuji. Memory Search Using Complex Dynamics in A Recurrent Neural Network Model, Neural Networks, vol. 6, 963-973, 1993.
23. R. Sole, L. Menendez, and D. Prida. Controlling Chaos in Discreteneural Networks, Physics Letters A, vol. 199, 65-69, 1995.
24. H. Sabbagh. Control of Chaotic Solutions of The Hindmarsh-Rose Equations, Chaos, Soliton, and Fractals, vol. 112, 1213-1218, 2000.
25. R. Kannan. Some Results on Fixed Points-II, American Mathematical Monthly, vol. 76, 405-408, 1969.
26. T. Shinbrot, C. Grebogi, E. Ott, and J. York. Using Small Perturbations to Control Chaos, Nature, vol. 363, 411-417, 1993.

Chaotic Modeling and Simulation (CMSIM) 1: 241-256, 2012

# Wave Fractal Dimension as a Tool in Detecting Cracks in Beam Structures 

Chandresh Dubey ${ }^{1}$ and Vikram Kapila ${ }^{2}$<br>${ }^{1}$ Polytechnic Institute of NYU, 6 Metrotech Center, Brooklyn, New York (E-mail: cdubey01@students.poly.edu)<br>${ }^{2}$ Polytechnic Institute of NYU, 6 Metrotech Center, Brooklyn, New York (E-mail: vkapila@poly.edu)


#### Abstract

A chaotic signal is used to excite a cracked beam and wave fractal dimension of the resulting time series and power spectrum are analyzed to detect and characterize the crack. For a single degree of freedom (SDOF) approximation of the cracked beam, the wave fractal dimension analysis reveals its ability to consistently and accurately predict crack severity. For a finite element simulation of the cracked cantilever beam, an analysis of spatio-temporal response using wave fractal dimension in frequency domain reveals distinctive variation vis-à-vis crack location and severity. Simulation results are experimentally validated.


Keywords: Chaotic excitation, Chen's oscillator, Wave fractal dimension.

## 1 Introduction

Vibration-based methods for crack detection in beam type structures continue to attract intense attention from researchers. Most often these methods use external forcing input, e.g., harmonic input, to cause the structure to vibrate. Typical vibration-based crack detection methods exploit modal analysis techniques to determine changes in beam's natural frequency $[4,11,13]$ and relate these changes to the crack severity and in some cases to crack location [17,23]. To quantify the crack depth and to detect crack location, vibration-based crack detection methods employ a variety of characterizing parameters, such as natural frequency [11], mode shape [19], mechanical impedance [2], statistical parameters [22], etc. In recent research, wave fractal dimension, originally introduced by Katz [12] to characterize biological signals, has been used to detect the severity and location of crack in beam [7] and plate structures [8].

Over the last decade, progress in chaos theory has led several researchers to consider the use of chaotic excitation in vibration-based crack detection $[15,18]$. A majority of these efforts necessitate the reconstruction of a chaotic attractor from the time series data corresponding to the vibration response of the structure $[15,18]$. Unfortunately, the reconstruction of a chaotic attractor is often tedious and may not always yield satisfactory results for crack detection even in
the SDOF approximation case. To detect and characterize cracks, the current chaos-based crack detection methods use a variety of chaos and statistics-based parameters, such as correlation dimension [18], Hausdorff distance [18], average local attractor variance ratio [15], etc. In this paper, we study the use of wave fractal dimension as a characterizing parameters to predict the severity and location of a crack in a beam that is made to vibrate using a chaotic input.

## 2 Beam Excitation Methods

In this section, we consider three methods to excite the cracked beam. We begin by producing and analyzing the beam response to a non-zero initial condition which facilitates our understanding of the behavior of wave fractal dimension as a characterizing parameter for crack detection. We consider a unit displacement initial condition. Various references $[16,22]$ have already indicated various reasons for the wide use of harmonic input in vibration-based crack detection. Thus, we next consider the use of both sub-harmonic ( $\omega<\omega_{n}$ ) and super-harmonic $\left(\omega>\omega_{n}\right)$ inputs to vibrate the cracked beam model and study its behavior. Finally, we use the chaotic solution of autonomous dissipative flow type Chen's attractor [20] as an input excitation force to vibrate the SDOF model of cracked beam. The Chen's system in state space form is expressed as

$$
\begin{equation*}
\dot{y_{1}}=a_{1}\left(y_{2}-y_{1}\right), \quad \dot{y_{2}}=\left(a_{3}-a_{1}\right) y_{1}-y_{1} y_{3}+a_{3} y_{2}, \quad \dot{y_{3}}=y_{1} y_{2}-a_{2} y_{3} \tag{1}
\end{equation*}
$$

where $a_{1}, a_{2}$, and $a_{3}$ are constant parameters. Figure 1 shows the time series $y_{1}$ and the 2 D phase portrait of Chen's system corresponding to a chaotic solution. For the indicated values of constants $a_{1}, a_{2}$, and $a_{3}$ (see Figure 1), the solution $y_{1}$ is expected to be non-periodic. We restricted our attention to Chen's system because its solutions $y_{1}$ and $y_{2}$ are approximately symmetric about the time axis, producing the mean of $\approx 0$. Furthermore, in a detailed analysis of several popular chaotic attractors [20], we found that the Chen's system produced one of the largest wave fractal dimension (see Figure 2). Moreover, our analysis has revealed that chaotic attractors possessing these two properties produce large changes in wave fractal dimension with increasing or decreasing crack depths. These advantages will become more apparent in the following sections.

## 3 Wave Fractal Dimension

Waveforms are common patterns that arise frequently in scientific and engineering phenomena. A waveform can be produced by plotting a collection of ordered $(x, y)$ pairs, where $x$ increases monotonically. The concept of wave fractal dimension [12] is used to differentiate one waveform from another.

For waveforms, produced using a collection of ordered point pairs $\left(x_{i}, y_{i}\right)$, $i=1, \ldots, n$, the total length, $L$, is simply the sum of the distances between successive points, i.e., $L=\sum_{i=1}^{n-1} \sqrt{\left(x_{i+1}-x_{i}\right)^{2}+\left(y_{i+1}-y_{i}\right)^{2}}$. Moreover, the diameter $d$ of a waveform is considered to be the farthest distance between the


Fig. 1. The chaotic input (Chen's attractor) with $a_{1}=35, a_{2}=3, a_{3}=28, y_{1}(0)=$ $-10, y_{2}(0)=0$, and $y_{3}(0)=37$. (a) Time series of $y_{1}$ and (b) phase portrait projected onto the $\left(y_{1}, y_{2}\right)$ plane.
starting point (corresponding to $n=1$ ) and some other point (corresponding to $n=i, i=2, \ldots, n$ ), of the waveform, i.e., $d=\max _{i=2, \ldots, n} \sqrt{\left(x_{i}-x_{1}\right)^{2}+\left(y_{i}-y_{1}\right)^{2}}$. Next, by expressing the length of a waveform $L$ and its diameter $d$ in a standard unit, which is taken to be the average step $\alpha$ of the waveform, the wave fractal dimension can be expressed as [12]

$$
\begin{equation*}
D=\frac{\log (L / \alpha)}{\log (d / \alpha)}=\frac{\log (n)}{\log (n)+\log (d / L)} \tag{2}
\end{equation*}
$$

where $n=L / \alpha$, denotes the number of steps in the waveform. We use (2) to estimate the wave fractal dimension.

Using (2), wave fractal dimension is calculated for various chaotic attractors and results are shown in Figure 2 only for one waveform $\left(y_{1}, y_{2}\right.$ or $\left.y_{3}\right)$ of each attractor having maximum wave fractal dimension. Waveforms are normalized before calculating wave fractal dimension to maintain parity among various attractors. It is found that Chen's attractor has the largest fractal dimension and this was the reason for using Chen's attractor in current study.

## 4 Modeling of a Cracked Beam as a SDOF System with Force Input

Following $[1,18]$, a cracked beam is modeled as a SDOF switched system which emulates the opening and closing of the surface crack by switching the effective stiffness $k_{s}=k-\Delta k$, where $k$ is the stiffness of the beam without crack, $k_{s}$ is stiffness during stretching and $\Delta k$ is stiffness difference. For a SDOF model with a relatively small crack, the ratio of $\Delta k$ to $k$ is equal to the ratio of the crack depth $a$ to the thickness $h$ of the beam [1,18]. Next, we consider that the $y_{1}$ solution of (1) is applied as a force to the mass of the SDOF system. The equations of motion for this piecewise continuous SDOF system are

$$
\begin{align*}
M \ddot{x}+c \dot{x}+k x & =F(t), & & \text { for } x \geq 0 \\
M \ddot{x}+c \dot{x}+k_{s} x & =F(t), & & \text { for } x<0 \tag{3}
\end{align*}
$$



Fig. 2. Wave fractal dimension of chaotic attractor waveforms. (a) Vanderpol attractor $y_{2}$ component; (b) Ueda attractor $y_{2}$ component; (c) Duffing's two well attractor $y_{2}$ component; (d) Lorenz attractor $y_{2}$ component; (e) Chen's attractor $y_{1}$ component; (f) ACT attractor $y_{1}$ component; (g) Chua's attractor $y_{3}$ component; and (h) Burkeshaw attractor $y_{3}$ component.
where $M$ is the mass of the cantilever beam, $c$ is the damping coefficient, and $x$ is the displacement of the beam. The physical parameters of the problem data used in our simulations are as follows: mass $m=0.18 \mathrm{~kg}$, nominal stiffness $k=295 \mathrm{~N} / \mathrm{m}$, and damping $c=0.03 \mathrm{Ns} / \mathrm{m}$.

## 5 SDOF Results

For the three excitation methods of Section 2, the system responses for the SDOF model of section 4 are recorded and analyzed to carefully examine the influence of different excitation methods and signal characteristics on the behavior of wave fractal dimension (2). Moreover, we consider alternative ways to efficiently compute the wave fractal dimension.

### 5.1 SDOF results of wave fractal dimension for non-zero initial condition

We begin by simulating the SDOF system of (3) with a unit displacement initial condition and $F(t)=0$, for $t \geq 0$. The simulation is performed for various
values of small crack depths and the resulting time series data is provided in Figure 3. In each case, the vibration starts with unit displacement and eventually settles to zero due to damping. Even though all the curves look quite similar, the damped vibration frequency decreases with increasing crack depth [6]. Next, for each time series, we compute the corresponding wave fractal dimension and plot normalized crack depth versus the wave fractal dimension in Figure 4, which shows the wave fractal dimension decreases with increasing crack depth. As indicated above, increasing crack depth leads to lowering of the waveform frequency, thereby reducing the wave fractal dimension. Furthermore, note that the trend shown in Figure 4 is quite monotonic and can be used to detect small cracks. Unfortunately, the rate of change of wave fractal dimension vis-à-vis crack depth is very small.


Fig. 3. Time response of the SDOF system to non-zero initial displacement

### 5.2 SDOF results of wave fractal dimension with harmonic input

For a SDOF model (3) emulating a cracked beam, the natural frequency of the resulting model depends on the crack depth and will not be known prior to crack characterization. Thus, we consider the use of sub-harmonic $\left(\omega<\omega_{n}\right)$ and super-harmonic $\left(\omega>\omega_{n}\right)$ force inputs to vibrate the SDOF model for various values of crack depths. Figure 5 provides the resulting time series


Fig. 4. Change of the wave fractal dimension with normalized crack depth for unit initial displacement
plots for the sub-harmonic input case with various normalized crack depths. Following the initial transient response, in each plot, a steady state sinusoidal response is observed. Moreover, these responses reveal that the amplitude of the output waveform increases with increasing crack depth.


Fig. 5. Time response of the SDOF system to sub-harmonic ( $\omega<\omega_{n}$ ) input

Next, for each time series of Figure 5, we compute the corresponding wave fractal dimension and plot normalized crack depth versus the wave fractal dimension in Figure 6(a), which shows that the wave fractal dimension mono-
tonically increases with increasing crack depth. Note that, as indicated above, increasing crack depth leads to increasing amplitude of the waveform, leading to an increase in the wave fractal dimension. Next, we apply a super-harmonic $\left(\omega>\omega_{n}\right)$ forcing input to vibrate the SDOF model for various values of crack depths. From the resulting time series, we compute the corresponding wave fractal dimension and plot normalized crack depth versus the wave fractal dimension in Figure 6(b), which shows that the wave fractal dimension monotonically decreases with increasing crack depth. The results of this subsection indicate that in order to accurately predict the crack depth, we need to know the approximate natural frequency of the cracked system so that the correct graph (Figure $6(\mathrm{a})$ versus $6(\mathrm{~b})$ ) can be used. This is not very satisfactory since, as noted above, the natural frequency of the cracked beam depends on the crack depth and is not known a priori.


Fig. 6. Change of the wave fractal dimension with normalized crack depth for (a) sub-harmonic ( $\omega<\omega_{n}$ ) and (b) super-harmonic ( $\omega>\omega_{n}$ ) input

### 5.3 SDOF results of wave fractal dimension with chaotic input

We now consider the application of the chaotic forcing input of section 2 to vibrate the SDOF model for various values of crack depths. Figure 7 provides the resulting time series plots for the chaotic input with various normalized crack depths. Since the resulting waveforms are non-periodic, no obvious trends can be discerned from these plots. Next, for each time series of Figure 7, we compute the corresponding wave fractal dimension and plot normalized crack depth versus the wave fractal dimension in Figure 8, which shows that the wave fractal dimension monotonically increases with increasing crack depth. Note that, in contrast to the harmonic forcing input case, when using a chaotic excitation we do not need a priori knowledge of the natural frequency of the cracked beam. This feature is facilitated by the fact that the chaotic excitation signal has a broad frequency content.

