Chaotic Modeling and Simulation (CMSIM) 1: 3–26, 2015

Error Diffusion on Acute Simplices: Geometry of invariant sets

R. Adler[1], T. Nowicki[1], G. Świrszcz[1], C. Tresser[1], and S. Winograd[1]

IBM T.J. Watson Research Center
(E-mail: rla,tnowicki,swirszcz,swino@us.ibm.com;
charlestresser@yahoo.com)

Abstract. We study the absorbing invariant set of a dynamical system defined by a map derived from Error Diffusion, a greedy online approximation algorithm that minimizes the (Euclidean) norm of the cumulated error. This algorithm assigns a sequence of outputs (each a vertex of the polytope) to any sequence of inputs in the polytope. Here, the polytope is assumed to be a simplex that is acute, meaning that the pairs of distinct external normal vectors to the codimension-one faces form obtuse angles. The input is assumed to be constant. The map is a system which consists of piecewise translations acting on the partition of an affine space into the Voronoï regions defined (once tie-breaking is resolved) by the vertices of the polytope. The translation vector in each partition piece is the difference between the input modified by adding the cumulated error and the corresponding vertex.

When the polytope is a simplex such piecewise translation projects onto a translation of a torus. It is known that if the projected translation is ergodic, then the invariant absorbing set of the piecewise translation is a fundamental set for the lattice generated by the simplex. In this paper we study some sets which are important in estimation the shape of this invariant set, and thus in the control of the rather erratic behaviour of the algorithm in the invariant set.

1 Introduction

We shall investigate a dynamical system derived from an algorithm called *error diffusion* used, *e.g.,* in digital color printing (for motivation and background see Section 5).

Main result

Let A be an d dimensional real affine space whose associated vector space V is equipped with the usual scalar product and Euclidean metric. Let $\mathbf{P}$ be a polytope in A with the set $\mathcal{C} = \{v_0, \ldots, v_r\}$ of vertices (extreme points) of which it is the convex hull. From the sequence of points $g(N) \in \mathbf{P}$ called *inputs* the *Error Diffusion Algorithm* (EDA) provides a sequence of vertices $v(N) \in \mathcal{C}$

Received: 18 September 2014 / Accepted: 10 December 2014

ISSN 2241-0503

with close (due to boundedness) Cesaro means as follows. Define the *cumulative error* $e_N = \sum_{n=1}^{N}(g(n) - v(n))$. Then $e(N) = x(N) - v(N)$, where $x(N) = e(N-1) + g(N)$, $e(0) = 0$ and $v(N) = v(x(N))$ is chosen to be the vertex that minimizes $||e(N)||$. It follows that $x(N+1) = x(N) + g(N) - v(x(N))$. If there are several minimizing vertices v_i we apply some tie-breaking rules to assign $v(x(N))$. One such rule consists in taking the vertex of smallest index among all that minimize the norm.

All points $x \in A$ with the same $v(x) = v_k \in \mathcal{C}$ form a *Voronoï* domain $\mathbf{V}_k$, with well defined interior and closure, while the boundary points are decided by the tie-breaking rules.

$$\overline{\mathbf{V}}_k = \{y \in A : ||y - v_k||^2 \leq ||y - v_i||^2, i = 0, \ldots, r\}\,. \tag{1}$$

The Voronoï domains form a partition of A. The EDA can be represented by a dynamical system $\mathcal{F}$ on the space A, which in case of the the case of a constant input $g \in \mathbf{P}$ is defined by:

$$\mathcal{F}(x) = \mathcal{F}_g(x) = g + x - v(x) \quad \text{where} \quad v(x) = v_k \quad \text{iff} \quad x \in \mathbf{V}_k\,. \tag{2}$$

The map $\mathcal{F}$ is a piecewise translation with the parameter g.[1]

In this paper we will consider the case of a constant input $g \in \mathbf{P}$, where $\mathbf{P}$ is a simplex with $d+1$ vertices such that the vectors $(v_i - v_0)_{i\neq 0}$ form a basis of the vector space V.

We shall assume that this simplex $\triangle(v)$ is *acute* (Definition 24), *i.e.,* that any pair of *external normal vectors* to distinct codimension-one *faces* forms an obtuse angle.

The simplex generates a lattice $\mathbf{L} = \{\sum_i n_i v_i : \sum_i n_i = 0, n_i \in Z\} \subset V$ and the space $A/\mathbf{L}$ of equivalence classes $[x] = \{y \in A : x - y \in \mathbf{L}\}$ is a d-dimensional torus. The map $\mathcal{F}$ projects to a well defined translation $[\mathcal{F}]$ of the torus. We will call the input parameter g *ergodic* if the toral translation $[\mathcal{F}]$ is ergodic (*cf.* [15]). We say that the set $\mathbf{T} \subset A$ is a *tile* for the lattice $\mathbf{L}$ if it projects bijectively to the torus, or equivalently if for each $x \in A$ there is a unique $y \in \mathbf{T}$ such that $y - x \in \mathbf{L}$.

We are interested in the asymptotic behavior of the iterates $\mathcal{F}^N$ of $\mathcal{F}$, where $\mathcal{F}^0 = \mathcal{F}$ and $\mathcal{F}^{N+1} = \mathcal{F} \circ \mathcal{F}^N$. The set $\mathbf{Q}$ is *absorbing* if each trajectory eventually enters it and then never leaves. We have proven elsewhere [6] that:

Theorem A (Ergodic Inputs) *The error diffusion map $\mathcal{F}$ with ergodic input on acute simplex has an absorbing invariant bounded tile for the lattice* $\mathbf{L}$ *generated by the simplex.*

Remark 11 *We conjectured that both acuteness and ergodicity assumptions are irrelevant. Our approach relied on geometry of the problem, which is different in acute and obtuse cases. Already in two dimensions obtuse triangles*

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[1] The case of nonconstant inputs g_N can be represented by a map on the much larger space $A \times \mathbf{P}^Z$, with

$$\overline{\mathcal{F}}(x, \overline{g}) = (x + \overline{g}_0 - v(x), \sigma(\overline{g}))\,,$$

where σ is the shift operator on the sequence $\overline{g} = (\cdots, g_{-1}, g_0, g_1, \cdots)$.

produce sometimes disconnected absorbing invariant set (it seems they are still tiles), while acute ones produce only connected tiles. Ergodicity, a typical condition, enables an elegant proof that the invariant set is indeed one tile, without it our approach can only prove that this set is a finite union of disjoint tiles.

In the follow up work [7] we described the invariant tile in more details and in particular proved:

Theorem B (Sub Tiles) *Under the assumptions of Theorem A each intersection of the invariant tile* $\mathbf{Q}$ *with the Voronoï domain* $\mathbf{V}_i$ *and each union of such intersections is a tile for another lattice (expressed explicitly in terms of* g *and the collection of vertices* v_i*). More precisely if we assume* $0 \in I \subset \{0, 1 \ldots, d\}$ *then* $\mathbf{Q}_I = \mathbf{Q} \cap \bigcup_{i \in I} \mathbf{V}_i$ *is a tile for the lattice* $Z((g - v_j)_{|j \notin I}, (v_i - v_0)_{|i \in I})$.

Remark 12 *The ergodicity assumption is not needed in dimension* $d = 2$*, see [5]. In higher dimensions, even in a non-ergodic case, if the trajectories on the torus are sufficiently dense then the proofs of both Theorems remain valid.*

For the sake of completeness we provide the sketches of the proofs of those Theorems at the end of Section 2

Here, assuming acuteness but not ergodicity, we study the following, natural in context, set:

$$\mathcal{H} = \overline{\bigcap_i \mathcal{F}(\mathbf{V}_i)} \,. \tag{3}$$

The set $\mathcal{H}$ in the ergodic case is mapped in finitely many steps onto the invariant tile and thus provides the following Theorem, the main result of this paper:

Theorem C *Under the same assumptions the invariant tile is a (possibly non convex) polytope.*

We also present the connections between the set $\mathcal{H}$ and the Voronoï cells, *i.e.,* the Voronoï regions of the lattice $\mathbf{L}$.

2 Preliminaries

We recall that in the affine space A the difference of two points is a vector (of the vector space V associated to the affine space) whence the sum of a point and a vector is a point.

2.1 Simplices and barycentric coordinates

Besides the *standard simplex*

$$\triangle = \{\xi = (\xi^0, \ldots, \xi^d) \in R^{d+1} : \quad \sum \xi^i = 1,\ \xi^i \geq 0\} \tag{4}$$

in the affine space $\{\xi \in R^{d+1} : \quad \sum \xi^i = 1\}$ modeled over the vector space $\{\xi \in R^{d+1} : \quad \sum \xi^i = 0\}$ we will also consider more general full dimensional simplices in a d dimensional real affine space A.

