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Asymmetry Coefficients and Chaos Determination. Application to the Henon - Heiles Hamiltonian

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Abstract: Recently, a simple, very fast and easy to compute qualitative indicator of the chaotic or ordered nature of orbits in dynamical systems was proposed by Waz et al (2009), the so-called "Asymmetry coefficients". The indicator has been obtained from an analysis of the statistical behavior of an ensemble derived from the time dependence of selected quantities characterizing the system's motion. It was found that for an ordered orbit the indicator converges to zero while for a chaotic orbit no sign of convergence can be observed. Using the Henon-Heiles Hamiltonian system and the Smaller Alignment Index method, in our paper we proposed a numerical criterion in order to quantify the results obtained by the "Asymmetry coefficients" method. This criterion helped us to define threshold values between regularity and chaoticity and to construct detailed phase-space portraits, where the ordered and chaotic regions are clearly distinguished. Additionally, exploiting the rapidity of the method, we showed how it can be used to identify "sticky" orbits or tiny regions of order and chaos.
Keywords: asymmetry coefficients, ordered and chaotic orbits, hamiltonian systems.

1. Introduction

A long-standing fundamental issue in nonlinear dynamics is to determine whether an orbit is regular or chaotic. This distinction is of great interest because in the case of regular orbits we have predictability in time whereas for chaotic orbits we are unable to predict the time evolution of the dynamical system after a short time period. There are many methods and indicators for chaotic motions. The well-known are the phase space method, the time series method, bifurcation diagram, the Poincare section of surface, Frequency-map analysis, Lyapunov characteristic exponents, and most recently the Fast Lyapunov indicator, the 0-1 test, the Dynamic Lyapunov indicator, and the Smaller alignment Index [1-6]. However, none of the methods has the merits to be beyond any doubt. Most of them, especially the so-called "traditional" tools, work hard in systems with many degrees of freedom, where phase space visualization is no longer easily accessible. The recent tools seem to be more efficient and faster than the older ones, but each of them has its weak points. This is the reason that motivates the researchers in the field to search better methods.

In 2009, Waz et al. proposed an alternative, very simple and related to the observational data, statistical indicator of chaos [7]. In their approach the values

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of a time dependent function describing the studied motion are recorded in a sequence of time intervals and each of these recordings are considered statistical distributions. Then, the "asymmetry coefficients" of these distributions are defined and their behavior for ordered and chaotic orbits is analyzed. Their qualitative indicator was applied only in the simple case of the damped driven pendulum. In present paper we have attempted to improve their work by proposing a numerical criterion associated to asymmetry coefficients, which helped us to reveal the detailed structure of the dynamics in the phase space of the Henon-Heiles Hamiltonian system.
The organization of rest of the paper is as follows. Section 2 contains that information strictly required for understanding the "Asymmetry coefficients" and SALI methods. All calculations and numerical results are given in Section 3. The final remarks and conclusions are presented in Section 4.

2. Description of methods

For the sake of completeness let us briefly recall the definition of the "Asymmetry coefficients" and of the "Smaller Alignment Index (SALI)" and their behavior for regular and chaotic orbits. The interested reader can consult [7, 8] to have a more detailed description of the methods.

2.1. Method of the Asymmetry Coefficients

Let $X(t)$ be a function characterizing the motion we are going to analyze. Usually, in practical applications, $X(t)$ is known as a part of the solution of a differential system of equations or from experimental measurements, so its values are given in a discrete set of points $\{X_i\}$. Let us define a time series $X_k(t) = \{X(t), t \in (T_0, T_{f\,k}) / \ k = 1, 2, ..., K\}$ with a fixed T_0 and $T_{f\,1} < T_{f\,2} < ... < T_{f\,K}$. The terms of the series are treated as statistical distributions. The starting time T_0 and the final one $T_{f\,K}$ denote the beginning and the end of the *k*- th distribution $X_k(t)$.
The asymmetry coefficients of the discrete *k*- th distribution X_k are defined as

$$A_q(k, N_k) = S(k, N_k) \cdot \sum_{i=1}^{N_k} \left(X_{t_i^k} + c \right) \left[\frac{t_i^k - M_1(k, N_k)}{\sqrt{M_2(k, N_k) - M_1^2(k, N_k)}} \right]^q$$

$$S(k, N_k) = \left[\sum_{i=1}^{N_k} \left(X_{t_i^k} + c \right) \right]^{-1} \qquad (1)$$

$$M_n(k, N_k) = S(k, N_k) \cdot \sum_{i=1}^{N_k} \left(X_{t_i^k} + c \right) \cdot \left(t_i^k \right)^n, \ n \in \{1,2\}$$

