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Dynamics of multiple pendula without gravity

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Abstract. We present a class of planar multiple pendula consisting of mathematical pendula and spring pendula in the absence of gravity. Among them there are systems with one fixed suspension point as well as freely floating joined masses. All these systems depend on parameters (masses, arms lengths), and possess circular symmetry $\mathbb{S}^1$. We illustrate the complicated behaviour of their trajectories using Poincaré sections. For some of them we prove their non-integrability analysing properties of the differential Galois group of variational equations along certain particular solutions of the systems.

Keywords: Hamiltonian systems, Multiple pendula, Integrability, Non-integrability, Poincaré sections, Morales-Ramis theory, Differential Galois theory.

1 Introduction

The complicated behaviour of various pendula is well known but still fascinating, see e.g. books [2,3] and references therein as well as also many movies on youtube portal. However, it seems that the problem of the integrability of these systems did not attract sufficient attention. According to our knowledge, the last found integrable case is the swinging Atwood's machine without massive pulleys [1] for appropriate values of parameters. Integrability analysis for such systems is difficult because they depend on many parameters: masses m_i, lengths of arms a_i, Young modulus of the springs k_i and unstretched lengths of the springs.

In a case when the considered system has two degrees of freedom one can obtain many interesting information about their behaviour making Poincaré cross-sections for fixed values of the parameters.

However, for finding new integrable cases one needs a strong tool to distinguish values of parameters for which the system is suspected to be integrable. Recently such effective and strong tool, the so-called *Morales-Ramis* theory [5] has appeared. It is based on analysis of differential Galois group of variational equations

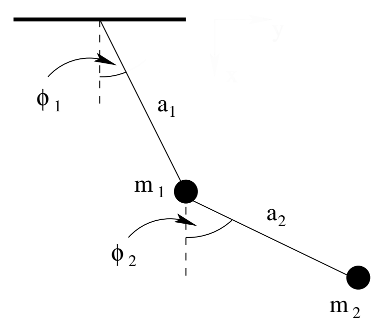

Fig. 1. Simple double pendulum.

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obtained by linearisation of equations of motion along a non-equilibrium particular solution. The main theorem of this theory states that if the considered system is integrable in the Liouville sense, then the identity component of the differential Galois group of the variational equations is Abelian. For a precise definition of the differential Galois group and differential Galois theory, see, e.g. [6].

The idea of this work arose from an analysis of double pendulum, see Fig. 1. Its configuration space is $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, and local coordinates are (ϕ_1, ϕ_2) mod 2π. A double pendulum in a constant gravity field has regular as well as chaotic trajectories. However, a proof of its non-integrability for all values of parameters is still missing. Only partial results are known, e.g., for small ratio of pendulums masses one can prove the non-integrability by means of Melnikov method [4]. On the other hand, a double pendulum without gravity is integrable. It has $\mathbb{S}^1$ symmetry, and the Lagrange function depends on difference of angles only. Introducing new variables $\theta_1 = \phi_1$ and $\theta_2 = \phi_2 - \phi_1$, we note that θ_1 is cyclic variable, and the corresponding momentum is a missing first integral.

The above example suggests that it is reasonable to look for new integrable systems among planar multiple-pendula in the absence of gravity when the $\mathbb{S}^1$ symmetry is present. Solutions of such systems give geodesic flows on product of $\mathbb{S}^1$, or products of $\mathbb{S}^1$ with $\mathbb{R}^1$. For an analysis of such systems we propose to use a combination of numerical and analytical methods. From the one side, Poincaré section give quickly insight into the dynamics. On the other hand, analytical methods allow to prove strictly the non-integrability.

In this paper we consider: two joined pendula from which one is a spring pendulum, two spring pendula on a massless rod, triple flail pendulum and triple bar pendulum. All these systems possess suspension points. One can also detach from the suspension point each of these systems. In particular, one can consider freely moving chain of masses (detached multiple simple pendula), and free flail pendulum. We illustrate the behaviour of these systems on Poicaré sections, and, for some of them, we prove their non-integrability. For the double spring pendulum the proof will be described in details. For others the main steps of the proofs are similar.