Since wave fractal dimension is a characteristic of the waveform only, we consider the wave fractal dimension analysis of the time series of Figure 7 in frequency domain. To do so, we use the Fast Fourier Transform (FFT) [10]


Fig. 7. Time response of the SDOF system with chaotic input


Fig. 8. Change of the wave fractal dimension with normalized crack depth for chaotic input
technique to convert the time domain data of Figure 7 to frequency domain. The resulting frequency domain data in Figure 9 provides the power spectrum of the response of the SDOF cracked beam. Whereas the time response plots of Figure 7 do not reveal any trend, the power spectrum illustrates that the portion of FFT in the vicinity of beam's natural frequency $\omega_{n}$ experiences significant changes. Thus, we now concentrate in the neighborhood of $\omega_{n}$ as our window for computing the wave fractal dimension. Using this technique, in Figure 10(a), we plot normalized crack depth versus the wave fractal dimension for the windowed waveforms of Figure 9. From Figure 10(a), we observe that
the wave fractal dimension monotonically increases with increasing crack depth and this curve exhibits a significant rate of change. Thus, in the following analysis, we use the wave fractal dimension of power spectrum as a natural choice for crack detection and crack characterization.

Finally, we also plot wave fractal dimension versus normalized crack depth plots for power spectrum constructed from the FFT of non-zero initial condition response and the harmonic input response corresponding to Figures 3 and 5, respectively. The resulting plots are provided in Figures 10(b) and 10(c) and demonstrate that the frequency domain wave fractal dimension analysis is an effective way to characterize crack depth in a SDOF system.


Fig. 9. Power spectrum of the time response of SDOF system with chaotic input

## 6 Continuous Model Case

We now extend the results of section 5 to the continuous model case. To do so, as in $[19,21]$, we consider a continuous model of the dynamical behavior of the beam with a surface crack in two parts. Specifically, when the beam moves away from the neutral position so that the crack remains closed, the beam behaves as a typical continuous beam $[6,19,21]$. However, when the beam moves in the other direction from the neutral position, causing the crack to open, the


Fig. 10. Frequency domain change of the wave fractal dimension with normalized crack depth for (a) chaotic input, (b) unit initial displacement, and (c) harmonic input
resulting dynamics require the modeling of crack with a rotational spring whose stiffness is related to the crack depth [2,6,19,21].

Next, we used the ANSYS software [14] to simulate the dynamics of a cracked beam under external excitation. We modeled the beam as a 2-D elastic object using a beam3 element [14] which has tension, compression, and bending capabilities. The crack is simulated by inserting a torsional spring at the location of the crack and using the mathematical model described in [2,6,19,21]. The torsional spring is modeled using a combin14 element [14] which is a springdamper element used in 1-D, 2-D, and 3-D applications. In our finite element (FE) model, we used the combin14 element as a pure spring with 1-D (i.e., torsional) stiffness since the model of $[2,6,19,21]$ does not consider damping. The physical characteristics of the beam used in our FE model are as follows: material-Plexiglass, length -500 mm , width -50 mm , thickness -6 mm , modulus of elasticity- 3300 MPa , density $-1190 \mathrm{~kg} / \mathrm{m}^{3}$, and Poisson's ratio-0.35. This FE model was validated [6] by comparing the natural frequencies resulting from the FE simulations versus the natural frequencies computed in Matlab [5] for the dynamic model of $[6,19,21]$.

Next, we apply force input to the FE model using the time series $y_{1}$ of (1). In particular, using MATLAB, we simulate (1) and save 15,000 time steps of $y_{1}$ time series, which is applied as force input at 40 mm from the fixed end in ANSYS. The FE simulation is used to produce and record spatio-temporal responses for each node (corresponding to discretized locations along the beam span). The resulting data is imported in MATLAB for a detailed wave fractal dimension analysis.

To detect the presence of a crack in the beam, we only consider the time series data corresponding to the beam tip displacement. The time series for tip displacement is converted to the frequency domain using the FFT. The resulting power spectrum plot is provided in Figure 11 for various sizes of cracks located at $L_{1}=0.2 L$. From Figure 11, we observe significant changes around 6.4 Hz which corresponds to the first fundamental frequency of the beam. These changes in the power spectrum are due to changes in crack depth at $L_{1}=0.2 L$. To characterize the changes in crack depth, we now compute and plot the wave fractal dimension for cracks at various location along the beam. For example, Figure 12 provides wave fractal dimension curves for a crack
located at $L_{1}=0.2 L$ and, alternatively, at $L_{1}=0.4 L$. We term these curves as uniform crack location curves. We observe that a beam without a crack yields a wave fractal dimension of 1.1205 , and wave fractal dimension above this nominal value indicates presence of a crack in the beam. Unfortunately, this method can not provide a concrete answer about the severity and location of the crack. However, this method can be used to indicate a combination of size and location of crack or a region of the beam where crack may be present.


Fig. 11. Power spectrum of beam tip time response for a crack located at $L_{1}=0.2 L$


Fig. 12. Wave fractal dimension versus normalized crack depth-uniform crack location curves for $L_{1}=0.2 L$ and $L_{1}=0.4 L$

Next, to predict the severity and approximate location of the crack on the beam surface, we record the time series data of the beam response along its span for chaotic forcing input. Using the FFT, the time series data is converted to frequency domain. The resulting power spectrum plot is analyzed to identify a suitable window for computing the wave fractal dimension. Throughout this analysis, the frequency window used for computing the wave fractal dimension
is kept fixed for all crack depths considered. Figure 13(a) plots wave fractal dimension against normalized beam length for cracks of various severity located at $L_{1}=0.2 L$. These uniform crack depth curves yield the same wave fractal dimension till the crack location and their slopes change abruptly at the location of crack. In fact, past the crack location, the uniform crack depth curves exhibits a larger slope for a larger crack depth. Figure 13(b) shows similar behavior for crack location, $L_{1}=0.4 L$. The abrupt split in uniform crack depth curves at crack location and their increasing slope with increasing crack depth can be used to establish both the severity and location of crack.


Fig. 13. Wave fractal dimension versus normalized beam length-uniform crack depth curves for (a) $L_{1}=0.2 L$ and (b) $L_{1}=0.4 L$

## 7 Experimental Verification

A schematic of the experimental setup used is given in Figure 14. An aluminum base holds the shaker (Brüel \& Kjær Type 4810). To produce a base excitation, a test specimen is clamped on shaker. An accelerometer (Omega ACC 103) is mounted at the tip of the specimen using mounting bee wax. Our software environment consists of Matlab, Simulink, and Real Time Workshop in which the Chen's chaotic oscillator is propagated to obtain the time series corresponding to the $y_{1}$ signals of (1). Next, an analog output block in the Simulink program outputs the $y_{1}$ signal to a digital to analog converter of Quanser's Q4 data acquisition and control board which in turn is fed to a 12 volt amplifier (Kenwood KAC-8202) to drive the shaker. The accelerometer output is processed by an amplifier (Omega ACC PSI) and interfaced to an analog to digital converter of the Q4 board for feedback to the Simulink program. Properties of the specimen used in our experiments are same as in Section 6. To emulate a fine hair crack, we used a 0.1 mm saw to introduce cracks of several different desired depths. As noted in [3], sawed and cracked beams yield different natural frequencies wherein the frequency difference is dependent on the width of the cut.

Thus, it follows that the frequency characteristics of sawed and cracked beams may differ significantly for larger crack width and render the natural frequency based crack detection methods ineffective. The results of this effort are not significantly affected since, instead of relying on changes in natural frequency, our crack detection approach relies on measuring and comparing wave fractal dimension of chaotically excited vibration response. For specimen of different crack depth, all located at $L_{1}=0.2 L=100 \mathrm{~mm}$ from fixed end, the accelerometer measurement is recorded and used to produce the output response time series, which is used to perform our analysis. A total of six specimens were prepared with crack depth varying from $0 \%$ to $50 \%$ of the thickness. In all the specimen, saw crack was introduced on the top surface to match with the simulation condition.


Fig. 14. Experimental setup

The time series data obtained from the accelerometer suffered from general sensor errors (dc offset and ramp bias), causing the raw time series data to be unusable for further analysis. We used the Wavelet transformation toolbox [9] of MATLAB to filter the raw time series data and remove the errors. This filtering technique uses a moving average of the waveform to shift its mean to 0 [6]. Using this technique with Chen's input to the beam structure with various crack depth, we obtain Figure 15 that shows the corrected time series. Next, we use the time series data of Figure 15 to compute the wave fractal dimension and plot the result against the crack depth. Following the trends observed in our numerical study, in Figure 16(a), wave fractal dimension versus crack depth plot shows an increasing trend.

Finally, we perform FFT on the time series data of Figure 15 to obtain the power spectrum plots (see [6]) for various crack depths. Next, we compute the wave fractal dimension of the frequency domain data using a window from 0 to 20 Hz . Figure 16(b) shows that the wave fractal dimension of frequency domain data exhibits an increasing trend against increasing crack depth, matching the trend observed in our numerical study. Although the plots obtained from the experimental data are not as smooth as the ones resulting from numerical simulation, this may be the result of inaccuracies resulting from sample preparation or a variety of experimental errors [6].


Fig. 15. Filtered time series for different crack depths with Chen's input


Fig. 16. Wave fractal dimension for different crack depths at $L_{1}=0.2 L$ from (a) time series and (b) frequency domain data

## 8 Conclusion

In this paper, to detect and characterize a crack in a beam, we considered a SDOF and a FE model of the beam excited by a chaotic force input. We showed that for the SDOF model, crack severity can be easily and consistently predicted by using wave fractal dimension of power spectrum of time series data. Moreover, for the FE model, we showed that wave fractal dimension exhibits a trend that can be used to predict crack location and crack depth. Finally, the simulation results were validated experimentally.

## Acknowledgments

This work is supported in part by the National Science Foundation under an RET Site grant 0807286, a GK-12 Fellows grant 0741714, and the NY Space Grant Consortium under grant 48240-7887.

## References

1.N. Bouraou and L. Gelman. Theoretical bases of free oscillation method for acoustical non-destructive testing. Proceedings of Noise Conference, The Pennsylvania State University, 519-524, 1997.
2.Y. Bamnios, E. Douka, and A. Trochidis. Crack identification in beam structures using mechanical impedance. Journal of Sound and Vibration, 256(2):287-297, 2002.
3.P. Cawley and R. Ray. A comparision of natural frequency changes produced by cracks and slots. Transactions of the ASME, 110:366-370, 1998.
4.S. Chinchalkar. Determination of crack location in beams using natural frequencies. Journal of Sound and Vibration, vol. 247(3), pp. 417-429. 2001.
5.C. H. Duane and L. L. Bruce. Mastering Matlab 7. Upper Saddle River, New Jersey, 2005. Prentice Hall.
6.C. Dubey. Damage Detection in Beam Structures using Chaotic Excitation. Master's Thesis at Polytechnic Institute of NYU, New York, 2010.
7.L. J. Hadjileontiadis, E. Douka, and A. Trochidis. Fractal dimension analysis for crack identification in beam structures. Mechanical Systems and Signal processing, vol. 19, pp. 659-674. 2005.
8.L. J. Hadjileontiadis and E. Douka. Crack detection in plates using fractal dimension. Engineering Structures, vol. 29, pp. 1612-1625. 2007.
9.A. Jensen and la. CH. Anders. Ripples in Mathematics: The Discrete Wavelet Transform. New York, 2001. Cambridge University Press.
10.J. F. James. A Student's Guide to Fourier Transforms: With Applications in Physics and Engineering. New York, NY, 2003. Springer Verlag.
11.J. T. Kim and N. Stubbs. Crack detection in beam type structures using frequency data. Journal of Sound and Vibration, vol. 259(1), pp. 145-160. 2003.
12.M. J. Katz. Fractals and the analysis of waveforms. Computers in Biology and Medicine, vol. 18(3), pp. 145-156. 1998.
13.J. Lee. Identification of multiple cracks in a beam using natural frequencies. Journal of Sound and Vibration, vol. 320, pp. 482-490. 2009.
14.S. Moaveni. Finite Element Analysis Theory and Application with ANSYS. Upper Saddle River, NJ, 2007. Prentice Hall.
15.J. M. Nichols, S. T. Trickey, and Virgin. Structural health monitoring through chaotic interrogation. Meccanica, 38:239-250, 2003.
16.S. Orhan. Analysis of free and forced vibration of a cracked cantilever beam. NDTधE International, 40:443-450, 2007.
17.D. P. Patil and S. K. Maiti. Experimental verification of a method of detection of multiple cracks in beams based on frequency measurements. Journal of Sound and Vibration, vol. 281, pp. 439-451. 2005.
18.J. Ryue and P. R. White. The detection of crack in beams using chaotic excitations. Journal of Sound and Vibration, 307:627-638, 2007.
19.P. F. Rizos, N. Aspragathos, and A. D. Dimarogonas. Identification of crack location and magnitude in a cantilever beam from the vibration modes. Journal of Sound and Vibration, 138(3):381-388, 1990.
20.J. C. Sprott. Chaos and Time-Series Analysis. New York, NY, 2003. Oxford University Press.
21.M. Taghi, B. Vakil, M. Peimani, M. H. Sadeghi, and M. M. Ettefagh. Crack detection in beam like structures using genetic algorithms. Applied Soft Computing, 8:1150-1160, 2008.
22.I. Trendafilova and E. Manoach. Vibration-based damage detection in plates by using time series analysis. Mechanical Systems and Signal processing, 22:10921106, 2008.
23.E. Viola, L. Federici, and L. Nobile. Detection of crack using cracked beam element method for structural analysis. Theoretical and Applied Fracture Mechanics, vol. 36, pp. 23-35. 2001.