$q = 2j+1,\ j = 1, 2, 3, ...$ and c is a constant. N_k is the number of points in the k-th distribution, i.e. $t_i^k = \tau_i, i = 1, 2, ..., N_k, k = 1, 2, .., K$, with $t_1^k = T_0$, $t_{N_k}^k = T_{f\,k}$, $N_1 < N_2 < ... < N_K$. Since T_0 is the same for all k, the length of the k- th distribution is proportional to N_k.

Waz et al shown that the qualitative results are the same for all c chosen so $X_k(t) + c \geq 0$. Using the damped driven pendulum, they demonstrated that for a periodic motion the asymmetry coefficients approach zero while T_f approaches infinity. For a chaotic orbit no regular asymptotic behaviour was observed. It results a qualitative indicator regarding the nature of an orbit. We proceeded one step further by introducing a quantitative criterion. Calculating for about one thousand orbits the maximum value of $|A_q|, q = 3, 5, 7$ when $t \in [500\,s, 1000\,s]$, we proposed for every asymmetry coefficient a threshold value between regularity and chaoticity, as Section 3 will demonstrate.

2.2. Method of the Smaller Alignment Index

Consider a n- dimensional phase-space of a dynamical system and an orbit in that space. In order to determine if this orbit is ordered or chaotic we follow the evolution in time of two different initial deviation vectors $\xi_1(0), \xi_2(0)$. In every time step, we compute the parallel/ anti-parallel alignment index (ALI) , $d_-(t) = \left\| \frac{\xi_1(t)}{\|\xi_1(t)\|} - \frac{\xi_2(t)}{\|\xi_2(t)\|} \right\|$ and $d_+(t) = \left\| \frac{\xi_1(t)}{\|\xi_1(t)\|} + \frac{\xi_2(t)}{\|\xi_2(t)\|} \right\|$, where $\|\cdot\|$ denotes the Euclidean norm of a vector. The Smaller Alignment Index (SALI) is defined as the minimum value of the above alignment indices at any point in time

$$SALI(t) = \min(d_-(t), d_+(t)) \tag{2}$$

Skokos shows that the two deviation vectors tend to coincide or become opposite for chaotic orbits, i.e. the SALI tends to zero. For ordered orbits, which lie on a torus, the two deviation vectors eventually become tangent to the torus, but in general converge to different directions, so the SALI does not tend to zero. Its values fluctuate around a positive value.

3. Numerical results

We consider the two degrees of freedom Henon-Heiles Hamiltonian

$$H_2(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3} y^3 \tag{3}$$

where x, y, and p_x, p_y are the coordinate and conjugate moments respectively. The equations of motion derived from the Hamiltonian are

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x \quad , \quad \dot{y} = \frac{\partial H}{\partial p_y} = p_y \tag{4a}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -x - 2xy \quad , \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -y - x^2 + y^2 \tag{4b}$$

and yields solutions (orbits) of the system evolving in a four dimensional phase space. In our study we keep the value of the Hamiltonian fixed at $H_2 = 0.125$.
We consider first two representative orbits: an ordered (quasi-periodic) orbit with initial conditions $(x, y, p_x, p_y) = (0.0, 0.55, 0.2417, 0.0)$ and a chaotic orbit with initial conditions $(x, y, p_x, p_y) = (0.0, -0.016, 0.49974, 0.0)$.
Figure 1a shows the Poincare surface of section (PSS) of the two orbits defined by $x = 0, p_x \geq 0$. The points of the ordered orbit (blue points) form a set of smooth curves while the points of the chaotic orbit (red points) appear randomly scattered. The $\log_{10}(SALI)$ of the ordered orbit (blue line in Figure 1b) fluctuates around 0.05, indicating the regular character of the orbit, while the $\log_{10}(SALI)$ of the chaotic orbit (red line in Figure 1b) falls abruptly reaching the limit of the accuracy of the computer precision (10^{-16}) after about 1700 time units.