In order to apply the Morales-Ramis method we need an effective tool which allows to determine the differential Galois group of linear equations. For considered systems variational equations have two-dimensional subsystems of normal variational equations. They can be transformed into equivalent second order equations with rational coefficients.

For such equations there exists an algorithm, the so-called the Kovacic algorithm [7], determining its differential Galois groups effectively.

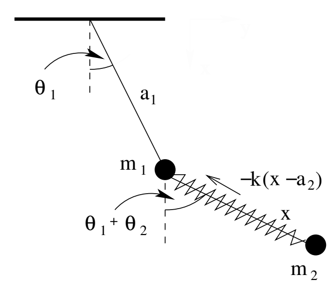


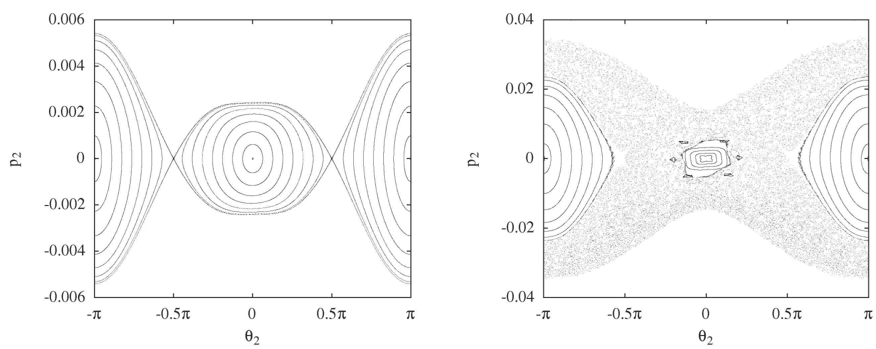
Fig. 2. Double spring pendulum.

2 Double spring pendulum

The geometry of this system is shown in Fig. 2. The mass m_2 is attached to m_1 on a spring with Young modulus k. System has $\mathbb{S}^1$ symmetry, and θ_1 is a cyclic coordinate.

The corresponding momentum p_1 is a first integral. The reduced system has two degrees of freedom with coordinates (θ_2, x), and momenta (p_2, p_3). It depends on parameter $c = p_1$.

The Poincaré cross sections of the reduced system shown in Fig. 3 suggest that the system is not integrable. The main problem is to prove that in fact



(a) E=0.00005 (b) E=0.001

Fig. 3. The Poincaré sections for double spring pendulum. Parameters: $m_1 = m_2 = a_1 = a_2 = 1, k = 0.1$, $p_1 = c = 0$ cross-plain $x = 1$.

the system is not integrable for a wide range of the parameters. In Appendix we prove the following theorem.

Theorem 1. *Assume that* $a_1 m_1 m_2 \neq 0$*, and* $c = 0$*. Then the reduced system descended from double spring pendulum is non-integrable in the class of meromorphic functions of coordinates and momenta.*

3 Two rigid spring pendula

The geometry of the system is shown in Fig. 4. On a massless rod fixed at one end we have two masses joined by a spring; the first mass is joined to fixed point by another spring. As generalised coordinates angle θ and distances x_1 and x_2 are used. Coordinate θ is a cyclic variable and one can consider the reduced system depending on parameter c - value of momentum p_3 corresponding to θ. The Poincaré cross sections in Fig. 5 and in Fig. 6 show the complexity of the system. We are able to prove non-integrability only under assumption $k_2 = 0$.