# Symmetry-Breaking of Interfacial Polygonal Patterns and Synchronization of Travelling Waves within a Hollow-Core Vortex 

Amr Mandour, Mohamed Fayed, Hamid Ait Abderrahmane, Hoi Dick Ng ${ }^{1}$, Lyes Kadem and Georgios H. Vatistas<br>${ }^{1}$ Concordia University, Montréal, QC, H3G 1M8, Canada<br>(E-mail: hoing@encs.concordia.ca)


#### Abstract

A hollow vortex core in shallow liquid, produced inside a cylindrical reservoir using a rotating disk near the bottom of the container, exhibits interfacial polygonal patterns. These pattern formations are to some extent similar to those observed in various geophysical, astrophysical and industrial flows. In this study, the dynamics of rotating waves and polygonal patterns of symmetry-breaking generated in a laboratory model by rotating a flat disc near the bottom of a cylindrical tank is investigated experimentally. The goal of this paper is to describe in detail and to confirm previous conjecture on the generality of the transition process between polygonal patterns of the hollow vortex core under shallow water conditions. Based on the image processing and an analytical approach using power spectral analysis, we generalize in this work - using systematically different initial conditions of the working fluids - that the transition from any $N$-gon to $(N+1)$-gon pattern observed within a hollow core vortex of shallow rotating flows occurs in an universal two-step route: a quasi-periodic phase followed by frequency locking (synchronization). The present results also demonstrate, for the first time, that all possible experimentally observed transitions from $N$-gon into ( $N+1$ )-gon occur when the frequencies corresponding to $N$ and $N+1$ waves lock at a ratio of $(N-1) / N$. Keywords: Swirling flow, patterns, transition, quasi-periodic, synchronization.


## 1 Introduction

Swirling flows produced in closed or open stationary cylindrical containers are of fundamental interest; they are considered as laboratory model for swirling flows encountered in nature and industries. These laboratory flows exhibit patterns which resemble to a large extent the ones observed in geophysical, astrophysical and industrial flows. In general, the dynamics and the stability of such class of fluid motion involve a solid body rotation and a shear layer flow. Because of the cylindrical confining wall, the shear layer flow forms the outer region while the inner region is a solid body rotation flow. The interface between the flow regimes can undergo Kelvin-Helmholtz instability because of the jump in velocity at the interface between the inner and outer regions, which manifests as azimuthal waves. These waves roll up into satellite vortices which impart the interface polygonal shape, e.g., see [5, 9, 11-13]. The inner solid body rotation region can also be subjected to inertial instabilities which manifest
as Kelvin's waves and it is this type of waves that will be investigated in this paper. In our experiment a hollow core vortex, produced by a rotating disk near the bottom of a vertical stationary cylinder, is within the inner solid body rotation flow region and acts as a wave guide to azimuthal rotating Kelvin's waves. The shape of the hollow core vortex was circular before it breaks into azimuthal rotating waves (polygonal patterns) when some critical condition was reached.
A fundamental issue that many research studies were devoted to the study of rotating waves phenomena is the identification and characterization of the transition from symmetrical to non-symmetrical swirling flows within cylindrical containers. Whether confined or free surface flow, the general conclusion from all studies confirmed that, the Reynolds number and aspect ratio (water initial height $H /$ cylinder container radius $R$ ) are generally the two dominant parameters influencing the symmetry breaking phenomenon's behavior. Escudier [7] and Vogel [16] studied the transitional process in confined flows and found that symmetry breaking occurs when a critical Reynolds number was reached for each different aspect ratio. Vogel [16] used water as the working fluids in his study where he observed and defined a stability range, in terms of aspect ratio and Reynolds number, for the vortex breakdown phenomenon which occurred in the form of a moving bubble along the container's axis of symmetry. Escudier [7] later extended the study by using an aqueous glycerol mixture ( 3 to 6 times the viscosity of water) and found that varying the working fluid viscosity caused changes in the critical Reynolds number values. He also observed that for a certain range of aspect ratio and viscosity, the phenomenon of vorticity breakdown has changed in behavior, revealing more vortices breakdown stability regions than the conventional experiments using water as the working fluid. Where in open free surface containers under shallow liquid conditions using water as the working fluid, Vatistas [14] studied the transitional flow visually and found that the range of the disc's RPM where the transitional process occurs shrinks as the mode shapes number increased. Jansson et al. [10] concluded that the end-wall shear layers as well as the minute wobbling of the rotating disc are the main two parameters influencing the symmetry breaking phenomenon and the appearance of the polygonal patterns. Vatistas et al. [15] studied the transition between polygonal patterns from $N$ to $N+1$, using image processing techniques, with water as the working fluid and found that the transition process from $N$ to a higher mode shape of $N+1$ occurs when their frequencies ratio locks at $(N-1) / N$, therefore following a devil staircase scenario which also explains the fact that the transition process occurs within a shorter frequency range as the mode shapes increase. Speculating the transition process as being a bi-periodic state, the only way for such system to lose its stability is through frequency locking [4]. From nonlinear dynamics consideration, Ait Abderrahmane et al. [2] proposed the transition between equilibrium states under similar configurations using classical nonlinear dynamic theory approach and found that the transition occurs in two steps being, a quasi-periodic and frequency locking stages, i.e., the transition occurs through synchronization of the quasi-periodic regime formed
by the co-existence of two rotating waves with wave numbers $N$ and $N+1$. Their studies however was built mainly on the observation of one transition, from 3gon to 4-gon.
In the present paper, we provide further details on the symmetry-breaking pattern transitions and confirm the generalized mechanism on the transition from $N$-gon into $(N+1)$-gon using power spectra analysis. This study systematically investigates different mode transitions, the effect of working fluid with varying viscosity, liquid initial height on the polygonal pattern instability observed within the hollow core.

## 2 Experimental Setup and Measurement Technique

The experiments were conducted in a 284 mm diameter stationary cylindrical container with free surface (see Figure 1). A disk, located at 20 mm from the bottom of the container, with radius $R_{\mathrm{d}}=126 \mathrm{~mm}$ was used and experiments with three initial water heights above the disk, $h_{o}=20,30$ and $h_{o}=40 \mathrm{~mm}$, were conducted. Similar experiment was conducted by Jansson et al. [10] within a container of different size where the distance of the disk from the bottom of the container is also much higher than in the case of our experiment. In both experiments similar phenomenon - formation of a polygonal pattern at the surface of the disk - was observed. It appears therefore that the dimension of the container and the distance between the disk and the container bottom do not affect the mechanism leading to the formation of the polygon patterns. In our experiment, the disk was covered with a thin smooth layer of white plastic sheet. It is worth noting that the roughness of the disk affects the contact angle between the disk and the fluid; this can delay the formation of the pattern. However, from our earlier observation in many experiments, roughness of the disk does not seem to influence prominently the transition mechanism.


Fig. 1. Experimental setup.


Fig. 2. The variation of dynamic viscosity as a function of glycerol concentration (by weight $\% \mathrm{wt}$ ).

The disk speed, liquid initial height and viscosity were the control parameters in this study. The motor speed, therefore the disc's speed, was controlled using a PID controller loop implemented on LABVIEW environment. Experiments with tape water and aqueous glycerol mixtures, as the working fluids, were conducted at three different initial liquid heights of 20,30 and 40 mm above the rotating disc. The viscosity values of the used mixtures were obtained through technical data provided by a registered chemical company - Dow Chemical Company 1995-2010 [6]. Eight different aqueous glycerol mixtures were used in the experiments with viscosity varying from 1 to 22 ( $0 \sim 75 \%$ glycerol) times the water's at room temperature $\left(21^{\circ} \mathrm{C}\right)$. The detailed points of study were: 1,2 , $4,6,8,11,15$ and 22 times the water's viscosity $\left(\mu_{\text {water }}\right)$ at room temperature. Although the viscosity of the mixture varied exponentially with the glycerol concentration (see Figure 2), closer points of study were conducted at low concentration ratios since significant effects have been recognized by just doubling the viscosity of water as it will be discussed later. The temperature variation of the working fluid was measured using a mercury glass thermometer and recorded before and just after typical experimental runs and was found to be stable and constant (i.e. room temperature). Therefore, the viscosity of the mixture was ensured to be constant and stable during the experiment. Phase diagrams had been conducted and showed great approximation in defining the different regions for existing patterns in terms of disc's speed and initial height within the studied viscosity range.
A digital CMOS high-speed camera (pco. 1200hs) with a resolution of 1280 x 1024 pixels was placed vertically above the cylinder using a tripod. Two types of images were captured: colored and 8 -bit gray scale images, at 30 frames per second, for the top view of the formed polygonal patterns (see Figure 3 for example). The colored images were used as illustration of the observed stratification of the hollow vortex core where each colored layer indicates a water depth within the vortex core. It is worth noticing that the water depth increases continuously as we move away from the center of the disk (due to the
applied centrifugal force). The continuous increase in the water depth, depicted in the Figure 3 by the colored layers, indicates momentum stratification in the radial direction (i.e., starting with the central white region which corresponds to a fully dry spot of the core and going gradually through different water depth phases until reaching the black color region right outside the polygonal pattern boundary layer). For subsequent quantitative analysis, the data was conducted with grey images as those are simpler for post-processing.
The transition mechanism is investigated using image processing techniques. First the images were segmented; the original 8-bit gray-scale image is converted into a binary image, using a suitable threshold, to extract the polygonal contours [8]. This threshold value is applied to all subsequent images in a given run. In the image segmentation process, all the pixels with gray-scale values higher than the threshold were assigned 1's (i.e., bright portions) and the pixels with gray-scale values lower than or equal to the threshold were assigned 0 's (i.e., dark portions). The binary image obtained after segmentation is filtered using a low-pass Gaussian filter to get rid of associated noises. In the next step, the boundaries of the pattern were extracted using the standard edge detection procedure. The pattern contours obtained from the edge detection procedure were then filtered using a zero-phase filter to ensure that the contours have no phase distortion. The transformations of the vortex core are analyzed using Fast Fourier Transform (FFT) of the time series of the radial displacement for a given point on the extracted contour, defined by its radius and its angle in polar coordinates with origin at the centroid of the pattern; see [1-3] for further details.


Fig. 3. Polygonal vortex core patterns. The inner white region is the dry part of the disk and the dark spot in the middle of the image is the bolt that fixes the disk to the shaft. The layers with different colors indicate the variation of water depth from the inner to the outer flow region.


Fig. 4. (a), (b), (c) Oval pattern progression and corresponding Power spectra; and (d) oval to triangular transition $N=2$ to $N=3$ and corresponding power spectrum.