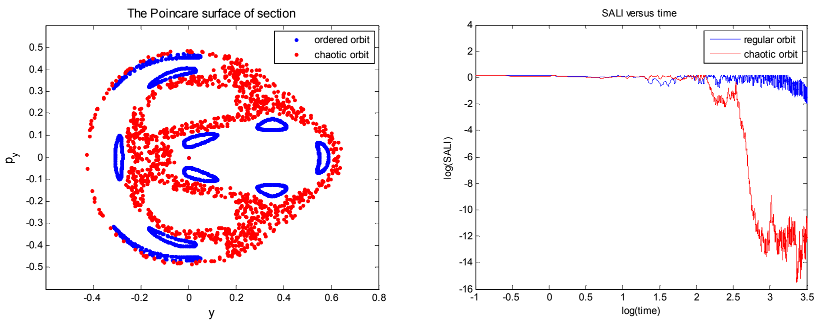

Fig.1. a) The PSS of an ordered orbit (blue points) and a chaotic orbit (red points); b) The time evolution of the SALI for the same orbits

The calculations of the asymmetry coefficients have been performed in equidistant points of the time interval $t \in [0s, 4000s]$. The origin of each distribution corresponds to the initial time $T_0 = 0$ whereas the final points of the distributions have been selected as $T_{f_k} = 0.2\,k, k = 1,2,..,20000$. The time step on each interval was equal to 0.02 s and c was taken as $-\min X_k(t)$. In addition, $X(t) = x(t)$.

Figure 2 depicts the asymmetry coefficients $A_q, q = 3,5,7$ as function of time. For the periodic orbit (blue lines) the coefficients A_q converge to 0, after a short transition period (about 300s). A irregular behaviour of A_q could be seen for the chaotic orbit (red lines). As it was proved in [8], the qualitative results are the same for all c that satisfy the condition $X_k(t) + c \geq 0$, for all t, and for any other component of the dynamical system (here, y, p_x or p_y).

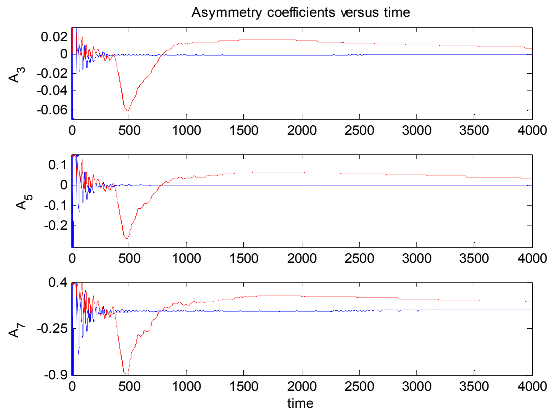

Fig. 2. The asymmetry coefficients $A_q, q = 3,5,7$ for the ordered orbit (blue lines) and for the chaotic orbit (red lines) discussed in Figure 1