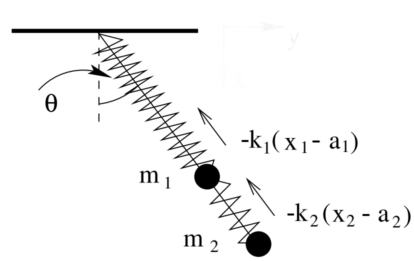


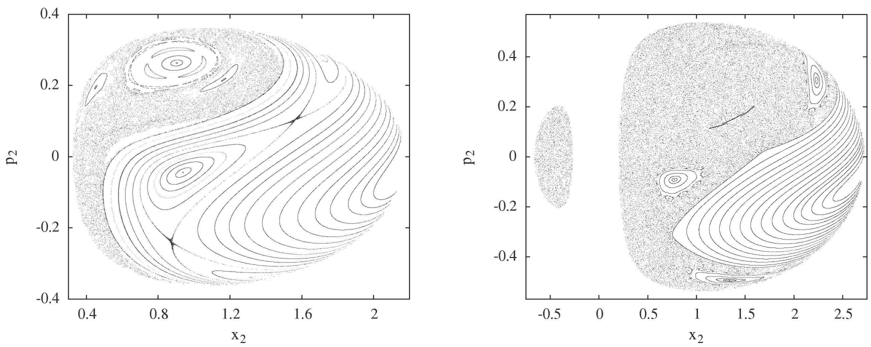
Fig. 4. Two rigid spring pendula.

Theorem 2. *If* $m_1 m_2 k_1 c \neq 0$*, and* $k_2 = 0$*, then the reduced two rigid spring pendula system is non-integrable in the class of meromorphic functions of coordinates and momenta.*

Moreover, we can identify two integrable cases. For $c = 0$ the reduced Hamilton equations become linear equations with constant coefficients and they are solvable. For $k_1 = k_2 = 0$ original Hamiltonian simplifies to

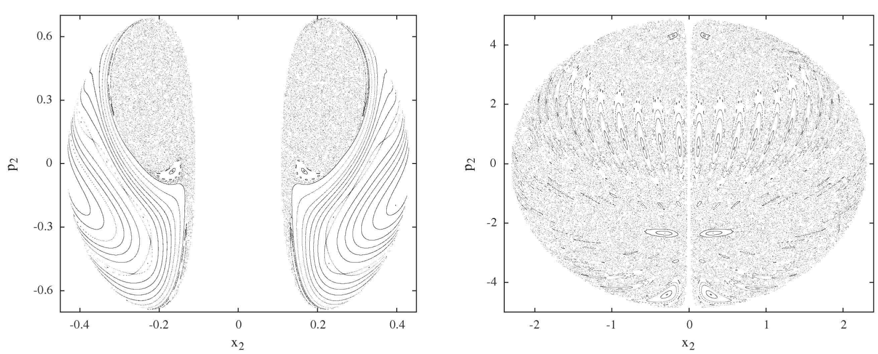
$$H = \frac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \frac{p_3^2}{m_1 x_1^2 + m_2 x_2^2}\right)$$

and is integrable with two additional first integrals $F_1 = p_3$, $F_2 = m_2 p_2 x_1 - m_2 p_1 x_2$.



(a) E=0.12, (b) E=0.2.

Fig. 5. The Poincaré sections for two rigid spring pendula. Parameters: $m_1 = m_2 = a_1 = a_2 = 1$, $k_1 = k_2 = 1/10, p_3 = c = 1/10$, cross-plain $x_1 = 0, p_1 > 0$.



(a) E=0.15, (b) E=4

Fig. 6. The Poincaré sections two rigid spring pendula. Parameters: $m_1 = 1$, $m_2 = 3$, $k_1 = 0.1$, $k_2 = 1.5$, $a_1 = a_2 = 0$, $p_3 = c = 0.1$, cross-plain $x_1 = 0, p_1 > 0$

4 Triple flail pendulum

In Fig. 7 the geometry of the system is shown. Here angle θ_1 is a cyclic coordinate. Fixing value of the corresponding momentum $p_1 = c \in \mathbb{R}$, we consider the reduced system with two degrees of freedom. Examples of Poincaré sections for this system are shown in Fig. 8 and 9. For more plots and its interpretations see [11]. One can also prove that this system is not integrable, see [9].