## 3 Results and Discussion

We first discuss results obtained at an initial height $h_{i}=40 \mathrm{~mm}$ where transitions from $N=2 \rightarrow N=3$ and $N=3 \rightarrow N=4$ were recorded and analyzed using power spectral analysis. Starting with stationary undisturbed flow, the disc speed was set to its starting point of 50 RPM and was then increased with increments of 1 RPM. Sufficient buffer time was allowed after each increment for the flow to equilibrate. At a disc speed of 2.43 Hz the first mode shape (oval) appeared on top of the disc surface. At the beginning of the $N=2$ equilibrium state, the vortex core is fully flooded. While increasing the disc speed gradually, several sets of 15008 -bit gray-scale images were captured and recorded. Recorded sets ranged 3 RPM in between. Systematic tracking of the patterns speed and shape evolution were recorded and the recorded images were processed. The evolution of the oval equilibrium state shape and rotating frequency is shown in Figures 4 a to 4 d . Starting with a flooded core at $f_{p}=$ 0.762 Hz in figure 4 a where the vertex of the inverted bell-like shape free surface barely touched the disc surface, Figure $4 b$ then shows the oval pattern after gaining more centrifugal force by increasing the disc speed by 9 RPM. The core became almost dry and the whole pattern gained more size both longitudinally and transversely with a rotating frequency of $f_{p}=0.791 \mathrm{~Hz}$. It is
clearly shown that at this instance, one of the two lobes of the pattern became slightly fatter than the other. Figure 4 c shows shape development and rotational speed downstream the $N=2$ range of existence. It is important to mention that once the oval pattern is formed, further increase in the disc speed, therefore the centrifugal force applied on the fluid, curved up the oval pattern and one of the lobes became even much fatter giving it a quasi-triangular shape. Figure 4d features the end of the oval equilibrium pattern in the form of a quasi-triangular pattern and therefore the beginning of the first transition process ( $N=2$ to $N=$ 3). The transition process is recorded, processed and the corresponding power spectrum was generated (see Figure 4d). The power spectral analysis revealed two dominant frequencies from the extracted time series function of the captured images; frequency $f_{\mathrm{m}}$ corresponds to the original oval pattern and frequency $f_{\mathrm{s}}$ corresponds to the growing subsequent wave $N=3$, which is a travelling soliton-like wave superimposed on the original oval pattern therefore forming the quasi-triangular pattern [2]. Further increase of the disc speed resulted in the forming and stabilizing of the triangular mode shape ( $N=3$ ) with a flooded core; both the troughs and apexes of the polygonal pattern receded and the core area shrank significantly.
Following the same procedure, the development of the triangular pattern and its transition to square ( $N=4$ ) shape were recorded, image processed and analyzed. Figures 5a to 5e show the power spectra plots and their corresponding sample image from the set recorded and used in generating each of the power spectra. The behaviour of the oval pattern's shape development and transition was also respected for the triangular pattern evolution.
Ait Abderrahmane et al. [2] described the transition process in the form of a rotating solid body $N$ shape associated with a traveling "soliton"-like wave along the vortex core boundary layer. The evidence of such soliton-like wave is revealed here. Figure 6 shows a sample set of colored RGB images during the transition process described above; these images feature the quasi-periodic state during $N=3$ to $N=4$ transition described earlier. Giving a closer look at the sequence of images, one could easily figure out the following: the three lobes or apexes of the polygonal pattern are divided into one flatten apex and two almost identical sharper apexes. Keeping in mind that the disc, therefore the polygonal pattern, is rotating in the counter clockwise direction and that the sequence of images is from left to right, by tracking the flatten lobe, one could easily recognize that an interchange between the flatten lobe and the subsequent sharp lobe (ahead) takes place (see third row of images). In other words, now the flattened apex receded to become a sharp stratified apex and the sharp lobe gained a more flattened shape. Such phenomenon visually confirms the fact that transition takes place through a soliton-like wave travelling along the vortex core boundary but with a faster speed than the parent pattern. This first stage of the transition process was referred to as the quasi-periodic stage by Ait Abderrahmane et al. [2]. The quasi-periodic stage takes place in all transitions until the faster travelling soliton-like wave synchronizes with the patterns rotational frequency forming and developing the new higher state of equilibrium pattern. Vatistas et al. [15] found that the synchronization process takes place
when the frequencies ratio of both pattern $(N)$ and the subsequent pattern developed by the superimposed soliton wave $(N+1)$ lock at a ratio of $(N-1) / N$. Therefore, for transition from $N=2$ to $N=3$, the synchronization takes place when the frequencies ratio is rationalized at $1 / 2$. And the transition $N=3$ to $N=$ 4 , takes place when the ratio between both frequencies are equal to $2 / 3$. In the above illustrated two transition processes, the frequency ratio for first transition was equal to $f_{N} / f_{N+1}=f_{m} / f_{s}=1.69 / 3.04=0.556 \approx 1 / 2$. On the other hand, the second transition took place when $f_{N} / f_{N+1}=f_{m} / f_{s}=3.28 / 4.92=0.666 \approx 2 / 3$.


Fig. 5. (a), (b), (c) Triangular pattern progression and corresponding power spectra; (d) Transitional process from triangular to square pattern; and (e) square pattern and corresponding power spectra.


Fig. 6. Quasi-periodic state during triangular to square transition.
Following the same trend, the second experiment was conducted using water at an initial height of 20 mm . At this low aspect ratio, transition between higher mode shapes was tracked and recorded. Using similar setup and experimental procedure, the transition from square mode $(N=4)$ to pentagonal pattern $(N=5)$ and from pentagonal to hexagonal pattern $(N=6)$ were recorded and imageprocessed for the first time in such analysis. Following the same behavior, the transition occurred at the expected frequency mode-locking ratio. Figure 7a shows the third polygonal transition, from $N=4$ to $N=5$. The frequency ratio of the parent pattern to the soliton-like wave is $f_{m} / f_{s}=4.102 / 5.449=0.753 \approx 3 / 4$. Similarly, Figure 7b shows the transition power spectrum for the last transition process observed between polygonal patterns, which is from $N=5$ to $N=6$ polygonal patterns. The frequency ratio $f_{m} / f_{s}=5.625 / 6.973=0.807$ which is almost equal to the expected rational value $4 / 5$. With these two experimental runs, the explanation of the transition process between polygonal patterns observed within hollow vortex core of swirling flows within cylinder containers under shallow water conditions is confirmed for all transitional processes.


Fig. 7. (a) Square to pentagonal transition; and (b) pentagonal to hexagonal transition.

| Initial height ( $\boldsymbol{h}_{\boldsymbol{i}}$ ) |  | $h_{i}=\mathbf{2 0} \mathrm{mm}$ |  |  | $h_{i}=\mathbf{3 0} \mathrm{mm}$ |  |  | $h_{i}=40 \mathrm{~mm}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Tra $(N)$ |  | 3-4 | 4-5 | 5-6 | 2-3 | 3-4 | 4-5 | 2-3 | 3-4 |
|  | 1 | 0.697 | 0.787 | 0.829 | 0.545 | 0.68 | 0.74 | 0.558 | 0.69 |
|  |  | 4.6\% | 4.9\% | 3.6\% | 9.0\% | 2.0\% | 7.5\% | 11.6\% | 3.5\% |
|  | 2 | 0.667 | 0.747 | -- | 0.558 | 0.671 | -- | 0.557 | 0.678 |
|  |  | 0.1\% | 0.4\% | -- | 11.6\% | 0.7\% | -- | 11.4\% | 1.7\% |
|  | 4 | 0.64 | -- | -- | -- | 0.671 |  | 0.557 | 0.686 |
|  |  | 4.0\% | -- | -- | -- | 0.7\% |  | 11.4\% | -- |
|  | 6 | -- | -- | -- | -- | 0.6667 |  | 0.55 | -- |
|  |  |  |  |  |  | 0.0\% |  | 10.0\% |  |
|  | 8 |  |  |  |  | -- |  | 0.536 |  |
|  |  |  |  |  |  |  |  | 7.2\% |  |
|  | 11 |  |  |  |  |  |  | 0.58 |  |
|  |  |  |  |  |  |  |  | 16.0\% |  |
|  | 15 |  |  |  |  |  |  | 0.552 |  |
|  |  |  |  |  |  |  |  | 10.4\% |  |
|  | 22 |  |  |  |  |  |  | 0.559 |  |
|  |  |  |  |  |  |  |  | 11.8\% |  |
|  |  |  |  | $\mathrm{fm} / \mathrm{fs}$ |  |  |  |  |  |
|  |  |  |  | \%error |  |  |  |  |  |

Table 1. Transition mode-locking frequencies for different liquid viscosities.


Fig. 8. Power spectrum for $N=2$ pattern replica
The influence of the liquid viscosity on the transitional process from any $N$ mode shape to a higher $N+1$ mode shape is also investigated. As described earlier, eight different liquid viscosities were used in this study ranging from 1 up to 22 times the viscosity of water. All transitional processes between subsequent mode shapes were recorded, and acquired images were processed. Using the same procedure as in the last section, the frequency ratio of the parent pattern $N$ and the subsequent growing wave $N+1$ has been computed and tabulated in Table 1. As shown in Table 1, the maximum deviation from the expected mode-locking frequency ratio $\left(f_{m} / f_{s}\right)$ always appeared in the first transition ( $N=2$ to $N=3$ ). A reasonable explanation for such induced error is the fact that, the higher the number of apexes per full pattern rotation, the more accurate is the computed speed of the pattern using the image processing technique explained before. Therefore, throughout the conducted analysis, the most accurate pattern's speed is the hexagon and the least accurate is the oval pattern. Apart from that significant deviation, one can confidently confirm that even at relatively higher viscous swirling flows, the transition between polygonal patterns instabilities takes place when the parent pattern ( $N$ ) frequency and the developing pattern $(N+1)$ frequency lock at a ratio of $(N-1) / N$, Vatistas et al. [15].
As explained earlier, transition has been found to occur in two main stages being the quasi-periodic and the frequency-locking stages [2]. It is also confirmed that frequency mode-locking does exist in polygonal patterns transition irrelative of the mode shapes, liquid heights and the liquid viscosity (within the studied region). In this section, the quasi-periodic phase will be further elucidated and confirmed. Earlier in this paper the quasi-periodic state in the transition of $N=3$ to $N=4$, using water as the working fluid, was observably described in Figure 6. To further analyze the quasi-periodic stage, a technique has been developed which animates the actual polygonal patterns instabilities but without the existence of the speculated travelling soliton-like wave along the patterns boundary layer. Using MAPLE plotting program, all mode shapes replica have been plotted and printed. Table 2 shows the plots and their corresponding plotting functions. Printed images were glued to the rotating disc under dry conditions one at a time. The disc was rotated with corresponding pattern's expected speeds under normal working conditions. Such technique
gave full control of the rotating pattern. Therefore, both speed and geometry of the patterns were known at all times. Sets of 15008 -bit images were captured and processed using similar computing procedure.

| $\mathbf{N}$ | Pattern plot | Plot function |
| :---: | :---: | :---: |
| $2-3$ | $r=1+0.2 \sin (2 \theta)$ |  |
| 3 |  | $r=1+0.2 \sin (2 \theta)+0.1 \sin (3 \theta+1)$ |
| $3-4$ |  | $r=1+0.1+15 \sin (3 \theta)$ |
|  |  |  |

Table 2. Patterns replica with corresponding functions.


Fig. 9. Power spectrum of transition processes using patterns replica (a) $N=2$ to $N=3$; and (b) $N=3$ to $N=4$

Power spectra of the processed sets of images revealed similar frequency plots. Starting with the oval-like shape, the disc was rotated at a constant speed of 1 Hz and the power spectrum was generated from the extracted images and plotted as shown in Figure 8. Since the oval pattern speed is controlled in this case (by disc speed), the frequency extracted could have been presumed to be double the disc frequency ( 2 Hz ). The actual frequency extracted is shown in Figure $8, f_{m}=1.934 \mathrm{~Hz}$ ( $3.3 \%$ error). Following the same procedure, other polygonal patterns replica were printed to the disc, rotated, captured and processed subsequently. Figures 9 a and 9 b show the power spectra generated from rotating the quasi-triangular and the quasi-square patterns, respectively.
Figure 9a shows a power spectrum generated from the set of pictures featuring a quasi-triangular pattern captured at 30 fps . The power spectrum revealed two dominant frequencies being $f_{m}=3.809 \mathrm{~Hz}$ and $f_{s}=5.742 \mathrm{~Hz}$ corresponding to the oval and triangular patterns, respectively. Since the quasi-triangular pattern is stationary and under full control, it could have been presumed that the frequency ratio would have a value of $2 / 3$ since the replica pattern is generated by superimposing the oval and triangular functions. The actual extracted frequency was $f_{m} / f_{s}=3.81 / 5.74=0.663 \approx 2 / 3$. Comparing this frequency ratio with the real polygonal patterns mode-locking ratio of $1 / 2$ described earlier, it is clear that the ratio is totally different which proves that both patterns are not behaving equivalently although having generally similar instantaneous geometry. Therefore, the actual rotating pattern does not rotate rigidly as the pattern replica does, but rather deforms in such a way that the ratio of the two frequencies is smaller which confirms the idea of the existence of the fast rotating soliton-like wave $\left(f_{s}\right)$. Moving to the second transition process, triangular to square, as shown in Figure 9b, the frequency ratio was found to be $3 / 4$ as expected since the function used to plot the quasi-square pattern is the superposition of both functions used in plotting the pure triangular and square patterns given in Table 2. Comparing this ratio with the actual mode-locking ratio of $2 / 3$ observed with real polygonal patterns, it is obvious that the ratio is
still smaller which respects the existence of a faster rotating wave along the triangular pattern boundary that eventually develops the subsequent square pattern as visualized earlier using the colored images. From these two experiments, along with the visual inspection discussed earlier, the existence of the fast rotating soliton-like wave $(N+1)$ along the parent pattern boundary layer $(N)$ is verified, therefore, the quasi-periodic stage.

## 4 Conclusions

Through the analysis of the present experimental results from different initial conditions, we confirmed with further evidences and generalized the mechanism leading to transition between two subsequent polygonal instabilities waves, observed within the hollow vortex core of shallow rotating flows. The transition follows the universal route of quasi-periodic regime followed by synchronization of the two waves' frequencies. We shows, for the first time, all observed transitions from N -gon to a subsequent $(N+1)$-gon occur when the frequencies corresponding to $N$ and $N+1$ waves lock at a ratio of $(N-1) / N$. The effect of varying the working fluid viscosity on the transitional processes between subsequent polygonal patterns was also addressed in this paper.
Both stages of the transitional process were further explored in this work. The quasi-periodic stage was first tackled using two different techniques, a visual method and an animated method. The deformation of the colored stratified boundary layers of polygonal patterns were inspected during transition process of polygonal patterns and the existence of a fast rotating wave-like deformation was recognized which confirms the idea of the co-existence of a soliton-like wave that initiates the quasi-periodic stage at the beginning of the transition. In order to further materialize this observation, experiments were re-conducted using fixed patterns replica featuring the quasi-periodic geometry of polygonal patterns under dry conditions. Such technique allowed full control of the patterns geometry and speed at all time, therefore working as a reference to the real experiment performed under wet conditions. The experiments revealed an interesting basic idea that was useful when addressing the significant difference in behavior associated with the real patterns transitions. The second part of the transition process included the frequency mode-locking ratio of subsequent patterns. Dealing with the first part of the transition process as being a biperiodic state or phase, in order for such state to lose its stability, a synchronization event has to occur [4]. This synchronization has been confirmed to occur when the frequency ratio of the parent pattern $N$ to the subsequent pattern $N+1$ rationalized at $(N-1) / N$ value [15]. The frequency mode-locking phenomenon was found to be respected even at relatively higher viscosity fluids when mixing glycerol with water.