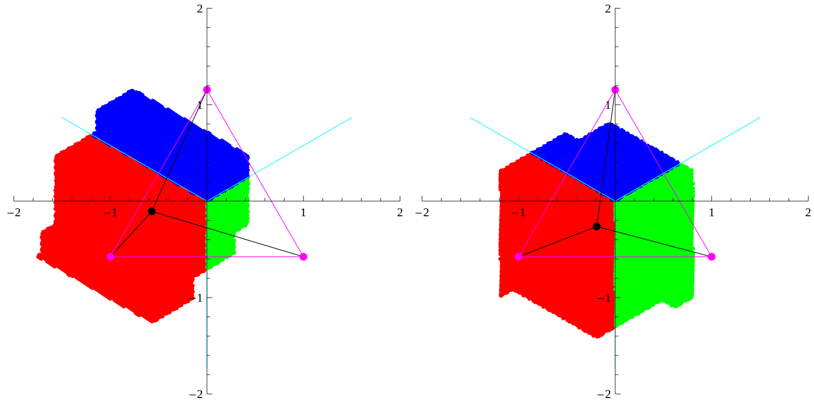

Fig. 1. Simulation of limit sets for two "inputs" g in an equilateral triangle. The parts ("sub-tiles") inside the three Voronoï regions are shaded differently. The limits sets are tiles (Theorem A) for the triangle generated lattice and each part and also each pair of parts tiles the space (Theorem B).

Remark 21 (Solving equations) *From independence of v it follows that for any k and any collection of d numbers β^i with $i \neq k$, the system of d equations: $r \cdot (v_i - v_k) = \beta^i$, $i \in \{0, 1, \dots, d\} \setminus \{k\}$, has a unique (vector) solution r. This solution can be found using dual basis. If b_i, $i \neq k$ are vectors dual to $v_i - v_k$, $i \neq k$ (that is $b_i \cdot (v_j - v_k)$ equals 0 for $j \neq i$, and equals 1,when $j = i$) then $r = \sum_{i \neq k} \beta^i b_i$. With the scalar product the dual vectors are naturally embedded in the same space as the basis $(v_i - v_k)_{i \neq k}$.*

Definition 21 (Simplex) *A (closed)* ***simplex*** *in A is the convex hull of a collection of points v_i, $i = 0, \dots, d$ such that $(v_i - v_0)_{i \neq 0}$ form a basis in V: $\triangle(v) = \{x = \sum \xi^i v_i : \quad \sum \xi^i = 1,\ \xi^i \geq 0\}$.*

By definition of the simplex A is a minimal affine space containing all the points v.

Remark 22 *Every point $x \in A$ can be uniquely represented by its* ***barycentric coordinates*** *ξ^i (*derived from the simplex $\triangle(v)$*): $x = \sum \xi^i v_i$, $\sum \xi^i = 1$.*

We shall represent the input, a fixed point $g \in \triangle^\circ(v)$ (interior) by its barycentric coordinates:

$$g = \sum \gamma^i v_i, \quad \gamma^i > 0, \quad \sum \gamma^i = 1\,. \tag{5}$$

Remark 23 *In the space $A = \{(\xi^0, \dots, \xi^d) \in R^{d+1} : \sum \xi^i = 1\}$ the barycentric coordinates derived from the standard simplex* (4) *are the same as the standard Cartesian ones.*

2.2 Acuteness

If $A_1 \subset A_2$ are affine spaces with respective dimensions $d_1 \leq d_2$, and Q is a polytope in A_1 not contained in any smaller affine space, then $d_2 - d_1$ is the *codimension of* Q *in* A_2, or simply the *codimension of* Q when the space A_2 is unambiguous. The dimension of the set is here understood as the dimension of the minimal affine subspace containing this set.

Definition 22 (Face) *The* ***face*** $\mathbf{F}_k$ *of the simplex* $\triangle(v)$ *opposite to the vertex* v_k *is the codimension one simplex* $\mathbf{F}_k = \{\sum \xi^i v_i : \quad \xi^k = 0, \quad \xi^i \geq 0, \quad \sum \xi^i = 1\}$ *lying in the affine space* $A_k = \{\sum \xi^i v_i : \quad \xi^k = 0, \ \sum \xi^i = 1\}$.

Definition 23 (External normal vector) *The* ***external normal*** *vector* s_k *to the face* $\mathbf{F}_k$ *is the unique vector such that the scalar products satisfy the* normalization *condition:*

$$s_k \cdot (v_j - v_k) = 1 \textit{ for any } j \neq k\,, \tag{6}$$

We choose this normalization to simplify future calculations. In particular for $i, j \neq k$ we have (*normality*) $s_k \cdot (v_j - v_i) = 0$. Trivially $s_k \cdot (v_j - v_k) \geq 0$ always holds.

Definition 24 (Acuteness) *We say that a simplex is* ***acute*** *(face-wise acute) if*

$$s_i \cdot s_j < 0 \ \textit{ whenever } i \neq j\,. \tag{7}$$

Remark 24 (Edge-wise acuteness) *There is a weaker condition to (face-wise) acuteness in Definition 24 that we call* edge-wise acuteness*:*

$$\forall k, l \neq i \qquad (v_j - v_i) \cdot (v_k - v_i) > 0\,, \tag{8}$$

which follows from the acuteness.

2.3 Forward invariance

Definition 25 (Invariant sets) *We say that a set is (forward)* ***invariant*** *with respect to the transformation* $\mathcal{F}$ *if* $\mathcal{F}(\mathbf{Q}) \subset \mathbf{Q}$.

Definition 26 (Absorbing sets) *We say that the set* $\mathbf{Q}$ *is* ***absorbing*** *if for every point* x *in* A *there is an* N $(= N(x))$ *such that for every* $M \geq N$ $\mathcal{F}^M(x) \in \mathbf{Q}$.

2.4 Lattices and Tiles

Definition 27 (Lattice) *For the points* $v_i \in A$ *the* ***lattice*** $\mathbf{L} = \mathbf{L}(v)$ *is a subgroup of the vector space modeling* A*, generated by the vectors* $v_i - v_j$*, that is for any* k*:* $\mathbf{L} = \mathbf{L}(v) = \{\sum_{i \neq k} n^i (v_i - v_k), n^i \in Z\}$.

In this definition the fact that the lattice does not depend on the choice of k can be expressed in a symmetric way as $\mathbf{L} = \{\sum n^i v_i : \quad n^i \in Z, \ \sum n^i = 0\}$.

Definition 28 *Given a set* $\mathbf{Q}$ *and a lattice* $\mathbf{L}$ *consider the map* $\mathbf{Q}+\mathbf{L} : \mathbf{Q} \times \mathbf{L} \to A$ *given by:* $(q, r) \mapsto q + r$*. We will say that* $\mathbf{Q} + \mathbf{L}$ *is:*

1. surjective *if this map is surjective, i.e., for every point* $x \in A$ *there are* $q \in \mathbf{Q}$ *and* $r \in \mathbf{L}$ *such that* $x = q + r$.
2. injective *if this map is injective, i.e., if* $q + r = q' + r'$ *then* $q = q'$ *and* $r = r'$.
3. bijective *if it is both surjective and injective.*

Definition 29 (Tile) *We say that a set* $\mathbf{Q}$ *is* ***a tile*** *for the lattice* $\mathbf{L}$ *if* $\mathbf{Q}+\mathbf{L}$ *is bijective.*

In other words all lattice translates of $\mathbf{Q}$ cover the whole space A without overlaps. If $\mathbf{Q}$ is a tile that has some minimal topological regularity it is sometimes called a *fundamental domain* of the group $\mathbf{L}$.

Definition 210 (Equivalent Points) *We say that the points* x *and* y *are* ***equivalent*** *(with respect to the group* $\mathbf{L}$*) if* $y - x \in \mathbf{L}$.

Remark 25 ($\mathcal{F}$ projects on the torus) *In the case of an independent and full dimensional collection of points the quotient space* $A/\mathbf{L}$ *of classes of* ***equivalent*** *points* $[x]$ *is a* d*-dimensional torus. The map* $\mathcal{F}$ *projects on a translation (in cube model) on this torus (or the rotation in exponential model on the product of unit circles),* $[\mathcal{F}]([x]) = [\mathcal{F}(x)]$. *In other words* x *and* y *are equivalent iff* $\mathcal{F}(x)$ *and* $\mathcal{F}(y)$ *are. This is due to the fact that all the translation vectors* $g - v_i$ *project on the same vector on the torus.*

Proof. Suppose that $[x] = [y]$ then $x - y \in \mathbf{L}$. Thus $\mathcal{F}(x) - \mathcal{F}(y) = (x + g - v(x)) - (y + g - v(y)) = (x - y) + (v(y) - v(x)) \in \mathbf{L}$ which means that $[\mathcal{F}(x)] = [\mathcal{F}(y)]$.