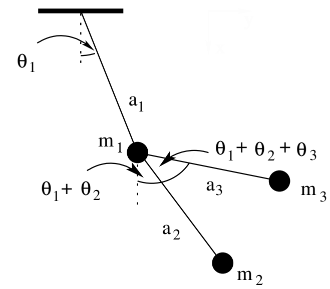
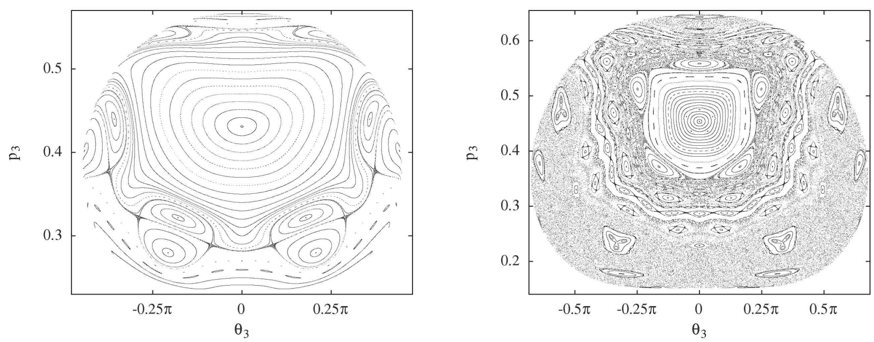


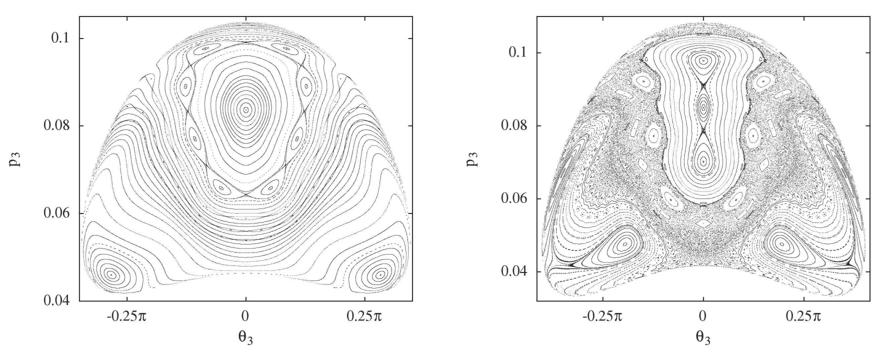
Fig. 7. Triple flail pendulum.



(a) E=0.01,

(b) E=0.012.

Fig. 8. The Poincaré sections for flail pendulum. Parameters: $m_1 = 1, m_2 = 3, m_3 = 2, a_1 = 1, a_2 = 2, a_3 = 3, p_1 = c = 1$, cross-plain $\theta_2 = 0, p_2 > 0$.



(a) E=0.0035,

(b) E=0.0363.

Fig. 9. The Poincaré sections for flail pendulum. Parameters: $m_1 = 1$, $m_2 = m_3 = 2, a_1 = 2, a_2 = a_3 = 1, p_1 = c = \frac{1}{2}$, cross-plain $\theta_2 = 0, p_2 > 0$.

Theorem 3. *Assume that $l_1 l_2 l_3 m_2 m_3 \neq 0$, and $m_2 l_2 = m_3 l_3$. If either (i) $m_1 \neq 0, c \neq 0$, $l_2 \neq l_3$, or (ii) $l_2 = l_3$, and $c = 0$, then the reduced flail system is not integrable in the class of meromorphic functions of coordinates and momenta.*