## Acknowledgment

This work is supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

## References

1. H. Ait Abderrahmane. Two Cases of Symmetry Breaking of Free Surface Flows. Ph.D. Thesis, Concordia University, Montreal, Canada, 2008.
2. H. Ait Abderrahmane, M.H.K. Siddiqui and G.H. Vatistas. Transition between Kelvin's equilibria. Phys. Rev. E 80, 066305, 2009.
3. H. Ait Abderrahmane, K. Siddiqui, G.H. Vatistas, M. Fayed and H.D. Ng. Symmetrization of polygonal hollow-core vortex through beat-wave resonance. Phys. Rev. E 83, 056319, 2011.
4. P. Bergé, Y. Pomeau and C. Vidal. Order Within Chaos Hermann, Paris, 1984.
5. J.M. Chomaz, M. Rabaud, C. Basdevant and Y. Couder. Experimental and numerical investigation of a forced circular shear layer J. Fluid Mech. 187:115-140, 1988.
6. Dow Chemical Company (1995-2010).
7. M.P. Escudier. Observations of the flow produced in a cylindrical container by a rotating endwall. Experiments in Fluids 2:189-196, 1984.
8. R.C. Gonzalez, R.E. Woods and S.L. Eddins. Digital Image Processing Using MATLAB $7^{\text {th }}$ edition. Prentice Hall, 2004.
9. R. Hide and C.W. Titman. Detached shear layers in a rotating fluid J. Fluid Mech. 29:39-60, 1967.
10. T.R.N. Jansson, M.P. Haspang, K.H. Jensen, P. Hersen and T. Bohr. Polygons on a rotating fluid surface Phys. Rev. Lett. 96: 174502, 2006.
11. H. Niino and N. Misawa. An experimental and theoretical study of barotropic instability J. Atmospheric Sciences 41:1992-2011, 1984.
12. S. Poncet and M.P. Chauve. Shear-layer instability in a rotating system J. Flow Visualization and Image Processing 14:85-105, 2007.
13. M. Rabaud and Y. Couder. Instability of an annular shear layer. J. Fluid Mech. 136:291-319, 1983.
14. G.H. Vatistas. A note on liquid vortex sloshing and Kelvin's equilibria J. Fluid Mech. 217:241-248, 1990.
15. G.H. Vatistas, H. Ait Abderrahmane and M.H.K. Siddiqui. Experimental confirmation of Kelvin's equilibria Phys. Rev. Lett. 100, 174503, 2008.
16. H.U. Vogel. Experimentelle Ergebnisse über die laminare Strömung in einem zylindrischen Gehäuse mit darin rotierender Scheibe MPI für Strömungsforschung Bericht 6, 1968.

# Compound Structures of Six New Chaotic Attractors in a Modified Jerk Model using Sinh ${ }^{-1}$ Nonlinearity 

Banlue Srisuchinwong, Teerachot Siriburanon, and Teera Nontapradit

Sirindhorn International Institute of Technology, Thammasat University, Thailand (E-mail: banlue@siit.tu.ac.th)


#### Abstract

Six new chaotic attractors in a modified single-coefficient jerk model are presented based on Sinh ${ }^{-1}$ nonlinearity and six new values of the single coefficient. Compound structures of such chaotic attractors are revealed through the use of a control parameter $n$ of a half-image operation. For an appropriate value of $n$, a positive $n$ isolates a right half-image attractor, whereas a negative $n$ isolates a left half-image attractor. Both images can be merged together as a compound structure.


Keywords: Chaos, Jerk moel, Compound structure, Sinh ${ }^{-1}$ nonlinearity,

## 1 Introduction

Studies of chaotic behavior in nonlinear systems and circuits have attracted great attention due to a variety of applications in science and technology. The best known electronic circuit exhibiting chaos is the Chua's circuit [1], [2], based on three first-order ordinary differential equations (ODEs). In contrast, Sprott [3] has alternatively proposed chaotic circuits based on a single thirdorder ODE in a "Jerk Model" with a single coefficient $K$, as shown in (1). The nonlinear component $G(x)$ has been summarized in (2).

$$
\begin{gather*}
\frac{d^{3} x}{d t^{3}}+K \frac{d^{2} x}{d t^{2}}+\frac{d x}{d t}=G(x)  \tag{1}\\
G(x)=\left\{\begin{array}{lll}
|x|-2 & ; K=0.6[3] \\
-6 \max (x, 0)+0.5 & ; K=0.6[3] \\
-4.5 \operatorname{sgn}(x)+1.2 x & ; K=0.6[3] \\
2 \operatorname{sgn}(x)-1.2 x & ; K=0.6[3] \\
2 \tanh (x)-x & ; K=0.19[5] \\
3 \sin (x)-x & ; K=1 & {[6]} \\
6 \tan ^{-1}(x)-2 x & ; K=1 & {[6]} \\
7 \tanh (x)-2 x & ; K=1 & {[6]} \\
\operatorname{sgn}(x)-2 x & ; K=1 & {[6]}
\end{array}\right\} \tag{2}
\end{gather*}
$$

The term "jerk" comes from the fact that in a mechanical system in which $x$ is
the displacement, successive time derivatives of $x$ are velocity, acceleration, and jerk [4]. Some of these jerk models have been implemented using currentfeedback op-amps [7], [8]. In addition, other values of the single coefficient $K$ have been presented using either $\mathrm{Tan}^{-1}$ nonlinearity [9] or $\mathrm{Sin}^{-1}$ nonlinearity [10]. Recently, compound structures of chaotic attractors based on the singlecoefficient jerk model [9], [10] and others [11], [12], [13] have been reported.

In this paper, six new chaotic attractors in a modified single-coefficient jerk model are proposed based on $\operatorname{Sinh}^{-1}$ nonlinearity and six new values of the single coefficient. In addition, compound structures of the six chaotic attractors are also demonstrated.

## 2 A Modified Single-Coefficient Jerk Model

Figure 1 shows an implementation of the jerk model described in (1) and (2) where the single coefficient $K$ and the nonlinearity $G(x)$ can now be modified. By using new nonlinearity $\operatorname{Sinh}^{-1}(x)$, six new values of $K$ and $G(x)$ are proposed, as shown in (3).


Fig. 1. A Single-Coefficient Jerk Model.

$$
G(x)=\left\{\begin{array}{l}
G_{1}(x)=+4 \operatorname{Sinh}^{-1}(x)-x ; K=0.24  \tag{3}\\
G_{2}(x)=+5 \operatorname{Sinh}^{-1}(x)-x ; K=0.26 \\
G_{3}(x)=+6 \operatorname{Sinh}^{-1}(x)-x ; K=0.32 \\
G_{4}(x)=-4 \operatorname{Sinh}^{-1}(x)+x ; K=0.19 \\
G_{5}(x)=-5 \operatorname{Sinh}^{-1}(x)+x ; K=0.21 \\
G_{6}(x)=-6 \operatorname{Sinh}^{-1}(x)+x ; K=0.23
\end{array}\right\}
$$

## 3 Compound Structures of New Chaotic Attractors

In the new systems shown in (1) and (3), compound structures [9]-[13] can be demonstrated using a half-image operation to obtain either a left- or a right-half-image attractor, each of which can be merged together as a compound structure. Such a half-image attractor can be revealed through the use of a control parameter $n$ of the form:

$$
\begin{equation*}
\frac{d^{3} x}{d t^{3}}+K \frac{d^{2} x}{d t^{2}}+\frac{d x}{d t}=G(x)+n \tag{4}
\end{equation*}
$$

For an appropriate value of $n$, a negative $n$ results in an isolation of the left-half image of the original attractor, whereas a positive $n$ results in an isolation of the right-half image of the original attractor.

## 4. Numerical Results

### 4.1. New Chaotic Attractors

By using the single-coefficient jerk model described in (1) and (3) based on Fig. 1, six new chaotic attractors are displayed either on an $X-Y$ phase plane as shown in Figs. 2(A1), 2(B1), 2(C1), 2(D1), 2(E1) and 2(F1), or on an $X$-Z phase plane as shown in Figs. 2(A2), 2(B2), 2(C2), 2(D2), 2(E2) and 2(F2), respectively. It appears that the new attractors exhibit complex behaviors of chaotic dynamics.

### 4.2. Compound Structures

For the nonlinearity $G_{1}(x)$ and $n=-0.09$, a left-half image of the original attractor shown in Figs. 2(A1) and 2(A2) can be isolated as illustrated in Figs. 2(A3) and 2(A4), respectively. In contrast, for $n=0.09$, another right-half image of Figs. 2(A1) and 2(A2) can be isolated as illustrated in Figs. 2(A5) and 2(A6), respectively. For the nonlinearity $G_{2}(x)$ and $n=-0.45$, a left-half image of the original attractor shown in Figs. 2(B1) and 2(B2) can be isolated as illustrated in Figs. 2(B3) and 2(B4), respectively. In contrast, for $n=0.45$, another right-half image of Figs. 2(B1) and 2(B2) can be isolated as illustrated in Figs. 2(B5) and 2(B6), respectively.

For the nonlinearity $G_{3}(x)$ and $n=-0.78$, a left-half image of the original attractor shown in Figs. 2(C1) and 2(C2) can be isolated as illustrated in Figs. 2(C3) and 2(C4), respectively. In contrast, for $n=0.78$, another right-half image of Figs. 2(C1) and 2(C2) can be isolated as illustrated in Figs. 2(C5) and 2(C6), respectively. For the nonlinearity $G_{4}(x)$ and $n=-0.15$, a left-half image of the original attractor shown in Figs. 2(D1) and 2(D2) can be isolated as illustrated in Figs. 2(D3) and 2(D4), respectively. In contrast, for $n=0.15$, another righthalf image of Figs. 2(D1) and 2(D2) can be isolated as illustrated in Figs. 2(D5) and 2(D6), respectively.

|  | New Original Attractors | Left-half-image Attractors | Right-half-image Attractors |
| :---: | :---: | :---: | :---: |
| $\begin{aligned} & \mathrm{G}_{1}(\mathrm{x})=4 \sinh ^{-1}(\mathrm{x})-\mathrm{x} \\ &(\mathrm{~K}=0.24) \end{aligned}$ |  | $\mathrm{n}=-\mathbf{0 . 0 9}$  <br> (A3) | $\mathrm{n}=0.09$  <br> (A5) |
|  |  <br> (A2) |  <br> (A4) |  <br> (A6) |
| $\begin{gathered} \mathrm{G}_{2}(\mathrm{x})=5 \sinh ^{-1}(\mathrm{x})-\mathrm{x} \\ (\mathrm{~K}=0.26) \end{gathered}$ |  <br> (B1) | $\mathrm{n}=-\mathbf{0 . 4 5}$  <br> (B3) | $\mathrm{n}=\mathbf{0 . 4 5}$ <br> (B5) |
|  |  <br> (B2) |  <br> (B4) |  <br> (B6) |

Figure 2. Six new chaotic attractors and the corresponding left- and right-halfimage attractors.

For the nonlinearity $G_{5}(x)$ and $n=-0.21$, a left-half image of the original attractor shown in Figs. 2(E1) and 2(E2) can be isolated as illustrated in Figs. 2(E3) and 2(E4), respectively. In contrast, for $n=0.21$, another right-half image of Figs. 2(E1) and 2(E2) can be isolated as illustrated in Figs. 2(E5) and 2(E6), respectively. Finally, for the nonlinearity $G_{6}(x)$ and $n=-0.29$, a left-half image of the original attractor shown in Figs. 2(F1) and 2(F2) can be isolated as illustrated in Figs. 2(F3) and 2(F4), respectively. In contrast, for $n=0.30$,
another right-half image of Figs. 2(F1) and 2(F2) can be isolated as illustrated in Figs. 2(F5) and 2(F6), respectively.


Figure 2. Six new chaotic attractors and the corresponding left- and right-halfimage attractors (continued).

278 B. Srisuchinwong et al.


Figure 2. Six new chaotic attractors and the corresponding left- and right-halfimage attractors (continued).

## 5. Conclusions

Six new chaotic attractors in a modified single-coefficient jerk model have been presented through the use of Sinh ${ }^{-1}$ nonlinearity and six new values of the single coefficient. In addition, a compound structure of each chaotic attractor has been demonstrated using a half-image operation to obtain either a left- or a right-halfimage attractor, each of which can be merged together as a compound structure.

## Acknowledgment

This work was supported by Telecommunications Research and Industrial Development Institute (TRIDI), NBTC, Thailand under Grant TARG 2553/002, and the National Research University Project of Thailand, Office of Higher Education Commission.