2.5 Sketch of the proof of Theorem A

We construct a large set, in fact a simplex $\triangle_B$, which, *due to acuteness*, turns out to be forward invariant and absorbing. Inside this large simplex there lies a smaller open set (also a simplex $\triangle_R$) the point of which do not have distinct equivalent points in the large simplex. We shall project the sets and the trajectories onto the torus and, *due to ergodicity of the input*, we shall conclude that there is indeed an absorbing invariant set inside the large simplex which projects bijectively on the the torus hence it is an invariant absorbing tile.

Define $\mathcal{O}$ to be the center of the sphere circumscribing the simplex $\triangle(v)$, that is the unique point satisfying the equations: $(v_i - \mathcal{O})^2 = (v_j - \mathcal{O})^2$ for all $i, j = 0, \dots, d$. Next define the points w:

$$w_k = \mathcal{O} + g - v_k \,, \tag{9}$$

which form the *Inverted Simplex*:

$$\triangle_R = \triangle(w) = \{\sum \xi^i w_i : \xi^i \geq 0, \sum \xi^i = 1\} \tag{10}$$

Finally define the points u:

$$u_k = w_k + \sum_i (w_i - w_k) \,, \tag{11}$$

which and form the *Big Simplex*:

$$\triangle_B = \triangle(u) = \{\sum \xi^i u_i : \xi^i \geq 0, \sum \xi^i = 1\} \quad (12)$$

We use the following results from [6]:

- Under acuteness condition for $\triangle(v)$ the Big Simplex $\triangle_B$ is invariant and absorbing under $\mathcal{F}_g$.
- The Inverted Simplex is contained in the Big Simplex.
- The points in the interior of the Inverted Simplex do not have distinct equivalent points in the Big Simplex.

Construction of the invariant set Q. Consider $\mathbf{Q}_0 = \triangle_R^\circ \subset \triangle_B$, the interior of the inverted simplex. Let $\mathbf{Q}_{N+1} = \mathbf{Q}_0 \cup \mathcal{F}(\mathbf{Q}_N)$. Then $\{\mathbf{Q}_N\}$ is an increasing family of sets that has a limit $\mathbf{Q} = \bigcup_{N=0}^{\infty} \mathbf{Q}_N = \bigcup_{N=0}^{\infty} \mathcal{F}^N(\mathbf{Q}_0)$. Due to to the invariance of $\triangle_B$ under $\mathcal{F}$ we have $\mathbf{Q} \subset \triangle_B$. By construction $\mathcal{F}(\mathbf{Q}) \subset \mathbf{Q}$, which makes $\mathbf{Q}$ invariant and not containing distinct equivalent points.

The set Q contains a tile. When we project $\mathbf{Q}$ onto the torus $A/\mathbf{L}$, we see that the projection is an invariant set containing an open set, hence by ergodicity the projection must contain the whole torus. That means that $\mathbf{Q}$ itself must contain a tile. We proved that $\mathbf{Q}$ contains a tile but no distinct equivalent points, thus it is a tile for the lattice $\mathbf{L}$, invariant with respect to $\mathcal{F}$.

The set Q absorbs all trajectories. Any trajectory must enter the absorbing set $\triangle_B$. Once there, because of ergodicity, the projection of every trajectory must pass through the projection of $\mathbf{Q}_0$, the interior of $\triangle_R$. Because there are no points in $\triangle_B$ which project there, other than $\triangle_R$ itself, we conclude that every trajectory from $\triangle_B$ must pass through $\mathbf{Q}_0$, and once there, it remains in $\mathbf{Q}$.

2.6 Sketch of the proof of Theorem B

We show here only the case of $I = \{0\}$, all other cases have similar philosophy but the calculations are more difficult. We want to show that for any point $P \in A$ there is a point $q \in \mathbf{Q}_I$ and a vector $r \in L_I = Z(g - v_k, k \neq 0)$ such that $P = q + r$ uniquely. Because $\mathbf{Q}$ is a tile then $P = x + \sum_k m_k(v_k - v_0)$, uniquely where $x \in \mathbf{Q}$, $m_k \in Z$. For any $z\mathbf{Q}_I = \mathbf{Q} \cap \mathbf{V}_i \subset \mathbf{Q}$ and any $n \in Z$ there is a point $q = q(z, n) \in \mathbf{Q}_I$ and $N \in N$ such that if $n > 0$ then $\mathcal{F}^N(q) = z$ and $\text{card}\{i < N : \mathcal{F}^i(w) \in \mathbf{Q}_I\} = n$ and if $n < 0$ then $\mathcal{F}^N(z) = q$ and $\text{card}\{i < N : \mathcal{F}^i(z) \in \mathbf{Q}_I\} = |n| = -n$. Then for some nonnegative integers p_k in the first case $z = q + n(g - v_0) + \sum_{k\neq 0} p_k(g - v_k)$ and in the second case $q = z + |n|(g - v_0) + \sum_{k\neq 0} p_k(g - v_k)$. This follows the properties of the ergodic toral translations and the fact that $\mathcal{F}$ i sa bijection on the tile $\mathbf{Q}$. Choose z and M such that $\mathcal{F}^M(z) = x$ and $F^i(z) \notin \mathbf{Q}_I$, $0 < i < M$, then $x = z + g - v_0 + \sum_{k\neq 0} n_k(g - v_k)$. Then $P = x + \sum_{k\neq 0} m_k(v_k - g + g - v_0) = z + \sum_{k\neq 0} n_k(g - v_k) + \sum_{k\neq 0} m_k(v_k - g + g - v_0) = z + \sum_{k\neq 0} n_k - m_k(g - v_k) + \sum_{k\neq 0} m_k(g - v_0)$. Let $n = -\sum_{k\neq 0} m_k$ and take $w = w(z, n)$. Then in any case $P = q - \sum_{k\neq 0} m_k(g - v_0) \pm \sum_{k\neq 0} p_k(g - v_k) + \sum_{k\neq 0} n_k - m_k(g - v_k) + \sum_{k\neq 0} m_k(g - v_0) = q + \sum_{k\neq 0} n_k - m_k \pm p_k(g - v_k) \in \mathbf{Q}_I + L_I$. Uniqueness follows from uniqueness of $q(z, n)$, x and m_k.

3 Proof of Theorem C

The set $\mathcal{H} = \overline{\bigcup_i \mathcal{F}(\mathbf{V}_i)}$, is a convex polytope with boundaries parallel to the co-dimension one planes which contains the boundaries of the Voronoï regions. More precisely, $\overline{\mathbf{V}_i}$ is an intersection of closed half spaces bounded by hyperplanes $M_{ij} = \{x : (x - (v_i + v_j)/2) \cdot (v_j - v_i) = 0\}$ orthogonal to the edges. The common point of all M_{ij}'s exists and is equal to the center $\mathcal{O}$. Therefore $\mathcal{H}$, as $\mathcal{F}_{|\mathbf{V}_i}$ consists of one translation, is the intersection of half spaces $A_{i,j} = \{x : (x - w_i) \cdot (v_j - v_i) \leq 0\}$, as w_i is the image of $\mathcal{O}$ under the translation by $g - v_i$, which acts on $\mathbf{V}_i$. Each boundary M_{ij} is mapped by two vectors $g - v_i$ and $g - v_j$ producing two hyperplanes passing respectively through w_i and w_j and forming a band of width $||v_i - v_j||$. The intersection of all such bands is bounded, as the edges span the space V, and has $d(d+1)$ codimension 1 faces pairwise parallel to M_{ij}'s. For $d = 1$ it is an invariant interval, for $d = 2$ it is a hexagon, a tile for the lattice $\mathbf{L}$, but not necessarily invariant. For $d > 2$ the set $\mathcal{H}$ contains a tile, but is usually larger than one.

Lemma 31 *For any n, $\overline{\mathcal{F}^n(\mathcal{H})}$ is a polytope with boundaries parallel to hyperplanes M_{ij}.*

Proof. The boundaries of $\overline{\mathcal{F}^n(\mathcal{H})}$ are formed by translation of either boundaries of $\overline{\mathcal{F}^{n-1}(\mathcal{H})}$ or by translations of discontinuities that is boundaries of $\mathbf{V}_i$'s which lie in M_{ij}'s.

Lemma 32 *Under ergodicity assumption there is an N such that $\mathcal{F}^N(\mathcal{H}) = \mathbf{Q}$.*

Proof. If follows from the general fact that for any bounded set with non empty interior after (a) a finite number of steps this set is mapped into $\triangle_B$ and then (b), by ergodicity it is mapped onto $\mathbf{Q}$.

