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New vortex invariants in magneto-hydrodynamics and a related helicity theorem

Anatoliy K. Prykarpatsky[1] and Denis Blackmore[2]

[1] AGH University of Science and Technology, Krakow, Poland and
The Ivan Franko State Pedagogical University, Drohobych, Lviv region, Ukraine
(E-mail: pryk.anat@ua.fm)
[2] The NJIT, University Heights, Newark, 07102, NJ USA
(E-mail: deblac@m.njit.edu)

Long ago it was stated [7,5] that quantum vortices in superfluid helium can be studied either as open lines with their ends terminating on free surfaces of walls of the container or as closed curves. Nowadays the closed vortices are treated as topological objects equivalent to circles. The existence of structures such as knotted and linked vertex lines in the turbulent phase is almost obvious [12] and has forced researchers to develop new mathematical tools for their detailed investigation. In this proposed direction Z. Peradzyński [8] proved a new version of the Helicity theorem, based on differential-geometric methods applied to the description of the collective motion in the incompressible superfluid. The Peradzyński helicity theorem describes in a unique way, both the superfluid equations and the related helicity invariants, which are, in the conservative case, very important for studying the topological structure of vortices.

By reanalyzing the Peradzyński helicity theorem within the modern symplectic theory of differential-geometric structures on manifolds, we propose a new unified proof and give a magneto-hydrodynamic generalization of this theorem for the case of an incompressible superfluid flow. As a by-product, in the conservative case we construct a sequence of nontrivial helicity type conservation laws, which play a crucial role in studying the stability problem of a superfluid under suitable boundary conditions.

1 Symplectic and symmetry analysis

We consider a quasi-neutral superfluid contained in a domain $M \subset \mathbf{R}^3$ and interacting with a "frozen" magnetic field $B : M \longrightarrow \mathbf{E}^3$, where $\mathbf{E}^3 := (\mathbf{R}^3, <.,.>)$ is the standard three-dimensional Euclidean vector space with the scalar $<.,.>$ and vector "$\times$" products. The magnetic field is considered to be source-less and to satisfy the condition $B = \nabla \times A$, where $A : M \longrightarrow \mathbf{E}^3$ is

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some magnetic field potential. The corresponding electric field $E : M \longrightarrow \mathbf{E}^3$, related with the magnetic potential, satisfies the necessary superconductivity conditions

$$E + u \times B = 0, \qquad \partial E/\partial t = \nabla \times B, \tag{1}$$

where $u : M \longrightarrow T(M)$ is the superfluid velocity.

Let ∂M denote the boundary of the domain M. The boundary conditions $\langle n, u\rangle|_{\partial M} = 0$ and $\langle n, B\rangle|_{\partial M} = 0$ are imposed on the superfluid flow, where $n \in T^*(M)$ is the vector normal to the boundary ∂M, considered to be almost everywhere smooth.

Then adiabatic magneto-hydrodynamics (MHD) quasi-neutral superfluid motion can be described, using (1), by the following system of evolution equations:

$$\begin{gathered} \partial u/\partial t = -\langle u, \nabla\rangle u - \rho^{-1}\nabla P + \rho^{-1}(\nabla \times B) \times B, \\ \partial\rho/\partial t = -\langle \nabla, \rho u\rangle, \qquad \partial\eta/\partial t = -\langle u, \nabla\eta\rangle, \qquad \partial B/\partial t = \nabla \times (u \times B), \end{gathered} \tag{2}$$

where $\rho : M \longrightarrow \mathbf{R}_+$ is the superfluid density, $P : M \longrightarrow \mathbf{E}^3$ is the internal pressure and $\eta : M \longrightarrow \mathbf{R}$ is the specific superfluid entropy. The latter is related to the internal MHD superfluid specific energy function $e = e(\rho, \eta)$ owing to the first law of thermodynamics:

$$T\, d\eta = de(\rho, \eta) - P\rho^{-2} d\rho, \tag{3}$$

where $T = T(\rho, \eta)$ is the internal absolute temperature in the superfluid. The system of evolution equations (2) conserves the total energy