In order to present the effectiveness of the quantitative indicator proposed in Section 2 (the maximum value of $\left|A_q\right|, q = 3, 5, 7$ when $t \in [500\, s, 1000\, s]$, hereafter noted by $\max\left|A_q\right|$) in detecting regions of chaos and order we computed it for a large grid of equally distributed initial conditions on the axis of PSS (y, p_y) of the Henon-Heiles system. To do this, we chose 440 initial conditions on the line $p_y = 0$ of the PSS, between $y = -0.43$ and $y = 0.67$ with step $\Delta y = 0.0025$, and 400 initial conditions on the semi-line $y = 0, p_y > 0$ (because of symmetry) of the PSS, between $p_y = 0$ and $p_y = 0.5$ with step $\Delta p_y = 0.00125$. Figure 3 shows the SALI values for these orbits. The running time for every orbit was T = 1,000 time units. We assigned a coloured circle to every individual initial condition according to the value of the SALI: if it was smaller than 10^{-8} the circle was coloured red (the orbit is chaotic beyond any doubt). If SALI $\in \left[10^{-8}, 10^{-4}\right)$ the circle was coloured yellow (the orbit is probably "sticky" chaotic) and finally, if SALI $\in \left[10^{-4}, 2\right)$ it was coloured bleu (the orbit is ordered). To clear up the nature of the orbits having SALI $\in \left[10^{-8}, 10^{-4}\right)$ and to verify if the running time T = 1,000 time units is sufficient for asymmetry coefficients to reveal the type of the orbits we computed the $\max\left|A_q\right|, q = 3,5,7$ for T = 1,000 and T = 4,000, respectively. The results for $\max\left|A_7\right|$ only are presented in Figure 4 (for semi-line $y = 0, p_y > 0$) and Figure 5 (for line $p_y = 0$).

There are some observations that are worth mentioning. Firstly, the CPU time needed to obtain the results plotted in Figure 3 was twenty times greater than for the results depicted in Figures 4a and 5a. Secondly, comparing Figure 3 with Figures 4b and 5b a similitude between them is easy to observe. In fact, every orbit with $\max\left|A_7\right| \geq 0.1$ has $SALI \leq 10^{-4}$ (therefore is chaotic) and all orbits having $\max\left|A_7\right| < 0.1$ are characterized by $SALI > 10^{-4}$ (they are ordered). Finally, we point out that a too short running time (here, T = 1,000) might give erroneous results concerning the "sticky" orbits.

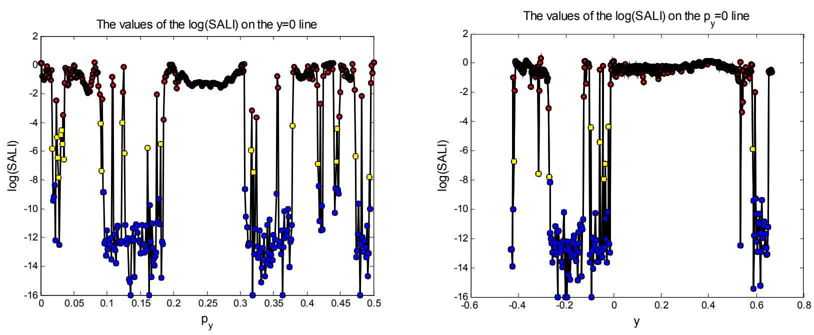

Fig. 3. The SALI values for initial conditions chosen on the semi-line $y = 0, p_y > 0$ (left panel) and on the line $p_y = 0$ (right panel)

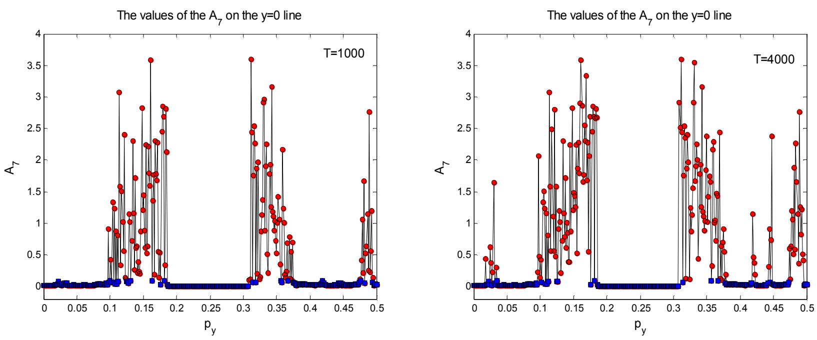

Fig. 4. The $\max\left|A_7\right|$ values for initial conditions chosen on the semi-line $y = 0, p_y > 0$ (T=1,000 - left panel; T=4,000 - right panel)