The two previous Lemmata prove Theorem C.

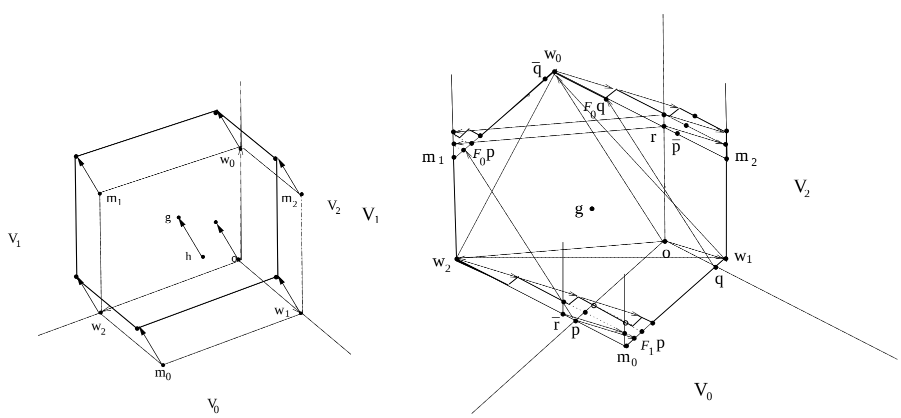

Fig. 2. The set $\mathcal{H}$. For different inputs g and f the sets are translates of each other by $f - g$. The iterates $\mathcal{F}^N\mathcal{H}$ make the set non-convex.

4 Properties of the intersection of the images of the Voronoï regions

In this section we investigate further properties of the set $\mathcal{H}$, assuming only acuteness of the simplex but not ergodicity of the input.

4.1 The Voronoï cell of a lattice

Given a lattice $\mathbf{L}$ we call the Voronoï cell of a point g a set of all points which are closer to g than to any its lattice translates.

$$\mathcal{V} = \mathcal{V}(g) = \{x : (x - g)^2 \leq (x - (g + r))^2 \, \forall r \in \mathbf{L}\} = \{x : (x - (g + \frac{1}{2}r)) \cdot r \leq 0 \forall r \in \mathbf{L}\} .$$

The cell $\mathcal{V}$ is closed and convex and it is a closure of a tile for $\mathbf{L}$ with which it shares the interior, in other words the tile differs from $\mathcal{V}$ by some boundary points.

4.2 Arbitrary dimension *d*

As $\overline{\mathbf{V}_i} = \{x : (x - v_i)^2 - (x - v_j)^2 = 2(x - \mathcal{O}) \cdot (v_j - v_i) \leq 0, \forall j\}$, we have $\mathcal{F}(\mathbf{V}_i) = \{x : ((\mathcal{F}_{|\mathbf{V}_i})^{-1}(x) - \mathcal{O}) \cdot (v_j - v_i) = (x - g + v_i - \mathcal{O}) \cdot (v_j - v_i) \leq 0, \forall j\}$ and finally

$$\overline{\mathcal{H}} = \{x : (x - w_i) \cdot (v_j - v_i) \leq 0, \forall j, i\} . \tag{13}$$

Lemma 41

$$\overline{\mathcal{H}} = \{x : (x - g)^2 \leq (x + (v_i - v_j) - g)^2, \forall i, j\}$$

Proof.

$$(x - g)^2 - (x + (v_i - v_j) - g)^2 = 2(x - g) - v_i - v_j + 2v_i) \cdot (v_j - v_i) = 2(x - g - \mathcal{O} + v_i) \cdot (v_j - v_i)$$

Corollary 42 *The set $\overline{\mathcal{H}}$ contains a tile, which is a Voronoï cell of the point g in the lattice $g + \mathbf{L}$.*

Proof. The set $\mathcal{H}$ is contained in a set defined by the same inequalities as $\mathcal{V}$ but with r restricted to the edges $r = v_i - v_j$.

Each $\mathbf{V}_k$ is a cone with vertex $\mathcal{O}$ and edges $s_i, i \neq k$ and has d co-dimension 1 faces with external normal vectors $v_i - v_k$, $i \neq k$.

Lemma 43 *For every i, j we have $w_j \in \mathcal{F}(\mathbf{V}_i)$.*

Proof. Consider the point $w_{ij} = w_j - g + v_i = \mathcal{O} + v_i - v_j$. Then $(w_{ij} - v_i)^2 = (\mathcal{O} - v_j)^2$ and $(w_{ij} - v_i)^2 - (w_{ij} - v_k)^2 = (2(v_i - v_j)(v_k - v_i)$, where we used the properties of $\mathcal{O}$. By edgewise acuteness that proves $w_{ij} \in \mathbf{V}_i$, but then $\mathcal{F}(w_{ij}) = w_{ij} = g - v_i = w_j$.

Each $\mathcal{F}(\mathbf{V}_k)$ is an affine cone with the vertex at $w_k = \mathcal{O} + g - v_k$ and as $w_k \in \mathcal{F}(\mathbf{V}_i)$ for all i it is a vertex of the set $\mathcal{H}$.

Lemma 44 ($\mathcal{H}$ is symmetric) *The convex set $\mathcal{H}$ is centrally symmetric with respect to the point g. In particular $g \in \mathcal{H}$.*

Proof. Convexity is trivial. If $x \in \overline{\mathcal{H}}$ then $(x - w_i) \cdot (v_i - v_j) \leq 0$, for all i, j. For the symmetric point $2g - x$ we have

$$\begin{aligned}(2g - x - w_i) \cdot (v_j - v_i) &= (g + \mathcal{O} - v_j - x - 2\mathcal{O} + v_j + v_i) \cdot (v_j - v_i) \\ &= (w_j - x) \cdot (v_j - v_i) = (x - w_j) \cdot (v_i - v_j) \leq 0\end{aligned}$$

because we can switch the indices i and j in the condition for $x \in \overline{\mathcal{H}}$.

By symmetry each point $m_k = -\mathcal{O} + g + v_k$ is also a vertex of $\mathcal{H}$. Let us formulate it and prove directly.

Lemma 45 *In an edgewise acute simplex every point w_l and every point m_i is a vertex of the set $\mathcal{H}$.*

Proof. We have $(w_l - w_j) \cdot (v_k - v_j) = (v_j - v_l) \cdot (v_k - v_j) \leq 0$ and $(m_i - w_j) \cdot (v_j - v_k) = (v_i + v_j - 2\mathcal{O} + v_k - v_k) \cdot (v_j - v_k) = (v_i - v_k) \cdot (v_k - v_j) \leq 0$.

In dimension $d = 1$ we have $\mathcal{H} = [m_0, m_1] = [w_1, w_0]$, which is an invariant tile. In dimension $d = 2$ of $\mathcal{H}$ is a centrally symmetric hexagon with alternating vertices w and m (a degeneration to a rectangle is possible if the triangle is not strictly acute). It is a tile but usually not invariant. In higher dimensions there are many more vertices of $\mathcal{H}$ and we shall describe the shape of it in some details.

For fixed two indices

$$k, j \in J = \{0, \ldots, d\} \text{ define } J_{kj} = J \setminus \{k, j\}$$

Proposition 46 (Faces of $\mathcal{H}$) *In edgewise simplices each co-dimension 1 face of $\mathcal{H}$ is uniquely determined by a pair of points w_k and m_j with $k \neq j$ and is included in a co-dimension 1 parallelepiped P_{kj} given by the intersection of two cones:*

$$\begin{aligned}P_{kj} &= \{w_k + \sum_{i \in J_{kj}} \lambda^i s_i, \lambda_i \geq 0\} \cap \{m_j - \sum_{i \in J_{kj}} \mu^i s_i, \mu_i \geq 0\} \\ &= \{w_k + \sum_{i \in J_{kj}} \lambda^i s_i, 0 \leq \lambda^i \leq \Lambda^i_{kj} = (v_k - v_i) \cdot (v_j - v_i)\}\,.\end{aligned}$$

We shall call the face of $\mathcal{H}$ contained in P_{kj} by $\mathcal{H}_{kj}$.

Proof. The faces of $\mathcal{H}$ are contained in the translates of the faces of $\mathbf{V}_k$. For each k there are d such faces, with an external normal vectors $v_j - v_k$, $j \neq k$. More precisely, with an equivalent characterization of Voronoï regions as normal cones, such a face of $\mathbf{V}_k$ is a cone itself $\{\mathcal{O} + \sum_{i \in J_{kj}} \lambda^i s_i, \lambda^i \geq 0\}$. Hence the co-dimension 1 faces of $\mathcal{H}$ are contained in the cones $\{w_k + \sum_{i \in J_{kj}} \lambda^i s_i, \lambda^i \geq 0\}$. There are no other co-dimension 1 faces created by the intersections of $\mathcal{F}(\mathbf{V}_k)$. By symmetry the faces are also included in the cones $\{m_j - \sum_{i \neq j,k} \mu^i s_i, \mu^i \geq 0\}$.