$$H := \int_M \left[\frac{1}{2\rho}|\mu|^2 + \rho e(\rho, \eta) + \frac{1}{2}|B|^2\right] d^3x, \tag{4}$$

called the Hamiltonian, since the dynamical system (2) is a Hamiltonian system on the functional manifold $\mathcal{M} := C^\infty(M; T^*(M) \times \mathbf{R}^2 \times \mathbf{E}^3)$ with respect to the following [4] Poisson bracket:

$$\begin{aligned} \{f, g\} := \textstyle\int_M \Big\{ &\langle \mu, [\tfrac{\delta f}{\delta\mu}, \tfrac{\delta g}{\delta\mu}]_c\rangle + \rho\left(\langle \tfrac{\delta g}{\delta\mu}, \nabla\tfrac{\delta f}{\delta\rho}\rangle - \langle \tfrac{\delta f}{\delta\mu}, \nabla\tfrac{\delta g}{\delta\rho}\rangle\right) \\ &+ \eta\langle \nabla, (\tfrac{\delta g}{\delta\mu}\tfrac{\delta f}{\delta\eta} - \tfrac{\delta f}{\delta\mu}\tfrac{\delta g}{\delta\eta})\rangle + \langle B, [\tfrac{\delta g}{\delta\mu}, \tfrac{\delta f}{\delta B}]_c\rangle \\ &+ \langle \tfrac{\delta f}{\delta B}, \langle B, \nabla\rangle\tfrac{\delta g}{\delta\mu}\rangle - \langle \tfrac{\delta g}{\delta B}, \langle B, \nabla\rangle\tfrac{\delta f}{\delta\mu}\rangle \Big\}\, dx, \end{aligned} \tag{5}$$

where we denoted by $\mu := \rho u \in T^*(M)$ the specific momentum of the superfluid motion and by $[.,.]_c$ the canonical Lie bracket of variational gradient vector fields:

$$[\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\mu}]_c := \langle \frac{\delta f}{\delta\mu}, \nabla\rangle\frac{\delta g}{\delta\mu} - \langle \frac{\delta g}{\delta\mu}, \nabla\rangle\frac{\delta f}{\delta\mu} \tag{6}$$

for any smooth functionals $f, g \in \mathcal{D}(M)$ on the functional space $\mathcal{M}$. Moreover, as was shown in [4], the Poisson bracket (5) is, in reality, the canonical Lie–Poisson bracket on the dual space to the Lie algebra $\mathcal{G}$ of the semidirect product of vector fields on M and the direct sum of functions, densities and differential one-forms on M. Namely, the specific momentum $\mu = \rho u \in T^*(M)$ is dual to vector fields, ρ is dual to functions, η is dual to densities and B is dual to the space of two-forms on M. Thus, the set of evolution equations (2) can be equivalently recast as follows:

$$\begin{aligned} \partial u/\partial t &= \{H, u\}, \qquad \partial \rho/\partial t = \{H, \rho\}, \\ \partial \eta/\partial t &= \{H, \eta\}, \qquad \partial B/\partial t = \{H, B\}. \end{aligned} \tag{7}$$

The Poisson bracket (5) can be rewritten for any $f, g \in \mathcal{D}(M)$ as

$$\{f, g\} = (Df, \vartheta\, Dg), \tag{8}$$

with $Df := \left(\frac{\delta f}{\delta \mu}, \frac{\delta f}{\delta \rho}, \frac{\delta f}{\delta \eta}, \frac{\delta f}{\delta B}\right) \in T^*(\mathcal{M})$ and $\vartheta : T^*(\mathcal{M}) \longrightarrow T(\mathcal{M})$, being the corresponding (modulo the Casimir functionals of bracket (5)) invertible [3] co-symplectic operator, satisfying the standard [10,2] properties

$$\vartheta^* = -\vartheta, \qquad \delta(\delta w, \wedge\, \vartheta^{-1} \delta w) = 0, \tag{9}$$

where the differential variation complex condition $\delta^2 = 0$ is assumed, the differential variation vector $\delta w := (\delta\mu, \delta\rho, \delta\eta, \delta B) \in T^*(\mathcal{M})$ and the symbol "$*$" denotes the conjugate mapping with respect to the standard bilinear convolution $(.,.)$ of the spaces $T^*(\mathcal{M})$ and $T(\mathcal{M})$. Note here that the second condition of (9) is equivalent [2,10] to the fact that the Poisson bracket (5) satisfies the Jacobi commutation condition. Thus, one can define the closed generalized variational differential two-form on $\mathcal{M}$