5 Triple bar pendulum

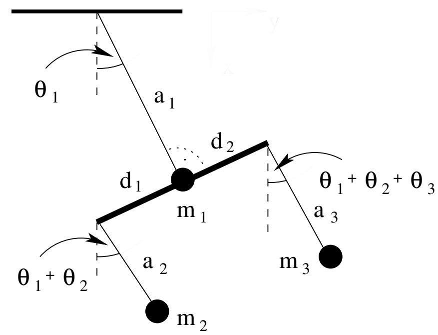
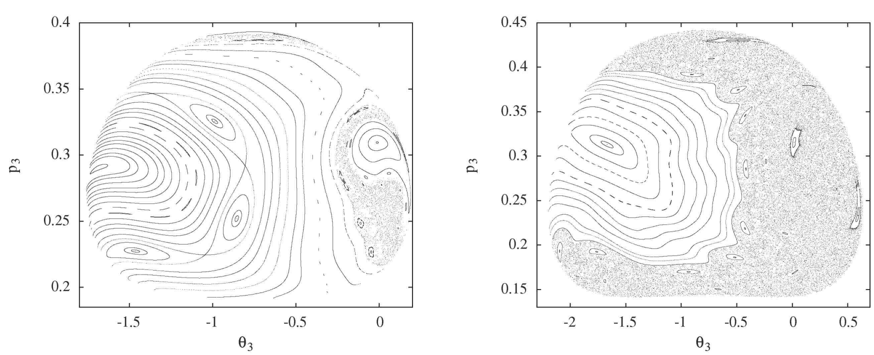


Fig. 10. Triple bar pendulum.

Triple bar pendulum consists of simple pendulum of mas m_1 and length a_1 to which is attached a rigid weightless rod of length $d = d_1 + d_2$. At the ends of the rod there are attached two simple pendula with masses m_2, m_3, respectively, see Fig.10. Like in previous cases fixing value for the first integral $p_1 = c$ corresponding to cyclic variable θ_1, we obtain the reduced Hamiltonian depending only on four variables $(\theta_2, \theta_3, p_2, p_3)$. Therefore we are able to make Poincaré cross sections, see Fig. 11, and also to prove the following theorem [10].:



(a) E=0.008, (b) E=0.009.

Fig. 11. The Poincaré sections for bar pendulum. Parameters: $m_1 = m_2 = 1$, $m_3 = 2, a_1 = 1, a_2 = 2, a_3 = 1, d_1 = d_2 = 1, p_1 = c = \frac{1}{2}$, cross-plain $\theta_2 = 0, p_2 > 0$.

Theorem 4. *Assume that $l_2 l_3 m_1 m_2 m_3 \neq 0$, and $m_2 l_2 = m_3 l_3$, $d_1 = d_2$. If either (i) $c \neq 0, l_2 \neq l_3$ or (ii) $l_2 = l_3$, and $c = 0$, then the reduced triple bar system governed by Hamiltonian is not integrable in the class of meromorphic functions of coordinates and momenta.*

6 Simple triple pendulum

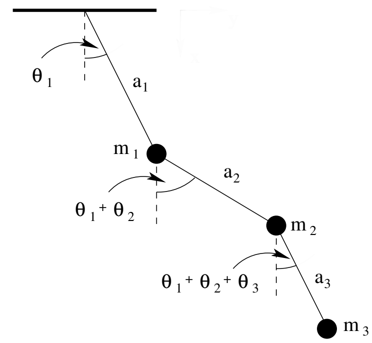
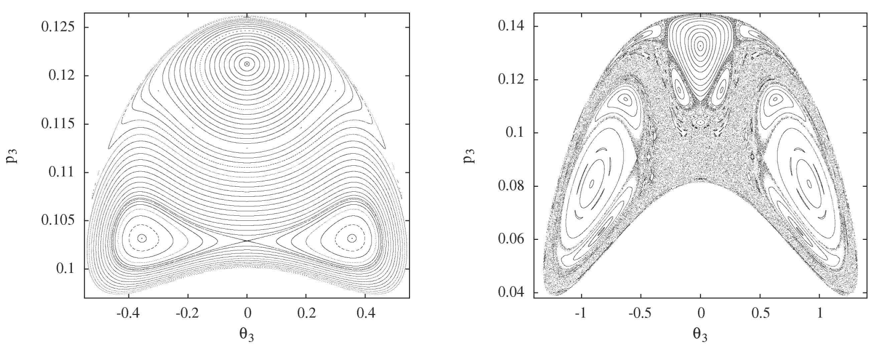


Fig. 12. Simple triple pendulum.