As $w_k - m_j = 2\mathcal{O} - v_k - v_j$ is orthogonal to $v_j - v_k$ it follows that both w_k and m_j lie on the same face which is included in the intersection of the two cones. If $x \in P_{kj}$ then the two representations of x give rise to the equation $\sum_{i\in J_{kj}}(\lambda^i(x) + \mu^i(x))s_i = m_j - w_k = (v_k + v_j - 2\mathcal{O})$. After multiplying both sides by $v_k - v_l$ we get $\lambda^l + \mu^l = (v_j - v_l + v_l + v_k - 2\mathcal{O}) \cdot (v_k - v_l) = (v_j - v_l) \cdot (v_k - v_l) = \Lambda^l_{kj}$, and the estimate on λ (and μ) follows from $\lambda, \mu \geq 0$.

Corollary 47 *The 2^{d-1} vertices of P_{kj} are given by all subsets of indices $I \subset J_{kj}$, namely:*

$$P^I_{kj} = w_k + \sum_{i\in I} \Lambda^i_{kj} s_i \,.$$

If we denote by $I^{\mathbf{c}} = J_{kj} \setminus I$ then also:

$$P^I_{kj} = m_k - \sum_{i\in I^{\mathbf{c}}} \Lambda^i_{kj} s_i \,.$$

In particular $P^{\emptyset}_{kj} = w_k$ and $P^{J_{kj}}_{kj} = P^{\emptyset^{\mathbf{c}}}_{kj} = m_j$. The vertices adjacent to w_j are $P^{\{i\}}_{kj} = w_k + \Lambda^i_{kj} s_i$ and the vertices adjacent to m_j are $P^{\{i\}^{\mathbf{c}}}_{kj} = m_j - \Lambda^i_{kj} s_i$. Not for all subsets I the points P^I_{kj} are the vertices of $\mathcal{H}$.

Lemma 48 *The vertices $P^{\{i\}}_{kj}$ adjacent to w_k belong to $\mathcal{H}$ if and only if $\Lambda^i_{kj} = \min_l \Lambda^i_{kl}$. Similarly the vertices $P^{\{i\}^{\mathbf{c}}}_{kj}$ adjacent to m_j belong to $\mathcal{H}$ if and only if $\Lambda^i_{kj} = \min_l \Lambda^i_{lj}$. In particular $P^{\{i\}}_{kj} \notin \mathcal{H}$, $i \in J_{kj} = J \setminus \{k, j\}$ if and only if for some $l \in J \setminus \{k, j, i\}$, $P^{\{l\}^{\mathbf{c}}}_{kj} \notin \mathcal{H}$*

Proof. The set $\mathcal{H}$ near w_k has d edges in the directions of s_i, $i \neq k$. If $P = w_k + \lambda s_i \in \mathcal{H}$, each edge belonging to $d-1$ faces P_{kj}. Then $P \in \cap_j P_{kj}$ and therefore $0 \leq \lambda \leq \Lambda^i_{kj}$ for all $j \in J_{ki}$. On the other hand the maximal such λ produces a point lying below (or on) all the faces and hence in $\mathcal{H}$. The statement for m_j follows from symmetry. Hence if $P^{\{i\}}_{kj} \notin \mathcal{H}$ then for some l:

$$\begin{aligned} 0 < \Lambda^i_{kj} - \Lambda^l_{kj} &= (v_k - v_i) \cdot (v_j - v_i) - (v_k - v_i) \cdot (v_l - v_i) \\ &= (v_k - v_i) \cdot (v_j - v_l) = (v_j - v_l) \cdot (v_k - v_l) - (v_j - v_l) \cdot (v_i - v_l) \\ &= \Lambda^l_{kj} - \Lambda^l_{ij} \,. \end{aligned}$$

which means that Λ^l_{kj} was not minimal hence $P^{\{i\}}_{kj} = m_j - \Lambda^l_{kj} s_l \notin \mathcal{H}$.

4.3 Dimension $d = 3$

In dimension 3 previous Lemma can be checked by calculations.

Corollary 49 *When $d = 3$ the point $P^{\{i\}}_{kj}$ is a vertex of $\mathcal{H}$ if and only if $(v_k - v_i)(v_j - v_l) \leq 0$.*

Proof. We will check the conditions (13) for arbitrary indices $a \neq b$:

$$(P^i_{kj} - w_a)(v_b - v_a) = (w_k - w_a + \Lambda^i_{kj} s_i)(v_b - v_a)$$
$$= (v_a - v_k)(v_b - v_a) + \begin{cases} \Lambda^i_{kj} & \text{when } a = i \\ -\Lambda^i_{kj} & b = i \\ 0 & a, b \neq i \end{cases}$$

If $a \neq i$ then both terms are non positive. In case $a = i$, and then $b \neq i$, we have $(v_i - v_k)(v_b - v_i) + (v_k - v_i)(v_j - v_i) = (v_i - v_k)(v_b - v_j)$ which can be positive only when $b = l$ which means that $(v_i - v_k)(v_l - v_j) > 0$ is the only condition when $P^{\{i\}}_{kj} \notin \mathcal{H}$.

Remark 41 *Previous Lemma has the following geometric meaning in dimension $d = 3$. The twelve faces $\mathcal{H}_{kj}$ lie on the parallelograms P_{kj} with edges in directions s_i and s_l, with $\{i, j, k, l\} = \{0, 1, 2, 3\} = J$. If $\mathcal{H}_{kj} \neq P_{kj}$ then one of the points $P^{\{i\}}_{kj}$ or $P^{\{l\}}_{kj}$ was cut off, suppose it was the former. But then we have also $P^{\{l\}^{\mathbf{c}}}_{kj} \notin \mathcal{H}$ (as there is no other choice of index left). Incidentally in dimension $d = 3$ we have $P^{\{i\}}_{kj} = P^{\{l\}^{\mathbf{c}}}_{kj}$. That means that this corner of P_{kj} was cut off and an additional edge of $\mathcal{H}$ was created. But we know the endpoints of this edge, those are $P^{\{i\}}_{kl}$ and $P^{\{l\}^{\mathbf{c}}}_{ij}$.*

We see that the condition of the face $\mathcal{H}_{kj}$ to have an extra edge is expressed as either by

- $(v_k - v_i) \cdot (v_l - v_j) < 0$,
  in which case the vertex $P^{\{i\}}_{kj} = P^{\{l\}^{\mathbf{c}}}_{kj}$ of P_{kj} is cut off by the additional edge $[P^{\{i\}}_{kl}, P^{\{l\}^{\mathbf{c}}}_{ij}]$, or by
- $(v_k - v_l) \cdot (v_i - v_j) < 0$,
  in which case the vertex $P^{\{l\}}_{kj} = P^{\{i\}\mathbf{c}}_{kj}$ of P_{kj} is cut off by the additional edge $[P^{\{l\}}_{ki}, P^{\{i\}^{\mathbf{c}}}_{kj}]$.

In dimension $d = 3$ for any k there are three products which indicate the length of edges in the direction of s_k, namely $(v_i - v_k) \cdot (v_j - v_k)$, $(v_j - v_k) \cdot (v_l - v_k)$ and $(v_l - v_k) \cdot (v_i - v_k)$, depending on the order of those numbers they determine the number of the edges of the faces of $\mathcal{H}$. Suppose that $(v_i - v_k) \cdot (v_j - v_k) < (v_j - v_k) \cdot (v_l - v_k) < (v_l - v_k) \cdot (v_i - v_k)$. Then the edge in the direction of s_k between the faces $\mathcal{H}_{ij}$ and $\mathcal{H}_{il}$ has length $(v_i - v_k) \cdot (v_j - v_k)$ and hence the face $\mathcal{H}_{ij}$ contains the point $P^{\{k\}}_{ij}$ and no additional edges at this side, while the face $\mathcal{H}_{il}$ contains an additional edge and this side.