$$\omega^{(2)} := (\delta w, \wedge \vartheta^{-1}\, \delta w), \tag{10}$$

which provides the symplectic structure on the functional factor manifold $\mathcal{M}$ (modulo the Casimir functionals of bracket (5)). Owing now to the commutation property

$$[\partial/\partial t + L_u, L_v] = 0, \tag{11}$$

equivalent to the subgroup $\mathcal{D}_t$ and $\mathcal{D}_\tau$ commuting for any suitable $t, \tau \in \mathbf{R}$, from the invariance condition

$$\partial \rho/\partial \tau = 0, \tag{12}$$

we deduce that the quantities

$$\gamma_n := L_v^n \gamma \tag{13}$$

for all $n \in \mathbf{Z}_+$ are invariants of the MHD superfluid flow (2) if the density $\gamma \in \Lambda^3(M)$ is also an invariant on M.

We construct the following new functionals on the functional manifold $\mathcal{M}$

$$\tilde{H}_n := \int_M \tilde{\gamma}_n\, d^3x = \int_M \rho L_v^n (\rho^{-1} \langle B, A\rangle)\, d^3x \tag{14}$$

for all $n \in \mathbf{Z}_+$, which are invariants of our MHD superfluid dynamical system (2). In particular, when $n = 0$ we obtain the well-known [4] magnetic helicity invariant

$$\tilde{H}_0 = \int_M \langle A, \nabla \times A \rangle \, d^3x, \tag{15}$$

which exists independently of boundary conditions, imposed on the MHD superfluid flow equations (2).

The result obtained above can be formulated as the following theorem.

Theorem 1. *The functionals (14), where the Lie derivative L_v is taken along the magnetic vector field $v = \rho^{-1}B$, are global invariants of the system of compressible MHD superfluid and superconductive equations (2).*

Below we proceed to a symmetry analysis of the incompressible superfluid dynamical system and construct the related local and global new helicity invariants. The case of superfluid hydrodynamical flows [9] is of great interest for many applications owing to the very nontrivial dynamical properties of so-called vorticity structures appearing in the motion.

2 The incompressible superfluid: symmetry analysis and conservation laws

The helicity theorem result of [8], where the kinematic helicity invariant

$$H_0 := \int_M \langle u, \nabla \times u \rangle \, d^3x \tag{16}$$

was derived, employed differential-geometric tools in Minkowski space in the case of an incompressible superfluid in the absence of a magnetic field ($B = 0$). We shall now describe its general dynamical symmetry nature. The governing equations are

$$\partial u/\partial t = -\langle u, \nabla \rangle u + \rho^{-1}\nabla P, \qquad \partial \rho/\partial t + \langle u, \nabla \rho \rangle = 0, \qquad \langle \nabla, u \rangle = 0, \tag{17}$$

where the density conservation properties

$$(\partial/\partial t + L_u)\rho = 0, \qquad (\partial/\partial t + L_u)d^3x = 0 \tag{18}$$

hold for all suitable $t \in \mathbf{R}$. Define now the vorticity vector $\xi := \nabla \times u$ and find from (17) that it satisfies the vorticity flow equation

$$\partial \xi/\partial t = \nabla \times (u \times \xi). \tag{19}$$

Actually, the first equation of (17) can be rewritten as

$$\partial u/\partial t = u \times (\nabla \times u) - \rho^{-1}\nabla P - \frac{1}{2}\nabla |u|^2. \tag{20}$$

Then, applying the operation "$\nabla \times \cdot$" to (20), one easily obtains the vorticity equation (19). Moreover, equation (19) can be recast in the equivalent form

$$\partial \xi/\partial t + \langle u, \nabla \rangle \xi = \langle \xi, \nabla \rangle u, \tag{21}$$

which allows a new dynamical symmetry interpretation. Now, define $\beta^{(1)} \in \Lambda^1(M)$ as the one–form

$$\beta^{(1)} := \langle u, dx\rangle \tag{22}$$

and readily conclude that

$$(\partial/\partial t + L_u)\beta^{(1)} = -\rho^{-1}dP + \frac{1}{2}d|u|^2 = d(\rho^{-1}P + \frac{1}{2}|u|^2). \tag{23}$$