## References

1. T. Matsumoto. A Chaotic Attractor from Chua's Circuit, IEEE Trans. Circuits and Systems, vol. CAS-31, no.12, 1055-1058, 1984.
2. B. Srisuchinwong, and W. San-Um, A Chua's Chaotic Oscillator Based on a Coarsely Cubic-Like CMOS Resistor, Proceedings of Asia-Pacific Conference on Communications -APCC 2007, 47-49, 2007.
3. J. C. Sprott, A New Class of Chaotic Circuit, Physics Letters A, vol. 266, 19-23, 2000.
4. S. H. Schot, The time rate of change of acceleration, Am. J. Phys, vol. 46, 1090-1094, 1978.
5. J. C. Sprott, Simple chaotic systems and circuits, Am. J. Phys. 68(8), 758-763, 2000.
6. J. C. Sprott, Simplifications of the Lorenz Attractor, Nonlinear Dynamics, Psychology and Life Sciences, 13(3), 271-278, 2009.
7. B. Srisuchinwong, and C.-H. Liou, High-Frequency Implementation of Sprott's Chaotic Oscillators Using Current-Feedback Op Amps, Proceedings of International Symposium on Signals Circuits and Systems - ISSCS 2007, 1:97-99, 2007.
8. B. Srisuchinwong, C.-H. Liou, and T. Klongkumnuankan, Prediction of Dominant Frequencies of CFOA-Based Sprott's Sinusoidal and Chaotic Oscillators, Topics on Chaotic Systems: Selected Papers from CHAOS 2008 International Conference, edited by Christos H. Skiadas, Ioannis Dimotikalis and Charilaos Skiadas, Word Scientific Publishing, 331-337, 2009.
9. T. Siriburanon, B. Srisuchinwong, T. Nontapradit, Compound Structures of Six New Chaotic Attractors in a Solely-Single-Coefficient Jerk Model with Arctangent Nonlinearity, Proceedings of the $22^{\text {nd }}$ Chinese Control and Decision Conference (CCDC 2010), Xuzhou, China, 985-990, 26-28 May, 2010.
10. B. Srisuchinwong, T. Siriburanon, T. Nontapradit, Compound Structures of Five New Chaotic Attractors Based on $\operatorname{Sin}^{-1}(\mathrm{x})$ in a Modified Sprott's Jerk Model, Proceedings of the 2010 Electrical Engineering/Electronics, Computer, Telecommunications and Information Technology International Conference (ECTICON 2010), Chiang Mai, Thailand, 16-19, 19-21 May, 2010.
11. B. Munmuangsaen and B. Srisuchinwong, A New Five-Term Simple Chaotic Attractor, Physics Letters A, 373, 4038-4043, 2009.
12. B. Munmuangsaen and B. Srisuchinwong, A New Lorenz-Like Chaotic Attractor and Its Synchronization, Proceedings of Chinese Control and Decision Conference (CCDC 2009), Guilin, China, 1508-1512, 17-19 June, 2009.
13. J. Lu, G. Chen, S. Zhang, The Compound Structure of a new Chaotic Attractor, Chaos, Solitons and Fractals, 14, 669-672, 2002.

# Geodesics Revisited 

Pavel Pokorny<br>Prague Institute of Chemical Technology, Prague, Czech Republic<br>(E-mail: pavel.pokorny@vscht.cz)


#### Abstract

Metric tensor and Christoffel symbols are revised and the equation of geodesic is derived from two possible definitions: based on zero tangent acceleration and on minimal length. Geodesics on a torus are shown to split into two distinct classes. Dynamical systems approach is used to investigate these two classes. Application of geodesics in optics and in mechanics are given. Keywords: dynamical system, geodetic, torus.


## 1 Introduction

In Euclidean space a segment of a line is the shortest connection of two given points. The segment has also the property that a point moving along the segment with velocity of constant magnitude has zero acceleration. A geodesic (a geodesic curve) is a generalization of the term segment for spaces that are not Euclidean. An example of such a space is a two dimensional surface in a three dimensional Euclidean space.

Historically perhaps the oldest example of such a surface is the sphere (the surface of a ball) because our space for living was limited to the surface of the Earth for a long time. Here comes the origin of the word geodesic. Geo- is the first part of compound words meaning the Earth.

These curved surfaces can be studied in two ways. Either as subsets of an Euclidean space of higher dimension, or as independent curved spaces without any reference to a higher dimensional Euclidean space. The intrinsic geometry of such a curved space can be described by certain matrix depending on the point in the space. This matrix function is called the metric tensor. A space with a constant metric tensor is called a flat (Euclidean) space, while a space with a non-constant metric tensor is called a curved space.

In chapter 2 a metric tensor is introduced and its examples for a sphere and for a torus are given.

In chapter 3 the geodesic is defined as the curve such that a point moving along the curve with the velocity of constant magnitude (i.e. the velocity can change its direction but not its magnitude) has the acceleration vector perpendicular to the given surface, i.e. the acceleration component tangent to the given surface is zero. Such a motion can be expressed by a non-scientific
expression "follow your nose". Given this condition the equation of geodesic is derived. It is a second order differential equation for the functions that parametrically describe the curve.

In chapter 4 alternatively the geodesic is defined as the shortest curve between two given points. Given this condition the same equation of geodesic is derived.

In chapter 5 we show that the magnitude of velocity remains constant for the solution of the equation of geodesic.

In chapter 6 the first integral (i.e. a constant function of state variables) is derived for certain simplified cases.

In chapter 7 the equation of geodesic is applied for a sphere. We show that geodesics on a sphere are the great circles i.e. the circles with the center in the center of the sphere.

In chapter 8 geodesics on a torus are investigated. The geodesics on a torus fall into two classes. Roughly speaking, one class contains geodesics that remain mainly on the outer part of the torus (see Fig. 6) while the other class contains geodesics that wind around the tube of the torus along a spiral (see Fig. 7). Another difference between these two classes is that a geodesic in the first class is either closed or it has self-intersections, while a geodesic in the other class is either closed or it has no self-intersections.

In chapter 9 a physical application of geodesics is given, namely the propagation of light in optically non-homogeneous medium, i.e. where the index of refraction depends on the point in the space. We find the metric tensor appropriate for investigation of the shape of the light ray and the Snell law of refraction is derived from the equation of geodesic found in chapter 3 and 4. This example is interesting in that it is convenient to replace the three dimensional Euclidean space by a curved space described by a non-constant metric tensor for the study of the propagation of light (or in general any wave with varying speed).

In chapter 10 the results of chapter 9 are applied for the study of the shape of the path that brings a mass point from a given initial point to another given point in the shortest possible time (assuming a homogeneous gravitational field). This path is called a brachistochrone and we show that it can be found as a geodesic with appropriate metric tensor.

There are many more examples of geodesics. Besides being an interesting mathematical question of its own, they have many physical and technological application. Spanning from general relativity to cases that seem to have nothing in common with mathematics or physics such as winding a ribbon round handlebars of a bicycle or dressing an injured knee.

Geodesics are sometimes illustrated as the equilibrium position of a spring on a slippery surface. This is a good example for convex parts of the surface; near concave parts of the surface a real spring would go through the air while the geodesic must stay in the given surface. To see this, imagine a thin rubber around an apple. There is a little pit near the stem of the apple. The rubber crosses this pit through the air which the geodesic is not allowed to do.

## 2 Metric tensor

Consider a $M$-dimensional manifold embedded into a $N$-dimensional Euclidean space with parametric equations

$$
y=r(x)
$$

where $r: R^{M} \rightarrow R^{N}$.
E.g. a sphere with unit radius can be given by

$$
\begin{align*}
& y_{1}=r_{1}\left(x_{1}, x_{2}\right)=r_{1}(\vartheta, \varphi)=\sin \vartheta \cos \varphi  \tag{1}\\
& y_{2}=r_{2}\left(x_{1}, x_{2}\right)=r_{2}(\vartheta, \varphi)=\sin \vartheta \sin \varphi \\
& y_{3}=r_{3}\left(x_{1}, x_{2}\right)=r_{3}(\vartheta, \varphi)=\cos \vartheta
\end{align*}
$$

and a torus by

$$
\begin{align*}
& y_{1}=r_{1}\left(x_{1}, x_{2}\right)=r_{1}(u, v)=(a+\cos u) \cos v  \tag{2}\\
& y_{2}=r_{2}\left(x_{1}, x_{2}\right)=r_{2}(u, v)=(a+\cos u) \sin v \\
& y_{3}=r_{3}\left(x_{1}, x_{2}\right)=r_{3}(u, v)=\sin u
\end{align*}
$$

where $a>1$ is the radius of the axis of the tube; the radius of the tube being 1 .
The comma before an index will denote the partial derivative with respect to the variable given by the index after the comma. Thus e.g. for $r_{k}$ (the $k$-th component of the vector $r$ ) its partial derivative is

$$
r_{k, i}=\frac{\partial r_{k}}{\partial x_{i}}
$$

Then the differential of $y$ is

$$
d y_{k}=r_{k, i} d x_{i}
$$

(we sum over each index appearing twice in a product) and the square of its norm is

$$
\|d y\|^{2}=d y_{k} d y_{k}=r_{k, i} r_{k, j} d x_{i} d x_{j}=g_{i j} d x_{i} d x_{j}
$$

where

$$
\begin{equation*}
g_{i j}=r_{k, i} r_{k, j} \tag{3}
\end{equation*}
$$

are the components of the metric tensor.
E.g. for a sphere putting (1) into (3) gives

$$
g=\left(\begin{array}{cc}
1 & 0 \\
0 & \sin ^{2} \vartheta
\end{array}\right)
$$

and for a torus putting (2) into (3) gives

$$
g=\left(\begin{array}{cc}
1 & 0 \\
0 & (a+\cos u)^{2}
\end{array}\right)
$$

Later we will need another relation between $g$ and $r$. We can differentiate

$$
g_{i j}(x)=r_{m, i}(x) r_{m, j}(x)
$$

with respect to $x_{k}$ to yield

$$
g_{i j, k}=r_{m, i k} r_{m, j}+r_{m, i} r_{m, j k}
$$

We add and subtract to this equation its two cyclic permutations

$$
\begin{aligned}
g_{j k, i} & =r_{m, i j} r_{m, k}+r_{m, j} r_{m, i k} \\
g_{k i, j} & =r_{m, k j} r_{m, i}+r_{m, k} r_{m, i j}
\end{aligned}
$$

and we get

$$
g_{i j, k}+g_{j k, i}-g_{k i, j}=2 r_{m, i k} r_{m, j}
$$

## 3 Geodesic as the curve with zero tangent acceleration

Consider a curve

$$
\alpha=\alpha(t)=r(x(t))
$$

where $\alpha: I \rightarrow R^{N}$ is a sufficiently smooth function, $I$ is the interval $I=\left[t_{1}, t_{2}\right]$. When we call $t$ the time, we can call

$$
\dot{\alpha}_{k}(t)=r_{k, i} \dot{x}_{i}(t)
$$

the velocity and

$$
\ddot{\alpha}_{k}(t)=r_{k, i j} \dot{x}_{i} \dot{x}_{j}+r_{k, i} \ddot{x}_{i} .
$$

the acceleration. We want to find the shape of the curve, so that the acceleration has zero projection to the plane tangent to the given surface

$$
\begin{gathered}
\ddot{\alpha}_{k} r_{k, n}=0 \quad \text { for } n=1, \ldots, M \\
\left(r_{k, i j} \dot{x}_{i} \dot{x}_{j}+r_{k, i} \ddot{x}_{i}\right) r_{k, n}=0 \\
r_{k, i} r_{k, n} \ddot{x}_{i}+r_{k, i j} r_{k, n} \dot{x}_{i} \dot{x}_{j}=0 \\
g_{i n} \ddot{x}_{i}+\frac{1}{2}\left(g_{i n, j}+g_{n j, i}-g_{j i, n}\right) \dot{x}_{i} \dot{x}_{j}=0 .
\end{gathered}
$$

It is convenient to denote $g^{n m}$ the element of the inverse matrix to the matrix with elements $g_{i n}$ (i.e. $g_{i n} g^{n m}=\delta_{i m}$ is the element of the unit matrix). Then

$$
\ddot{x}_{m}+\frac{1}{2} g^{n m}\left(g_{i n, j}+g_{n j, i}-g_{j i, n}\right) \dot{x}_{i} \dot{x}_{j}=0
$$

and finally

$$
\begin{equation*}
\ddot{x}_{m}+\Gamma_{i j}^{m} \dot{x}_{i} \dot{x}_{j}=0 \tag{4}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma_{i j}^{m}=\frac{1}{2} g^{n m}\left(g_{i n, j}+g_{n j, i}-g_{j i, n}\right) \tag{5}
\end{equation*}
$$

is called the Christoffel symbol.
From (3) it follows that the metric tensor $g$ is symmetric, i.e.

$$
g_{i j}=g_{j i}
$$

and as a result the Christoffel symbol is also symmetric

$$
\Gamma_{i j}^{m}=\Gamma_{j i}^{m}
$$

We call (4) the equation of geodesic. In this equation the properties of the surface appear only through the metric tensor $g$ and its derivatives (via the Christoffel symbol $\Gamma_{i j}^{m}$ ). This allows us to work in the $M$-dimensional space with the metric $g$ without any reference to the $N$-dimensional Euclidean space.

If the metric tensor $g$ as a function of the point in the space is constant, its derivatives vanish and so do all the Christoffel symbols. The equation of geodesic is then

$$
\ddot{x}_{m}=0
$$

and the geodesic is the straight line in this special case.

## 4 Geodesic as the shortest curve

Consider a curve

$$
x=\alpha(t)
$$

where $\alpha: I \rightarrow R^{n}$ is a sufficiently smooth function and the interval $I$ is $I=$ $\left[t_{1}, t_{2}\right]$.