Lemma 410 *In dimension $d = 3$ the face $\mathcal{H}_{ij}$ of $\mathcal{H}$ is a tetragon (a quadrilateral, in fact a parallelogram) iff for $k, l \neq i, j$ if for any $a \neq b$*

$$(v_i - v_k) \cdot (v_j - v_k) = \min_{a, b \neq k} (v_a - v_k) \cdot (v_b - v_k) \ \text{ and } \ (v_i - v_l) \cdot (v_j - v_l) = \min_{a, b \neq l} (v_a - v_l) \cdot (v_b - v_l)$$

The face is a hexagon when we exchange the min *by the* max *and a pentagon if the products are both the middle numbers in the corresponding three products order.*

Proof. Below we consider only the edges in the direction of s_k. Consider the edge s_k on the face $\mathcal{H}_{ij}$ from the point w_i and the parallel one from the edge m_j, then the second one coincides (at least partially) with the edge of the face $\mathcal{H}_{lj}$ which has a parallel edge at point w_l a partial common to the face $\mathcal{H}_{li}$. Suppose that $(v_i - v_k)\cdot(v_j - v_k) < (v_j - v_k)\cdot(v_l - v_k) < (v_l - v_k)\cdot(v_i - v_k)$. Then the first inequality does not produce the additional edge on the face $\mathcal{H}_{ij}$ but does on the face $\mathcal{H}_{lj}$. Similarly the second inequality does not produce the additional edge on $\mathcal{H}_{lj}$ but does on $\mathcal{H}_{il}$. That makes the face $\mathcal{H}_{lj}$ a pentagon, while using the symmetric argument to the faces $\mathcal{H}_{ij}$, $\mathcal{H}_{il}$ and $\mathcal{H}_{jl}$ we deduce that the face $\mathcal{H}_{ij}$ has no additional edges and thus is a tetragon. By central symmetry (or by exchanging the role of i and j) the face $\mathcal{H}_{ji}$ is a tetragon, the face $\mathcal{H}_{jl}$ is a pentagon and the faces $\mathcal{H}_{il}$ and $\mathcal{H}_{li}$ acquire additional edges from both directions, that is they are both hexagonal.

It is interesting to see that one condition is enough, in fact if

$$(v_i - v_k)\cdot(v_j - v_k) < (v_j - v_k)\cdot(v_l - v_k) < (v_l - v_k)\cdot(v_i - v_k)$$

then

$$(v_i - v_l)\cdot(v_j - v_l) < (v_k - v_l)\cdot(v_i - v_l) < (v_j - v_l)\cdot(v_k - v_l)$$

First inequality of the top chain is equivalent to $(v_i - v_l)\cdot(v_j - v_k) < 0$ the second one is equivalent to $(v_l - v_k)(v_j - v_i) < 0$. First inequality of the bottom chain is equivalent to $(v_i - v_l)\cdot(v_j - v_k) < 0$ and the second one to $(v_k - v_l)\cdot(v_i - v_j) < 0$. That means that if $(v_i - v_k)\cdot(v_j - v_k)$ is minimal so is $(v_i - v_l)\cdot(v_j - v_l)$ and $\mathcal{H}_i j$ is a tetragon. Similarly if we reversed the inequalities we would have $\mathcal{H}_i j$ a hexagon. That implies that if $(v_i - v_k)\cdot(v_j - v_k)$ were in the middle so must have been also $(v_i - v_l)\cdot(v_j - v_l)$, and the face would be a pentagon.

The product condition involves a pair of edges of the original simplex that have no point in common. In dimension $d = 3$ there are three such pairs, and their products are not independent.

Lemma 411 (Opposite edges condition)

$$(v_i - v_j)(v_k - v_l) + (v_i - v_k)(v_l - v_j) + (v_i - v_l)(v_j - v_k) = 0$$

Proof. Adding $0 = v_j - v_j$ to each first factor we obtain $(v_i - v_j)(v_k - v_l + v_l - v_j + v_j - v_k) + (v_j - v_k)(v_l - v_j) + (v_j - v_l)(v_j - v_k) = 0$

Remark 42 (Opposite edges convention OE) *After permuting the indices we shall always assume the following opposite edge conditions.*

$$\begin{aligned} OE1: &\quad (v_0 - v_1)(v_2 - v_3) \geq 0 \\ OE2: &\quad (v_0 - v_2)(v_3 - v_1) \geq 0 \\ OE3: &\quad (v_0 - v_3)(v_1 - v_2) \leq 0 \end{aligned}$$

By Lemma 411 either all three products are zero or the last one is negative.

Remark 43 *The following gives a geometric interpretation of the orthogonality of opposite edges in dimension $d = 3$:*
Let h_i denote the altitude of a simplex from the vertex v_i, that is a segment from v_i to its orthogonal projection onto the affine subspace containing the face $\mathbf{F}_i$. If $(v_0 - v_1)(v_2 - v_3) = 0$ then $h_0 \cap h_1 \neq \emptyset$ and $h_2 \cap h_2 \neq \emptyset$. If two of the conditions (and thus all three) are zero then all the altitudes meet at one point.

Proof. *Let M be a two dimensional plane orthogonal to the edge (v_2, v_3) and passing through the point v_0. Then M contains all the segments orthogonal to this edge and passing through v_0 in particular it contains the edge (v_0, v_1) and the altitude h_0 which is orthogonal to the face $\mathbf{F}_0$ containing (v_2, v_3). It contains also h_1 which is orthogonal to $\mathbf{F}_1$ containing (v_2, v_3). The intersection of this plane with the simplex form a triangle, whose two altitudes are h_0 and h_1 meet at one point. The statement about h_2 and h_3 is proven in a similar way.*

From previous considerations there follows:

Proposition 412 (Structure of the faces of $\mathcal{H}$ in case $d = 3$)

1. *When all three opposite edge OE conditions are not zero the twelve faces of $\mathcal{H}$ consist of:*
   four tetragons (quadrilaterals, more precisely parallelograms)ruled by $OE2$ and $OE3$: $\mathcal{H}_{01}, \mathcal{H}_{23}, \mathcal{H}_{10}, \mathcal{H}_{32}$,
   four hexagons ruled by $OE1$ and $OE3$: $\mathcal{H}_{02}, \mathcal{H}_{13}, \mathcal{H}_{20}, \mathcal{H}_{31}$, and
   four pentagons ruled by $OE2$ and $OE1$:$\mathcal{H}_{03}, \mathcal{H}_{12}, \mathcal{H}_{30}, \mathcal{H}_{21}$.
   In this case the eight points (from the tetragons) with lower indices 01 and 23 (in both orders and both possible upper indices) and the four points (from pentagons) $P_{03}^{\{1\}}, P_{21}^{\{3\}}, P_{30}^{\{2\}}, P_{12}^{\{0\}}$ are the remaining vertices of $\mathcal{H}$.
   There are six edges (additional to the 24 edges in the direction of s vectors attached to the points w and m): two common faces of two pairs of pentagons $(P_{01}^{\{2\}}, P_{23}^{\{2\}})$ and $(P_{10}^{\{3\}}, P_{32}^{\{1\}})$ (ruled by $OE2$), and four edges common to pairs of hexagons: $(P_{30}^{\{2\}}, P_{21}^{\{3\}}), (P_{12}^{\{0\}}, P_{03}^{\{1\}})$ (ruled by $OE1$), and $(P_{32}^{\{0\}}, P_{01}^{\{3\}}), (P_{10}^{\{2\}}, P_{23}^{\{1\}})$ (ruled by $OE3$). Of the additional edges of the hexagons the first and the second connect the tetragons, while the third and the fourth connect the pentagons.
   We have 12 faces, 12 edges from the points w in the direction of the vectors s, 12 edges from the points m in the direction of the vectors $-s$ and 6 additional edges, making 30 edges total, and we have 4 w vertices, 4 m vertices and 12 P vertices making 20 vertices total. Each vertex has three edges, the P points have each one s edge, one $-s$ edge and one edge to another P point.
2. *A typical bifurcation can occur uniquely by changing the sign of either OE1 and then the hexagons become pentagons and pentagons become hexagons or by changing the sign of $OE2$ and then the pentagons become tetragons and tetragons become pentagons.*

3. *When the condition $OE1$ is zero then the hexagons become pentagons resulting in four tetragons and eight pentagons.*
   The third and fourth hexagonal edges collapse to one point each, producing two vertices with four edges collecting four pentagons around such a vertex. That gives 12 faces, 28 edges and 18 vertices. It is not a tile.
4. *When the condition $OE2$ is zero then the pentagons become tetragons resulting in eight tetragons and four hexagons.*
   The pentagonal edges collapse to one point each, producing two vertices with four edges collecting four tetragons.
   That gives 12 faces, 28 edges and 18 vertices. Then the set $\mathcal{H}$ is a tile. It is a hexa-rhombic dodecahedron.
5. *When all three condition are zero then each additional edge collapses to a point, leaving six vertices with four edges.*
   Each face becomes a tetragon. That gives 12 faces, 24 edges (no additional ones) and 14 vertices (6 points P). The set $\mathcal{H}$ is a tile. This is a rhombic dodecahedron.