Problem of dynamics of a simple triple pendulum in the absence of gravity field was numerically analysed in [8]. Despite the fact that θ_1 is again cyclic variable, and the corresponding momentum p_1 is constant, the Poincaré sections suggest that this system is also non-integrable, see Fig.13. One can think, that the approach applied to the previous pendula can be used for this system. However, for this pendulum we only found particular solutions that after reductions become equilibria and then the Morales-Ramis theory does not give any obstructions to the integrability.



(a) E=0.0097 (b) E=0.011

Fig. 13. The Poincaré sections for simple triple pendulum: $m_1 = 2$, $m_2 = 1$, $m_3 = 1, a_1 = 2, a_2 = a_3 = 1, p_1 = c = 1$, cross-plain $\theta_2 = 0, p_2 > 0$.

7 Chain of mass points

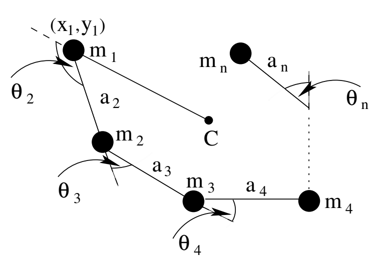
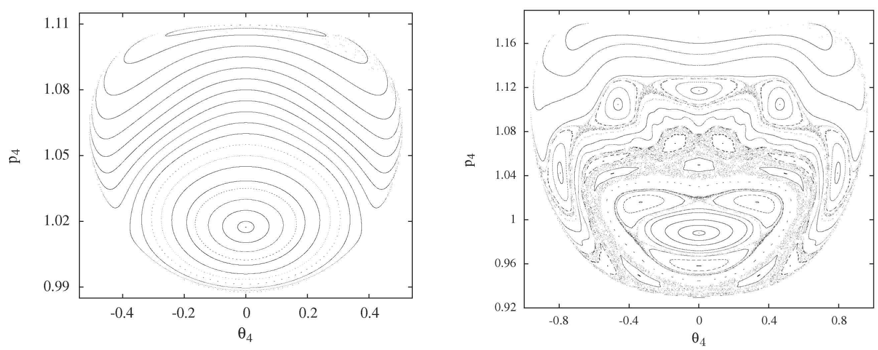


Fig. 14. Chain of mass points

We consider a chain of n mass points in a plane. The system has $n+1$ degrees of freedom. Let $\boldsymbol{r}_i$ denote radius vectors of points in the center of mass frame. Coordinates of these vectors (x_i, y_i) can be expressed in terms of (x_1, y_1) and relative angles θ_i, $i = 2, \ldots, n$. In the centre of mass frame we have $\sum m_i \boldsymbol{r}_i = \boldsymbol{0}$, thus we can expressed (x_1, y_1) as a function of angles θ_i. Lagrange and Hamilton functions do not depend on (x_1, y_1), $(\dot{x}_1, \dot{y}_1)$, and θ_2 is a cyclic variable thus the corresponding momentum p_2 is a first integral. The reduced system has $n-2$ degrees of freedom. Thus the chain of $n = 3$ masses is integrable. Examples of Poincaré sections for reduced system of $n = 4$ masses are given in Fig. 15. In the case when $m_3 a_4 = m_2 a_2$ a non-trivial particular solution is known and non-integrability analysis is in progress.



(a) E=0.04, (b) E=0.045.

Fig. 15. The Poincaré sections for chain of 4 masses. Parameters: $m_1 = m_3 = 1$, $m_2 = 2, m_4 = 3, a_2 = 1, a_3 = 1, a_4 = 3, p_2 = c = \frac{3}{2}$, cross-plain $\theta_3 = 0, p_3 > 0$.