If $g$ is the metric tensor, then the magnitude of the velocity of a point traveling along the curve $\alpha$ is

$$
v_{\alpha}(t)=\sqrt{g_{i j}(\alpha(t)) \dot{\alpha}_{i}(t) \dot{\alpha}_{j}(t)}
$$

Let us denote

$$
V\left(x_{1}, \ldots, x_{n}, \dot{x}_{1}, \ldots, \dot{x}_{n}\right)=\sqrt{g_{i j}\left(x_{1}, \ldots, x_{n}\right) \dot{x}_{i} \dot{x}_{j}}
$$

in short

$$
\begin{equation*}
V(x, \dot{x})=\sqrt{g_{i j}(x) \dot{x}_{i} \dot{x}_{j}} \tag{6}
\end{equation*}
$$

Similarly we will write $\alpha$ instead of $\alpha_{1}, \ldots, \alpha_{n}$ and $\dot{\alpha}$ instead of $\dot{\alpha}_{1}, \ldots, \dot{\alpha}_{n}$.
Then the length of the curve $\alpha$ is

$$
L(\alpha)=\int_{t_{1}}^{t_{2}} v_{\alpha}(t) d t=\int_{t_{1}}^{t_{2}} V(\alpha(t), \dot{\alpha}(t)) d t
$$

For $\epsilon \in R$ and $\beta: I \rightarrow R^{n}$ such that $\beta\left(t_{1}\right)=\beta\left(t_{2}\right)=0$ we denote

$$
\tilde{L}(\epsilon)=L(\alpha+\epsilon \beta)=\int_{t_{1}}^{t_{2}} V(\alpha(t)+\epsilon \beta(t), \dot{\alpha}(t)+\epsilon \dot{\beta}(t)) d t
$$

and

$$
\tilde{L}^{\prime}(\epsilon)=\frac{d \tilde{L}}{d \epsilon}
$$

We want

$$
\tilde{L}^{\prime}(0)=0
$$

meaning that a small change in the shape of the curve does not make it shorter. Thus

$$
0=\int_{t_{1}}^{t_{2}}\left(V_{x_{i}} \beta_{i}+V_{\dot{x}_{i}} \dot{\beta}_{i}\right) d t
$$

We integrate by parts and we use the assumption $\beta\left(t_{1}\right)=\beta\left(t_{2}\right)=0$ (meaning that the start point and the end point of the curve are fixed) to get

$$
0=\int_{t_{1}}^{t_{2}}\left(V_{x_{i}} \beta_{i}-\dot{V}_{\dot{x}_{i}} \beta_{i}\right) d t=\int_{t_{1}}^{t_{2}}\left(V_{x_{i}}-\dot{V}_{\dot{x}_{i}}\right) \beta_{i} d t
$$

This must hold for arbitrary functions $\beta_{i}$, thus the bracket must vanish

$$
V_{x_{i}}-\dot{V}_{\dot{x}_{i}}=0 \quad \text { for } i=1, \ldots, n
$$

thus

$$
\begin{equation*}
V_{x_{i}}-\left(V_{\dot{x}_{i} x_{k}} \dot{x}_{k}+V_{\dot{x}_{i} \dot{x}_{k}} \ddot{x}_{k}\right)=0 \tag{7}
\end{equation*}
$$

To get a unique solution we add the assumption of constant magnitude of the velocity

$$
\begin{equation*}
\dot{V}=0 \tag{8}
\end{equation*}
$$

thus

$$
\begin{equation*}
V_{x_{k}} \dot{x}_{k}+V_{\dot{x}_{k}} \ddot{x}_{k}=0 . \tag{9}
\end{equation*}
$$

When putting (6) into (7) and using (9) the same equation of geodesic (4) is derived. To do it by hand it is convenient to introduce $W$ by

$$
\begin{equation*}
W(x, \dot{x})=g_{i j}(x) \dot{x}_{i} \dot{x}_{j} \tag{10}
\end{equation*}
$$

i.e.

$$
\begin{equation*}
V=\sqrt{W} \tag{11}
\end{equation*}
$$

Putting (11) into (7) yields

$$
\begin{equation*}
2 W W_{\dot{x}_{m} x_{s}} \dot{x}_{s}+2 W W_{\dot{x}_{m} \dot{x}_{r}} \ddot{x}_{r}-W_{\dot{x}_{m}}\left(W_{x_{s}} \dot{x}_{s}+W_{\dot{x}_{r}} \ddot{x}_{r}\right)=2 W W_{x_{m}} \tag{12}
\end{equation*}
$$

where the bracket vanishes because of (9) and (11). When we substitute $W$ from (10) into (12) we get again the equation of geodesic

$$
\ddot{x}_{m}+\Gamma_{i j}^{m} \dot{x}_{i} \dot{x}_{j}=0
$$

where

$$
\Gamma_{i j}^{m}=\frac{1}{2} g^{n m}\left(g_{i n, j}+g_{n j, i}-g_{j i, n}\right)
$$

## 5 Constant magnitude of velocity

We have used the assumption of constant magnitude of velocity (8) to simplify (12). It is not clear, however, whether the solution of the resulting equation of geodesic (4) still satisfies the condition (8). We show it does. Let us write the equation of geodesic (4) as a system of first order ODE's

$$
\begin{align*}
\dot{x}_{m} & =X_{m}  \tag{13}\\
\dot{X}_{m} & =-\Gamma_{i j}^{m} X_{i} X_{j}
\end{align*}
$$

The condition of constant square of the magnitude of velocity

$$
W=g_{i j}(x) \dot{x}_{i} \dot{x}_{j}=g_{i j}(x) X_{i} X_{j}=\text { const. }
$$

is the equation of a hyper-surface in the state space (of twice the dimension). It is easy to show that the vector field $\boldsymbol{f}$ of (13) is orthogonal to the gradient of $W$

$$
\begin{gathered}
\boldsymbol{f} \cdot \nabla W=\binom{X_{m}}{-\Gamma_{i j}^{m} X_{i} X_{j}} \cdot\binom{g_{i j, m} X_{i} X_{j}}{2 g_{m k} X_{k}}= \\
=g_{i j, m} X_{i} X_{j} X_{m}-2 g_{m k} \Gamma_{i j}^{m} X_{i} X_{j} X_{k}= \\
=g_{i j, m} X_{i} X_{j} X_{m}-g_{m k} g^{n m}\left(g_{i n, j}+g_{n j, i}-g_{j i, n}\right) X_{i} X_{j} X_{k}= \\
=g_{i j, m} X_{i} X_{j} X_{m}-\left(g_{i n, j}+g_{n j, i}-g_{j i, n}\right) X_{i} X_{j} X_{n}=0 .
\end{gathered}
$$

Thus the square of the magnitude of velocity is constant and so is the magnitude itself.

## 6 First integral

In this chapter we rewrite the equation of geodesic (4) for special cases and then we find its first integral.

Assuming dimension 2, denoting $x_{1}=x, x_{2}=y$ and assuming a diagonal metric tensor $g$, i.e. $g_{12}(x, y)=g_{21}(x, y)=0$ we arrive at

$$
\begin{align*}
& \ddot{x}+\frac{g_{11,1}}{2 g_{11}} \dot{x} \dot{x}+\frac{g_{11,2}}{g_{11}} \dot{x} \dot{y}-\frac{g_{22,1}}{2 g_{11}} \dot{y} \dot{y}=0  \tag{14}\\
& \ddot{y}-\frac{g_{11,2}}{2 g_{22}} \dot{x} \dot{x}+\frac{g_{22,1}}{g_{22}} \dot{x} \dot{y}+\frac{g_{22,2}}{2 g_{22}} \dot{y} \dot{y}=0 . \tag{15}
\end{align*}
$$

Further, assuming that $g$ depends on $x$ only and not on $y$, formally written ${ }_{, 2}=0$ meaning $\frac{\partial}{\partial y}=0$ or $g_{11,2}=g_{22,2}=0$ we get even more simple equations

$$
\begin{align*}
\ddot{x}+\frac{g_{11,1}}{2 g_{11}} \dot{x} \dot{x}-\frac{g_{22,1}}{2 g_{11}} \dot{y} \dot{y} & =0  \tag{16}\\
\ddot{y}+\frac{g_{22,1}}{g_{22}} \dot{x} \dot{y} & =0 \tag{17}
\end{align*}
$$

We will use this result (assuming further $g_{11}=g_{22}$ ) in chapter 9 .

In chapters 7 and 8 we will work with the metric tensor where one of its diagonal elements is constant. Assuming $g_{11}(x, y)=1$ allows us to simplify the equation of geodetic even more

$$
\begin{align*}
& \ddot{x}-\frac{g_{22,1}}{2} \dot{y} \dot{y}=0  \tag{18}\\
& \ddot{y}+\frac{g_{22,1}}{g_{22}} \dot{x} \dot{y}=0 \tag{19}
\end{align*}
$$

Now we can find the first integral of this system of ODE's. Multiplying (18) by $\dot{y}$ and multiplying (19) by $\dot{x}$ and subtracting the second equation from the first one we get

$$
\ddot{x} \dot{y}-\ddot{y} \dot{x}-\frac{g_{22,1}}{2} \dot{y}^{3}-\frac{g_{22,1}}{g_{22}} \dot{x}^{2} \dot{y}=0 .
$$

When multiplying this equation by $\frac{2 \dot{x}}{g_{22}^{2} \dot{y}^{3}}$ we get (after simple manipulation)

$$
\frac{d}{d t}\left(\frac{1}{g_{22}(x(t))^{2}}\left(\frac{\dot{x}}{\dot{y}}\right)^{2}+\frac{1}{g_{22}(x(t))}\right)=0
$$

which is equivalent to

$$
\begin{equation*}
\frac{1}{g_{22}^{2}}\left(\frac{d x}{d y}\right)^{2}+\frac{1}{g_{22}}=\text { const. } \tag{20}
\end{equation*}
$$

We will use this first integral of the equation of geodesic in chapters dealing with the sphere and with the torus.

## $7 \quad$ Geodesics on a sphere

Putting $x_{1}=\vartheta, x_{2}=\varphi$ and

$$
g=\left(\begin{array}{cc}
1 & 0 \\
0 & \sin ^{2} \vartheta
\end{array}\right)
$$

into the equation of geodesic (4) yields

$$
\begin{aligned}
\ddot{\vartheta}-\sin \vartheta \cos \vartheta \dot{\varphi}^{2} & =0 \\
\ddot{\varphi}+2 \cot \vartheta \dot{\varphi} \dot{\vartheta} & =0
\end{aligned}
$$

Its first integral (20) is

$$
\begin{equation*}
\frac{1}{\sin ^{4} \vartheta}\left(\frac{d \vartheta}{d \varphi}\right)^{2}+\frac{1}{\sin ^{2} \vartheta}=\frac{1}{\sin ^{2} \vartheta_{0}} \tag{21}
\end{equation*}
$$

where $\vartheta_{0}=\min (\vartheta)$ and $\varphi_{0}$ are the coordinates of "the north most" point of the curve.

This is the equation of a circle lying in the plane going through the origin. Such a plane has the equation

$$
x \cdot x_{P}=0
$$

where

$$
x=\left(\begin{array}{c}
\sin \vartheta \cos \varphi \\
\sin \vartheta \sin \varphi \\
\cos \vartheta
\end{array}\right)
$$

and

$$
x_{P}=\left(\begin{array}{c}
\sin \vartheta_{P} \cos \varphi_{P} \\
\sin \vartheta_{P} \sin \varphi_{P} \\
\cos \vartheta_{P}
\end{array}\right)
$$

where

$$
\varphi_{P}=\varphi_{0}+\pi, \quad \vartheta_{P}=\frac{\pi}{2}-\vartheta_{0}
$$

are coordinates of the normal vector to the plane. After some manipulation

$$
\varphi=\varphi_{0}+\arccos \left(\tan \vartheta_{0} \cot \vartheta\right)
$$

Differentiating gives

$$
\frac{d \varphi}{d \vartheta}=\frac{1}{\sin ^{2} \vartheta} \frac{\tan \vartheta_{0}}{\sqrt{1-\left(\tan \vartheta_{0} \cot \vartheta\right)^{2}}}
$$

and

$$
\frac{1}{\sin ^{4} \vartheta}\left(\frac{d \vartheta}{d \varphi}\right)^{2}=\frac{1}{\sin ^{2} \vartheta_{0}}-\frac{1}{\sin ^{2} \vartheta}
$$

which agrees with(21).

## 8 Geodesics on a torus



Fig. 1. Geodesic (24).


Fig. 2. Geodesic (25).


Fig. 3. Geodesic (26).

Substituting $x_{1}=u, x_{2}=v$, and the metric tensor

$$
g=\left(\begin{array}{lc}
1 & 0  \tag{22}\\
0 & (a+\cos u)^{2}
\end{array}\right)
$$

into the equation of geodesic (4) gives

$$
\begin{aligned}
\ddot{u}+(a+\cos u) \sin u \dot{v}^{2} & =0 \\
\ddot{v}-2 \frac{\sin u}{a+\cos u} \dot{u} \dot{v} & =0
\end{aligned}
$$



Fig. 4. The graph of the function (29) shows the minimum for $(u, d)=(0,0)$ and a saddle for $(u, d)=(\pi, 0)$.

This system of two differential equations of the second order can be written as a system of four equations of the first order

$$
\begin{align*}
\dot{u} & =U  \tag{23}\\
\dot{U} & =-(a+\cos u) V^{2} \sin u \\
\dot{v} & =V \\
\dot{V} & =2 \frac{\sin u}{a+\cos u} U V
\end{align*}
$$

The 4 dimensional state space $(u, U, v, V)$ of this system is divided by the hyper-plane $V=0$ into two half-spaces. The hyper-plane $V=0$ contains the solution

$$
\begin{align*}
u & =k_{1} t+k_{2}  \tag{24}\\
U & =k_{1} \\
v & =k_{3} \\
V & =0
\end{align*}
$$

where $k_{1}, k_{2}, k_{3} \in R$.
Corresponding to each solution in one half-space there is one solution in the other half-space. These two solutions are symmetrical with respect to the hyper-plane $V=0$. The theorem of the uniqueness of solution implies that $V(t)$ is either always positive or always zero or always negative. Meaning the solutions neither cross nor touch the hyper-plane $V=0$. Thus we can limit our attention to solutions satisfying $V(t)=\dot{v}(t)>0$.