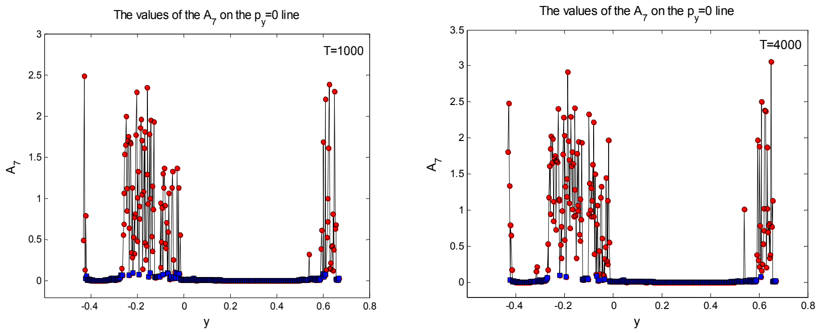

Fig. 5. The $\max|A_7|$ values for initial conditions chosen on the line $p_y = 0$ (T=1,000 - left panel; T=4,000 - right panel)

The same remarks are valid for the others asymmetry coefficients, A_3 and A_5. We propose as threshold values between regularity and chaoticity the value 0.005 for $\max|A_3|$ and 0.025 for $\max|A_5|$.

Let us now return to the "sticky" orbits that make the difference between the two panels of Figures 4 and 5. In order to illustrate the capability of the asymmetry coefficients to identify these kinds of orbits we considered a set of three orbits with very closely initial conditions on the axis $p_y = 0$ and computed the coefficient A_7 for T=12,000 time units. Figures 6 and 7 present our findings. When T=4,000 time units, one can see that the PSSs of these orbits are practically indistinguishable and indicate ordered orbits. The first visible deviations from these smooth curves appeared for $T \cong 5{,}000$ time units, as Figure 7 shown. When T=12,000 time units two of these orbits clearly entered in the chaotic sea, while the third remained ordered.

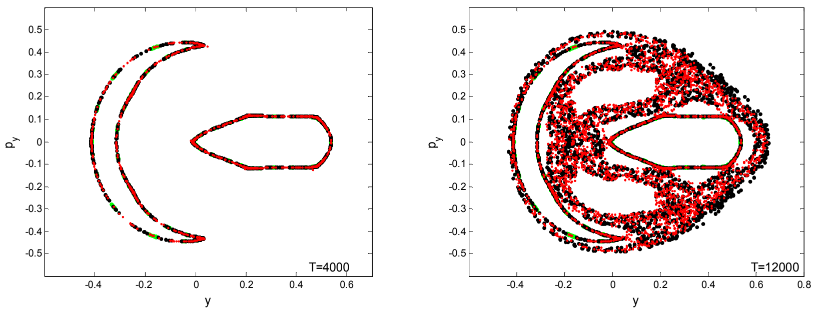

Fig. 6. The PSS of an ordered orbit (green points) and two "sticky" orbits (black and red points); T=4,000 – left panel, T=12,000 – right panel

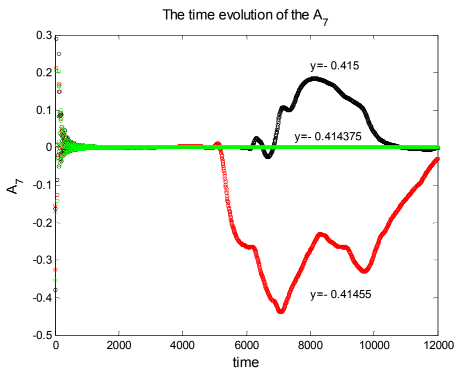

Fig. 7. The asymmetry coefficient A_7 for the ordered orbit (green line) and for the "sticky" orbits (black and red lines) discussed in Figure 6

3. Conclusions

In this paper we have illustrated the capability of the "Asymmetry coefficients" method in distinguishing between order and chaos in Henon-Heiles Hamiltonian system. Besides the fact that our calculations have validated the qualitative results obtained by Waz et al, we proposed a numerical criterion in order to quantify these results. Exploiting the rapidity of the method, we constructed detailed phase-space portraits and defined threshold values between regularity and chaoticity. Additionally, we showed how it can be used to identify "sticky" orbits or tiny regions of order and chaos.

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