4.4 The tile $\mathcal{T}$, the set $\mathcal{H}$ cut by two additional half-spaces

We are in dimension $d = 3$. For $k \neq i$ define

$$w_i^k = w_i + \lambda^k s_k \text{ and } m_i^k = m_i - \lambda^k s_k$$

Lemma 413 *If $\lambda_k = \min_{ab}(v_a - v_k) \cdot (v_b - v_k) = (v_i - v_k) \cdot (v_j - v_k)$ then*

$$\begin{aligned} m_j^l &= w_i^k & m_l^j &= w_i^k \\ m_j^k &= w_i^l & m_l^k &= w_i^l \\ w_j^l &= m_i^k & w_l^j &= m_i^k \\ w_j^k &= m_i^l & w_l^k &= m_i^l \,. \end{aligned}$$

Proof. Under the assumption by Lemma 410 the face $\mathcal{H}_{ij}$ is a tetragon hence $w_i^k = w_i + \lambda^k s_k = P_{ij}^{\{k\}} = P_{ij}^{\{l\}^c} = m_j - \lambda_l s_l = m_j^l$. The other equalities follow similarly.

Lemma 414 *Under Convention on OE (Remark 42) in dimension $d = 3$ we have:*

$$\begin{aligned} \lambda^0 &= \lambda^0_{23} = (v_2 - v_0) \cdot (v_3 - v_0) \\ \lambda^1 &= \lambda^1_{23} = (v_2 - v_1) \cdot (v_3 - v_1) \\ \lambda^2 &= \lambda^2_{01} = (v_0 - v_2) \cdot (v_1 - v_2) \\ \lambda^3 &= \lambda^3_{01} = (v_0 - v_3) \cdot (v_1 - v_3) \,. \end{aligned}$$

Proof. Direct computations from OE. Remark that those are the coefficients for the edges of two pairs of tetragonal faces $\mathcal{H}_{23}$ with $\mathcal{H}_{32}$ and $\mathcal{H}_{01}$ with $\mathcal{H}_{10}$.

Definition 415 (Separated indices) *Under the convention OE we partition the set of indices $\{0, 1, 2, 3\}$ into $\{0, 1\} \cup \{2, 3\}$. An upper index i and a lower index j are said* separated *if they belong to two different parts of the partition.*

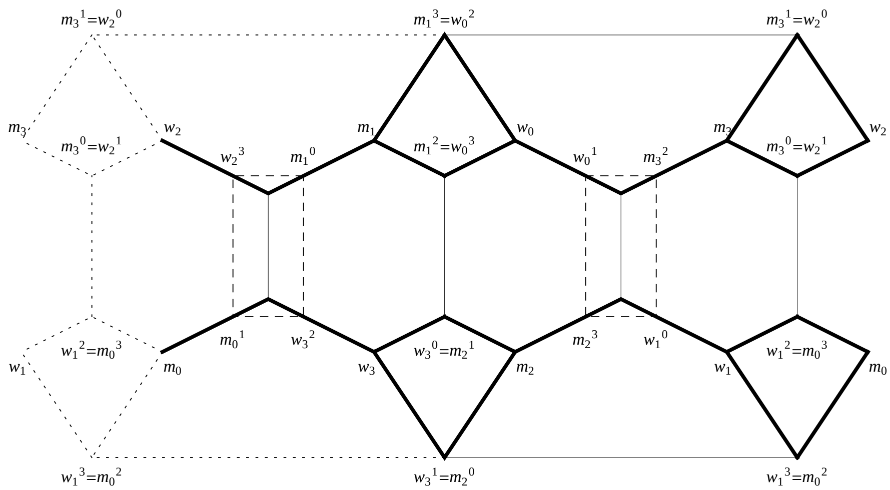

Fig. 3. The combinatorial structure of the set $\mathcal{T}$. The dotted lines are glued (identified) with corresponding continuous ones. The thick lines correspond to the edges in the direction of one of the s vectors. The thin lines are the remaining edges of $\mathcal{H}$. The dashed boxes show the edges of $\mathcal{T}$ which are not the edges of $\mathcal{H}$, they represent the two additional tetragones (hence the edges inside the boxes are cut off). The proportions are distorted: all horizontal edges of $\mathcal{T}$ have equal length and all vertical edges of $\mathcal{T}$ have equal length.

Lemma 416 *The points w_i^j (and by symmetry m_i^j) belong to tetragonal faces of $\mathcal{H}$ if they have separated indices and are additional vertices of $\mathcal{T}$ if their indices are not separated.*

Proof. By inspection.

Lemma 417 *Under the conventions $OE1$ and $OE2$, in the following four groups of all four points within the group are equivalent with respect to* $\mathbf{L}$.

$$\begin{aligned}
&w_3^0 \sim w_1^0 \sim w_2^0 = m_3^1 \sim m_0^1 \sim m_2^1 (= w_3^0)\\
&w_2^1 \sim w_0^1 \sim w_3^1 = m_2^0 \sim m_1^0 \sim m_3^0 (= w_2^1)\\
&w_1^2 \sim w_3^2 \sim w_0^2 = m_1^3 \sim m_2^3 \sim m_0^3 (= w_1^2)\\
&w_0^3 \sim w_3^2 \sim w_1^3 = m_0^2 \sim m_3^2 \sim m_1^2 (= w_0^3)\,.
\end{aligned}$$

Remark that the middle points w and m do not have their equal counterparts.

Proof. As $w_i - w_j = v_j - v_i$ all the w points are $\mathbf{L}$-equivalent, similarly $m_i - m_j = v_i - v_j$. Therefore all the three points $w_i^k = w_i + \lambda^k s_k$ with the same upper index are equivalent. Similarly all three m_j^l points with the same upper index are equivalent. Lemma 413 merges the groups.

Lemma 418 (All vertical and all horizontal segments are equal) *Under the conventions $OE1$ and $OE2$ there are following two groups of six equal vectors:*

Vertical in Figure 4.4

$$m_3^0 - w_1^2 \boxed{=} w_2^1 - m_0^3 = w_2^3 - m_0^1 = w_0^1 - m_2^3 = w_0^3 - m_2^1$$
$$m_3^0 - w_1^2 = m_3^2 - w_1^0 = m_1^0 - w_3^2 = m_1^2 - w_3^0 \boxed{=} w_0^3 - m_2^1$$

Horizontal in Figure 4.4

$$m_3^1 - w_0^2 \boxed{=} w_2^0 - m_1^3 = w_2^3 - m_1^0 = w_1^0 - m_2^3 = w_1^3 - m_2^0$$
$$m_3^1 - w_0^2 = m_3^2 - w_0^1 = m_0^1 - w_3^2 = m_0^2 - w_3^1 \boxed{=} w_1^3 - m_2^0$$

Remark that compared to Figure 4.4 some vectors seem to have reversed order. This is due to the fact that after gluing the solid together these vectors will go "behind" $\mathcal{T}$. The boxed equality $\boxed{=}$ refers to the fact that the endpoints are the same, for example in $m_3^0 - w_1^2 \boxed{=} w_2^1 - m_0^3$ we have $m_3^0 = w_2^1$ and $w_1^2 = m_0^3$, that is the equality happens in the affine space as well as in vector space.

Proof. Each of these vectors is written as a difference of an m and a w point. Each can be expressed in a symmetric way. The "vertical" non boxed equalities follows from:

$$2\mathcal{O} - v_2 - v_0 + \lambda^1 s_1 + \lambda^3 s_3 = w_2^1 - m_0^3 = w_2^3 - m_0^1 = w_0^3 - m_2^1 = w_0^1 - m_2^3$$

and similar expressions with lower indices $1, 3$ and upper $0, 2$. The link between the two is given by the boxed equalities which result from Lemma 413. Similarly one calculates the "horizontal" chain of equalities.

Corollary 419 *Assume the convention OE. The points with no separated indices: w_2^3, m_0^1, m_1^0, w_3^2 form a parallelogram and hence lie on the same two-dimensional plane. They form a face of $\mathcal{T}$. The same statement holds for the points $w_0^1, m_2^3, m_3^2, w_1^0$. For the first four points the plane is given by*

$$\{x : -(x-g)^2 = (x-(g+r))^2\} \quad r = v_2 + v_3 - v_1 - v_0\,,$$

for the second four points take $r = v_0 + v_1 - v_2 - v_3$.