8 Unfixed triple flail pendulum

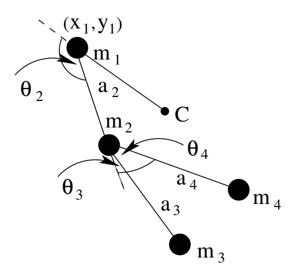
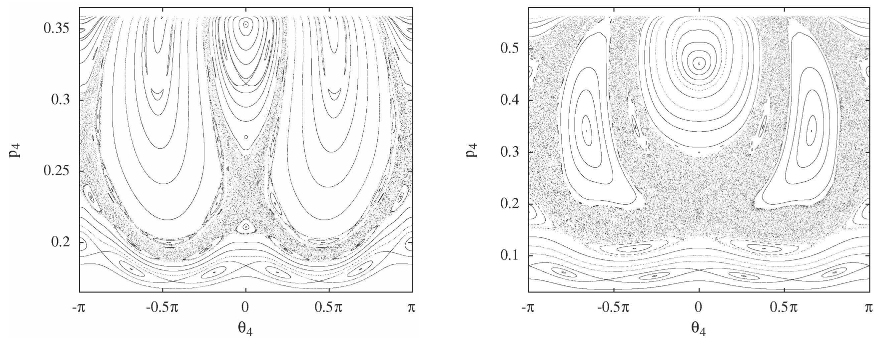


Fig. 16. Chain of mass points

One can also unfix triple flail pendulum described in Sec.4, and allow to move it freely. As the generalised coordinates we choose coordinates (x_1, y_1) of the first mass, and relative angles, see Fig. 16. In the center of masses frame coordinates (x_1, y_1), and their derivatives $(\dot{x}_1, \dot{y}_1)$ disappear in Lagrange function, and θ_2 is a cyclic variable. Thus we can also consider reduced system depending on the value of momentum $p_2 = c$ corresponding to θ_2. Its Poincaré sections are presented in Fig. 17. One can also find a non-trivial particular solution when $a_3 = a_4$. The non-integrability analysis is in progress.

9 Open problems

We proved non-integrability for some systems but usually only for parameters that belong to a certain hypersurface in the space of parameters. It is an open question about their integrability when parameters do not belong to these hypersurfaces. Another problem is that for some systems we know only very simple particular solutions that after reduction by one degree of freedom transform into equilibrium. There is a question how to find another particular solution for them.



(a) E=0.24, (b) E=0.3.

Fig. 17. The Poincaré sections for unfixed flail pendulum. Parameters: $m_1 = 2$, $m_2 = 1, m_3 = 2, m_4 = 1, a_2 = a_3 = a_4 = 1, p_2 = c = \frac{3}{2}$, cross-plain $\theta_3 = 0, p_3 > 0$.

10 Appendix: Proof of non-integrability of the double spring pendulum, Theorem 1

Proof. The Hamiltonian of the reduced system for $p_1 = c = 0$ is equal to

$$
\begin{aligned}
H = \big[&m_2 p_2^2 x^2 + 2a_1 m_2 p_2 x(p_2\cos\theta_2 + p_3 x\sin\theta_2) + a_1^2(m_1(p_2^2 + x^2(p_3^2 \\
&+ k m_2(x-a_2)^2)) + m_2(p_2\cos\theta_2 + p_3 x\sin\theta_2)^2\big]/(2a_1^2 m_1 m_2 x^2),
\end{aligned} \tag{1}
$$

and its Hamilton equations have particular solutions given by

$$
\theta_2 = p_2 = 0, \quad \dot{x} = \frac{p_3}{m_2}, \quad \dot{p}_3 = k(a_2 - x). \tag{2}
$$

We chose a solution on the level $H(0, x, 0, p_3) = E$. Let $[\Theta_2, X, P_2, P_3]^T$ be variations of $[\theta_2, x, p_2, p_3]^T$. Then the variational equations along this particular solution are following

$$
\begin{bmatrix}\dot{\Theta}_2\\ \dot{X}\\ \dot{P}_2\\ \dot{P}_3\end{bmatrix} = \begin{bmatrix} \frac{p_3(a_1+x)}{a_1 m_1 x} & 0 & \frac{a_1^2 m_1 + m_2(a_1+x)^2}{a_1^2 m_1 m_2 x^2} & 0\\ 0 & 0 & 0 & \frac{1}{m_2}\\ -\frac{p_3^2}{m_1} & 0 & -\frac{p_3(a_1+x)}{a_1 m_1 x} & 0\\ 0 & -k & 0 & 0\end{bmatrix}\begin{bmatrix}\Theta_2\\ X\\ P_2\\ P_3\end{bmatrix}, \tag{3}
$$