Fig. 5. The contour-lines of the function (29) are closed curves near a minimum. A separatrix (shown in red) going from the saddle separates a region with closed contour-lines from the region with non-closed contour-lines. Closed contour-lines correspond to geodesics that remain mainly in the outer part of the torus (see Fig. 6). Non-closed contour-lines correspond to geodesics that wind around the tube of the torus (see Fig. 7).

Among these solutions there are two special solutions satisfying $\dot{u}(t)=0$, namely

$$
\begin{align*}
u & =0  \tag{25}\\
U & =0 \\
v & =k_{1} t+k_{2} \\
V & =k_{1},
\end{align*}
$$

and

$$
\begin{align*}
u & =\pi  \tag{26}\\
U & =0 \\
v & =k_{1} t+k_{2} \\
V & =k_{1} .
\end{align*}
$$

The behavior of nearby trajectories can be studied by linear expansion. The Jacobi matrix of partial derivatives of the system (23) evaluated on the
trajectory (25) has two zero eigenvalues and two purely imaginary complex conjugate eigenvalues

$$
\begin{equation*}
\lambda_{3,4}= \pm i k_{1} \sqrt{a+1} \tag{27}
\end{equation*}
$$

This means it acts like a center; nearby trajectories rotate around it in the $u$ - $U$ plane spanned by the corresponding eigenvectors.

The Jacobi matrix of partial derivatives of the system (23) evaluated on the trajectory (26) has two zero eigenvalues and two real eigenvalues with opposite signs

$$
\lambda_{3,4}= \pm k_{1} \sqrt{a-1}
$$

Thus the trajectory (26) is a saddle with one stable and one unstable directions in the $u-U$ plane. The saddle itself is not a stationary point but rather a closed trajectory. In fact, there are no stationary points, the velocity has a constant magnitude.

The first integral (20) for torus is

$$
\begin{equation*}
\frac{1}{(a+\cos u)^{4}}\left(\frac{d u}{d v}\right)^{2}+\frac{1}{(a+\cos u)^{2}}=\text { const. } \tag{28}
\end{equation*}
$$

Thus geodesics on a torus can be described by contour-lines of the function

$$
\begin{equation*}
f(u, d)=\frac{1}{(a+\cos u)^{4}}(d)^{2}+\frac{1}{(a+\cos u)^{2}} \tag{29}
\end{equation*}
$$

The graph of the function (29) is shown in Fig. 4 and its contour-lines are shown in Fig. 5. For fixed $u$ it is a quadratic function of $d$ with positive coefficients, thus having a minimum. For $d=0$ it is a periodic function of $u$ with the period $2 \pi$ having a minimum for $u=0$ and a maximum for $u=\pi$. As a function of two variables $f$ has a minimum in $u=0, d=0$ and a saddle in $u=\pi, d=0$. The contour-lines near a minimum are closed curves, the contour-line leaving a saddle is a separatrix separating a region with closed contour-lines near a minimum (with bounded values of $u$ ) and a region with contour-lines which are neither closed not bounded (here $u(t)$ is a monotone function). This is shown in Fig. 4 depicting the graph of the function (29) and in Fig. 5 with its contour-lines.

We can find the equation of the separatrix. From

$$
\frac{d u}{d v}=0 \quad \text { for } u=\pi
$$

it follows

$$
\frac{1}{(a+\cos u)^{4}}\left(\frac{d u}{d v}\right)^{2}+\frac{1}{(a+\cos u)^{2}}=\frac{1}{(a-1)^{2}}
$$

Thus for the separatrix for $u=0$ it is

$$
\frac{d u}{d v}=2 \frac{a+1}{a-1} \sqrt{a} .
$$

The angle $\alpha_{C}$, formed by the critical geodesic and the plane $z=0$ in $u=0$ (i.e. on the outer edge of the torus) is

$$
\tan \alpha_{C}=\frac{d z}{d y}=\frac{1}{a+1} \frac{d u}{d v}=\frac{2 \sqrt{a}}{a-1}
$$



Fig. 6. If the angle $\alpha$, formed by the geodesic and the plane $z=0$, is less than a critical value $\alpha_{C}$, the geodesic remains mainly on the outer part of the torus. For this graph $\alpha=60^{\circ}, \alpha_{C} \doteq 64.6^{\circ}$ and $a=\frac{5}{2}$. Only a finite part of the geodesic is shown.
E.g. for $a=\frac{5}{2}$ the critical angle is $\alpha_{C} \doteq 64,6^{\circ}$. Fig. 6 shows an example of a geodesic for $\alpha<\alpha_{C}$ and Fig. 7 for $\alpha>\alpha_{C}$.


Fig. 7. If the angle $\alpha$ formed by the geodesic and the plane $z=0$ on the outer part of the torus is greater than the critical angle $\alpha_{C}$ then the geodesic winds around the tube of the torus. For this graph $\alpha=69^{\circ}, \alpha_{C} \doteq 64.6^{\circ}$ and $a=\frac{5}{2}$. Only a finite part of the geodesic is shown.

Are the geodesics on a torus closed curves? From (28) it follows that $u$ as a function of $v$ is periodic for small $u$ (i.e. for $u_{0}<\pi$ ). Let us denote its period $T$. If $T$ is a rational multiple of $2 \pi$ (as the increase of $v$ by $2 \pi$ corresponds to the same point) then the geodesic is a closed curve. The period $T$ can be evaluated as follows. From (28) we find

$$
d v=\frac{a+\cos u_{0}}{(a+\cos u) \sqrt{(a+\cos u)^{2}-\left(a+\cos u_{0}\right)^{2}}} d u
$$

and

$$
T=4 \int_{0}^{u_{0}} \frac{a+\cos u_{0}}{(a+\cos u) \sqrt{(a+\cos u)^{2}-\left(a+\cos u_{0}\right)^{2}}} d u
$$

It is sufficient to integrate over one quarter of the period because the function (29) is even in both $u$ and $d$.

The period $T$ is a continuous function of two variables: $a$ (the ratio of the radius of the axis of the tube of the torus and the radius of the tube of the torus) and $u_{0}$ (maximum of $u$ on the geodesic) thus

$$
T=T\left(a, u_{0}\right)
$$

When $a$ or $u_{0}$ is varied continuously then the ratio $\frac{T}{2 \pi}$ will achieve rational and irrational values and in every neighborhood of a closed geodesic there will be infinitely many non-closed ones and vice versa. Almost every geodesic will be non-closed.

It is possible to compute the period $T\left(a, u_{0}\right)$ for small amplitude $u_{0}$

$$
\lim _{u_{0} \rightarrow 0} T\left(a, u_{0}\right)=\frac{2 \pi}{\sqrt{a+1}}
$$

so that e.g. for $a=3$ the geodesic for small $u_{0}$ will almost close after two turns around the torus. This is in agreement with (27).

Our results in this chapter differ from the classical ones based on the intrinsic geometry of the torus $T^{2}=R^{2} / N$ inherited from the Euclidean plane $(u, v)$, where the geodesics are straight lines in the plane $(u, v)$ and thus having the constant slope. When wound on the torus a geodesic is either a closed curve or it fills densely the entire surface of the torus. We, however, assume the metric tensor (22) based on the geometry of the torus as embedded into a three dimensional Euclidean space. This non-constant metric tensor gives rise to two distinct classes of geodesic curves (cf. Fig. 6 and Fig. 7).

## 9 Geodesic as the light beam

If the optical index of refraction

$$
n=\frac{c}{v}
$$

where $c$ is the speed of light in vacuum and $v$ is the speed of light in the given medium, is independent of the point in space, then the light propagates along a straight line. If the index of refraction depends of the point in space

$$
n=n(x, y, z)
$$

refraction of light takes place. This is of fundamental importance for the human eye and for a large range of optical devices.

The light beam propagates along such a curve, to minimize the time necessary to reach a given point from another given point. This is called the Fermat principle. The element of time is

$$
d t=\frac{d l}{v}
$$

where $d l$ is the element of length. Then

$$
\begin{gathered}
c d t=\frac{c}{v} d l=n d l \\
(c d t)^{2}=n^{2}\left((d x)^{2}+(d y)^{2}+(d z)^{2}\right)
\end{gathered}
$$

meaning the shape of the beam is a geodesic in the space with metric

$$
g=\left(\begin{array}{ccc}
n^{2} & 0 & 0 \\
0 & n^{2} & 0 \\
0 & 0 & n^{2}
\end{array}\right)
$$

Let us consider a special case when the index of refraction depends on a single space variable (say, $y$ )

$$
n=n(y)
$$

Then it is sufficient to consider the shape of the beam in a plane. Using the equation of geodesic (4) for $x_{1}=x, x_{2}=y$ we get

$$
\begin{align*}
\ddot{x}+2 \frac{n^{\prime}}{n} \dot{x} \dot{y} & =0  \tag{30}\\
\ddot{y}+\frac{n^{\prime}}{n}(\dot{y} \dot{y}-\dot{x} \dot{x}) & =0 \tag{31}
\end{align*}
$$

Multiplying (30) by $\dot{y}$ and multiplying (31) by $\dot{x}$ and subtracting the first equation from the second one we get

$$
\ddot{y} \dot{x}-\ddot{x} \dot{y}-\frac{n^{\prime}}{n} \dot{x}\left(\dot{x}^{2}+\dot{y}^{2}\right)=0 .
$$

When multiplying this equation by $\frac{2 \dot{y} n^{2}}{\dot{x}^{3}}$ we get (after simple manipulation)

$$
\frac{d}{d t}\left(\frac{1}{n(y(t))^{2}}\left(\left(\frac{\dot{y}}{\dot{x}}\right)^{2}+1\right)\right)=0
$$

which is equivalent to

$$
\frac{1}{n^{2}}\left(\left(\frac{d y}{d x}\right)^{2}+1\right)=\text { const. }
$$

$$
\begin{aligned}
& \frac{1}{n^{2}} \frac{(d l)^{2}}{(d x)^{2}}=\text { const. } \\
& \frac{1}{n^{2}} \frac{1}{\sin ^{2} \alpha}=\text { const. }
\end{aligned}
$$

and finally

$$
\begin{equation*}
n \sin \alpha=\mathrm{const} \tag{32}
\end{equation*}
$$

where $\alpha$ is the angle formed by the beam and the normal vector to the plane of constant index of refraction.

A special case

$$
n(y)=\frac{1}{y}
$$

i.e.

$$
g=\left(\begin{array}{cc}
\frac{1}{y^{2}} & 0 \\
0 & \frac{1}{y^{2}}
\end{array}\right)
$$

for $y>0$ is called the Poincare metric. Then (32) gives

$$
\frac{1}{y} \sin \alpha=K
$$

Comparing with the equation of a circle with radius $R$

$$
\sin \alpha=\frac{y}{R}
$$

shows that the geodesics in Poincare metric are semicircles for $K=\frac{1}{R}>0$ and straight lines

$$
\alpha=0
$$

i.e.

$$
x=\text { const. }
$$

for $K=0$.

## 10 Brachistochrone

Brachi- is the first part of compound words meaning short and chronos means time. Brachistochrone is the name for the curve bringing a mass point from a given point to another given point in the shortest possible time (assuming homogeneous gravitational field). To find it we make use of the results from the previous chapter.

The conservation of mechanical energy

$$
\frac{1}{2} m v^{2}+m g h=m g h_{0}
$$

lets us introduce a quantity playing a similar role as the index of refraction for light

$$
n=\frac{c}{v}=\frac{\text { const }}{\sqrt{y_{0}-y}}
$$

Using the law of refraction

$$
n \sin \alpha=\mathrm{const}
$$

we get

$$
\frac{\sin \alpha}{\sqrt{y_{0}-y}}=\text { const. }
$$

When we describe the curve as the graph of a function

$$
y=y(x)
$$

we get the equation

$$
\left(1+y^{\prime 2}\right)\left(y_{0}-y\right)=\text { const. }
$$

It is easy to show that this is the cycloid. Starting with the parametric equation of cycloid

$$
\begin{aligned}
& x=R \omega t+R \cos \omega t \\
& y=R \sin \omega t
\end{aligned}
$$

and differentiating with respect to time

$$
\begin{aligned}
\dot{x} & =R \omega-R \omega \cos \omega t \\
\dot{y} & =R \omega \cos \omega t
\end{aligned}
$$

we find

$$
y^{\prime 2}=\left(\frac{\dot{y}}{\dot{x}}\right)^{2}=\frac{\cos ^{2} \omega t}{(1-\sin \omega t)^{2}}
$$

and

$$
\left(1+y^{\prime 2}\right)\left(y_{0}-y\right)=2 R=\text { const. }
$$

Meaning that the cycloid is also a geodesic with a suitable metric.

## References

1.Arnold V. I.: Ordinary Differential Equations. MIT Press 1998.
2.Horsky J. et al.: Mechanika ve fyzice. Academia Prague 2001.
3.Ljusternik L. A.: Kratcajsije linii. Gostechizdat Moscow 1955.