We have shown that the following generalized functionals

$$H_n := \int_M \rho L_v^n(u \times \xi)\; d^3x \tag{24}$$

for all $n \in \mathbf{Z}_+$are new helicity invariants for (17). Notice here that all of the constraints imposed above on the vorticity vector $\xi = \nabla \times u$ are automatically satisfied if the condition *supp* $\xi \cap \partial M = \emptyset$ holds. The result obtained can be summarized as follows.

Theorem 2. *Assume that an incompressible superfluid, governed by the set of equations (17) in a domain $M \subset \mathbf{E}^3$, possesses the vorticity vector $\xi = \nabla \times u$, which satisfies the boundary constraints $L^n_{\rho^{-1}\xi}\xi|_{\partial M}$ for all $n \in \mathbf{Z}_+$. Then all of the functionals (24) are generalized helicity invariants of (17).*

The results obtained above allow some interesting modifications. To present them in detail, observe that equality (23) can be rewritten as

$$(\partial/\partial t + L_u)\beta^{(1)} - dh = (\partial/\partial t + L_u)\tilde{\beta}^{(1)} = 0, \tag{25}$$

where, by definition,

$$h := \rho^{-1}P + \frac{1}{2}|u|^2, \qquad \tilde{\beta}^{(1)} := \langle u - \nabla\varphi, dx\rangle, \tag{26}$$

and the scalar function $\varphi : M \longrightarrow \mathbf{R}$ is chosen in such a way that

$$(\partial/\partial t + L_u)\varphi = \nabla h. \tag{27}$$

Then, obviously, one obtains the additional equation

$$(\partial/\partial t + L_u)d\tilde{\beta}^{(1)} = 0, \tag{28}$$

following from the commutation property $[d, \partial/\partial t + L_u] = 0$. Then, we see that the density $\tilde{\lambda} := \tilde{\beta}^{(1)} \wedge d\tilde{\beta}^{(1)} \in \Lambda^3(M)$ satisfies the condition

$$(\partial/\partial t + L_u)\tilde{\mu} = 0, \tag{29}$$

for all $t \in \mathbf{R}$. A similar result holds for densities $\tilde{\lambda}_n := L_v^n\tilde{\lambda} \in \Lambda^3(M)$, $n \in \mathbf{Z}_+$; namely,

$$(\partial/\partial t + L_u)\tilde{\lambda}_n = 0, \tag{30}$$

owing to the commutation property (11). Therefore, the following functionals on the corresponding functional manifold $\mathcal{M}$ are invariants of the superfluid flow (2):

$$\Upsilon_n := \int_M \tilde{\lambda}_n = \int_{D_t} \rho L^n_{\rho^{-1}\xi} \langle (u - \nabla\varphi), \xi \rangle \, d^3x \qquad (31)$$

for all $n \in \mathbf{Z}_+$ and an arbitrary domain $D_t \subset M$, independent of boundary conditions, imposed on the vorticity vector $\xi = \nabla \times u$ on ∂M. Notice here that only the invariants (31) strongly depend on the function $\varphi : M \longrightarrow \mathbf{R}$, implicitly depending on the velocity vector $u \in T(M)$. It should be mentioned here that the practical importance of the constructed invariants (31) remains to be fully clarified.

3 Conclusions

The symplectic and symmetry analysis of compressible MHD super-fluids developed above, appears to be an effective approach for constructing the related helicity type conservation laws, which are generally important for practical applications. In particular, these conserved quantities play a decisive role [4,1] when studying the stability of MHD superfluid flows under special boundary conditions. Some of the results in this direction can also be obtained making use of group-theoretical and topological tools developed in [1,13,11], where the importance of the basic group of diffeomorphisms $Diff(M)$ of a manifold $M \subset \mathbf{R}^3$ and its differential-geometric characteristics were shown in considerable detail.

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