where x and p_3 satisfy (2). Equations for Θ_2 and P_2 form a subsystem of normal variational equations and can be rewritten as one second-order differential equation

$$
\begin{aligned}
&\ddot{\Theta} + P\dot{\Theta} + Q\Theta = 0, \quad \Theta \equiv \Theta_2, \qquad P = \frac{2a_1 p_3(a_1(m_1+m_2) + m_2 x)}{m_2 x(a_1^2 m_1 + m_2(a_1+x)^2)},\\
&Q = \frac{k(a_1+x)(x-a_2)}{m_1 a_1 x} - \frac{2a_1^2 p_3^2}{m_2 a_1 x(a_1^2 m_1 + m_2(a_1+x)^2)}.
\end{aligned} \tag{4}
$$

The following change of independent variable $t \longrightarrow z = x(t) + a_1$, and then a change of dependent variable

$$\Theta = w \exp\left[-\frac{1}{2}\int_{z_0}^{z} p(\zeta)\,\mathrm{d}\zeta\right] \tag{5}$$

transforms this equation into an equation with rational coefficients

$$w'' = r(z)w, \qquad r(z) = -q(z) + \frac{1}{2}p'(z) + \frac{1}{4}p(z)^2, \tag{6}$$

where

$$\begin{aligned} p = {} & [a_1^2 m_1(-4E + k(2a_2^2 + 3a_1(a_1 - 2z) + 5a_2(a_1 - z)) + 3kz^2) + m_2 z(2a_1 a_2^2 k \\ & + a_1(-4E + a_1 k(2a_1 - 3z)) + kz^3 + a_2 k(a_1 - z)(4a_1 + z))]/[(a_1^2 m_1 + m_2 z^2) \\ & \times (z - a_1)(-2E + kz^2 - (a_1 + a_2)k(2z - a_1 - a_2))], \\ q = {} & \frac{m_2(a_1^2 m_1(4E - k(2(a_1 + a_2) - 3z)(a_1 + a_2 - z)) + km_2(a_1 + a_2 - z)z^3)}{a_1 m_1(-2e + k(a_1 + a_2 - z)^2)(z - a_1)(a_1^2 m_1 + m_2 z^2)}. \end{aligned}$$

We underline that both transformations do not change identity component of the differential Galois group, i.e. the identity components of differential Galois groups of equation (4) and (6) are the same.

Differential Galois group of (6) can be obtained by the Kovacic algorithm [7]. It determines the possible closed forms of solutions of (6) and simultanously its differential Galois group $\mathcal{G}$. It is organized in four cases: (I) Eq. (6) has an exponential solution $w = P\exp[\int\omega]$, $P \in \mathbb{C}[z]$, $\omega \in \mathbb{C}(z)$ and $\mathcal{G}$ is a triangular group, (II) (6) has solution $w = \exp[\int\omega]$, where ω is algebraic function of degree 2 and $\mathcal{G}$ is the dihedral group, (III) all solutions of (6) are algebraic and $\mathcal{G}$ is a finite group and (IV) (6) has no closed-form solution and $\mathcal{G} = \mathrm{SL}(2, \mathbb{C})$. In cases (II) and (III) $\mathcal{G}$ has always Abelian identity component, in case (I) this component can be Abelian and in case (IV) it is not Abelian.

Equation (6) related with our system can only fall into cases (I) or (IV) because its degree of inifinity is 1, for definition of degree of infinity, see [7]. Moreover, one can show that there is no algebraic function ω of degree 2 such that $w = \exp[\int\omega]$ satisfies (6) thus $\mathcal{G} = \mathrm{SL}(2, \mathbb{C})$ with non-Abelian identity component and the necessary integrability condition is not satisfied.

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