Proof. The parallelogram statement was proven in the previous Lemma. Using the properties of $\mathcal{O}, s_1$ and λ^1 we get for w_0^1:

$$\begin{aligned}
&(w_0^1 - g)^2 - (w_0^1 - g - r)^2 = (2\mathcal{O} - 2v_0 + 2\lambda^1 s_1 - r)\cdot r \\
&= 2\lambda^1 s_1 \cdot r + (2\mathcal{O} - 2v_0)\cdot(v_0 - v_2) + (2\mathcal{O} - 2v_0)\cdot(v_1 - v_3) + r^2 \\
&= -2\lambda^1 + (v_2 - v_0)\cdot(v_0 - v_2) + 2(v_1 - v_0)\cdot(v_1 - v_3) + (v_3 - v_1)\cdot(v_1 - v_3) + r^2 \\
&= -2(v_2 - v_1)\cdot(v_3 - v_1) + 2(v_1 - v_0)\cdot(v_1 - v_3) + 2(v_0 - v_2)\cdot(v_1 - v_3) \\
&= 2(v_2 - v_1 + v_1 - v_0 + v_0 - v_2)\cdot(v_1 - v_3) = 0\,.
\end{aligned}$$

The computation for all other points is similar and will be skipped.

Given $g \in A$, for any $r \in \mathbf{L}$ let $\mathbf{M}(r)$ be the (closed) half space of points closer to g than to $g + r$.

$$\mathbf{M} = \mathbf{M}(r) = \{x : (x-g)^2 \le (x-(g+r))^2\} = \{x : (2x - 2g - r)\cdot r \le 0\}\,.$$

Remark 44 *The set*

$$\bigcap_{r\in\mathbf{L}} \mathbf{M}(r)$$

consists of points which are closer to g then to any of it lattice translates. It is called a Voronoï cell of the point g with respect to the lattice $\mathbf{L}$. It is a closed, convex, bounded set. It is a closure of a (Voronoï) tile with which it shares the interior.

Lemma 420 *In any dimension d:*

$$\mathcal{H} = \bigcap_{ij} \mathbf{M}(v_i - v_j)$$

Proof. This is geometrically well understood. Each $\mathbf{V}_i$ is the cone, an intersection of half spaces of points closer to v_i than to any other vertex. After translation by a vector $g - v_i$ an intersecting all such translates we recover $\mathcal{H}$. Computation follows:

$$\begin{aligned}
\mathcal{H} &= \bigcap_i \mathcal{F}(\mathbf{V}_i) = \bigcap_i (\mathbf{V}_i + g - v_i)\\
&= \bigcap_i \left(\bigcap_j \left(\{y : (y - v_i)^2 - (y - v_j)^2 \le 0\}\right) + g - v_i \right)\\
&= \bigcap_i \left(\bigcap_j (\{y : (2y - v_i - v_j)\cdot(v_j - v_i) \le 0\}) + g - v_i \right)\\
&= \bigcap_i \left(\bigcap_j \{x = y + g - v_i : (2(x - g + v_i) - v_i - v_j)\cdot(v_j - v_i) \le 0\} \right)\\
&= \bigcap_{ij} \{x : (2x - 2g - (v_j - v_i))\cdot(v_j - v_i) \le 0 = \bigcap_{ij} \mathbf{M}(v_j - v_i)\,.
\end{aligned}$$

Lemma 421 *In dimension $d = 3$ assuming convention OE the two additional (as compared to $\mathcal{H}$) faces of $\mathcal{T}$ lie on the boundaries of $\mathbf{M}(r)$ and $\mathbf{M}(-r)$ with $r = v_3 + v_2 - v_1 - v_0$.*

Proof. This was proven in Corollary 419.

Corollary 422 *The points with no separated indices belong to $\mathbf{M}(r) \cap \mathbf{M}(-r)$, with r from the previous Lemma.*

Proof. By geometry. The two groups of points are symmetric to each other with respect to g, and so are $\mathbf{M}(r)$ and $\mathbf{M}(-r)$. But both half planes contain g therefore $\mathbf{M}(-r)$ contains the symmetric image of $\partial\mathbf{M}(r)$ and hence the first group of four points with separated indices and $\mathbf{M}(r)$ contains $\partial\mathbf{M}(-r)$ and hence the second group. Interested reader is welcome to perform the computation on inequalities.

Remark 45 *Remark that the faces of $\mathcal{H}$ which are adjacent to the additional face w_0^1,m_2^3,w_1^0,m_3^2 with external normal vector $r = v_3 + v_2 - v_1 - v_0$ are $\mathcal{H}_{12}, \mathcal{H}_{13}, \mathcal{H}_{03}, \mathcal{H}_{02}$ with external normal vectors $v_2 - v_1, v_3 - v_1, v_3 - v_1, v_2 - v_1$.*

From geometrical point of view we have just proven that the two additional half spaces $\mathbf{M}(r)$ and $\mathbf{M}(-r)$ cut off the edge of $\mathcal{H}$ joining two pentagonal faces along hexagonal ones. But to be sure that we do not rely too much on the intuition we provide an algebraic proof.

Lemma 423 (w and m inside additional cuts) *If $r = v_i + v_j - v_k - v_l$, then, in an edge-wise acute simplex, for every $t \in i, j, k, l$ we have $w_t, m_t \in \mathbf{M}$.*

Proof. First note that by symmetry for every t: $w_t - g = \mathcal{O} - v_t = g - m_t$ and the lengths of these two vectors are equal (and equal for all t). Moreover $(m_t-(g-r))^2 = (r-(g-m_t))^2 = (-(w_t-g)+r)^2 = (w_t-(g+r))^2$. Thus the statement about m_t and r follows from the statement about w_t and $-r$. We represent the inequality defining $\mathbf{M}$ as $0 \le (x-g+r)^2-(x-g)^2 = (2(x-g)+r)\cdot r$. As $w_t = \mathcal{O} + g - v_t$, we have:

$$
\begin{aligned}
&(2(w_t - g) + r) \cdot r \\
&= (2\mathcal{O} - v_i - v_k + v_i + v_k - 2v_t + r) \cdot (v_i - v_k) \\
&\quad +(2\mathcal{O} - v_j - v_l + v_j + v_l - 2v_t + r) \cdot (v_j - v_l) \\
&= (v_i + v_k - 2v_t + r) \cdot (v_i - v_k) + (v_j + v_l - 2v_t + r) \cdot (v_j - v_l) \\
&= 2(v_i - v_t) \cdot (v_i - v_k) + (v_j - v_l) \cdot (v_i - v_k) + 2(v_j - v_t) \cdot (v_j - v_l) \\
&\quad +(v_i - v_k) \cdot (v_j - v_l) \\
&= 2\left((v_i - v_t) \cdot (v_i - v_k) + (v_j - v_l) \cdot (v_i - v_k) + (v_j - v_t) \cdot (v_j - v_l)\right) \\
&= 2 \left\{ \begin{array}{l} t = i \ (v_j - v_l) \cdot (v_j - v_k) \\ t = j \ (v_i - v_l) \cdot (v_i - v_k) \\ t = k \ (v_i - v_l) \cdot (v_i - v_k) + (v_j - v_k) \cdot (v_i - v_k) + (v_j - v_k) \cdot (v_j - v_l) \\ t = l \ (v_i - v_l) \cdot (v_i - v_k) + (v_j - v_l) \cdot (v_i - v_l) + (v_j - v_l) \cdot (v_j - v_k) \end{array} \right\} \ge 0\,,
\end{aligned}
$$

by edgewise acuteness.

Lemma 424 *Under OE set $r = v_0 + v_1 - v_2 - v_3$. The points w_i^j (and by symmetry m_i^j) with separated indices belong to $\mathbf{M}(r)$ and to $\mathbf{M}(-r)$.*

Proof. Let us calculate for w_0^3 and r, other calculations are similar (or simpler).

$$\begin{aligned}
&(w_0^3 - g)^2 - (w_0^3 - (g + r))^2 = (2\mathcal{O} - 2v_0 + 2\lambda^3 s_3 - r) \cdot r \\
&= 2\lambda^3 s_3 \cdot r + (2\mathcal{O} - 2v_0) \cdot r - r^2 = 2\lambda^3 + (2\mathcal{O} - v_0 - v_2) \cdot (v_0 - v_2) - (v_0 - v_2)^2 \\
&\quad + (2\mathcal{O} - v_1 - v_3) \cdot (v_1 - v_3) + (v_1 + v_3 - 2v_0) \cdot (v_1 - v_3) - ((v_0 - v_2) + (v_1 - v_3))^2 \\
&= 2(v_1 - v_3) \cdot (v_0 - v_3) - 2(v_0 - v_2)^2 + (v_1 - v_3) \cdot (v_1 + v_3 - 2v_0 - (v_1 - v_3) - 2(v_0 - v_2)) \\
&= 2(v_1 - v_3)(v_2 - v_0) - 2(v_0 - v_2)^2 = 2((v_1 - v_0 + v_0 - v_3)(v_2 - v_0) - (v_0 - v_2)^2) \\
&= 2((v_0 - v_2) \cdot (-v_1 + v_0 - v_0 + v_2) + (v_0 - v_2) \cdot (-v_0 + v_3)) \\
&= 2((v_0 - v_2) \cdot (v_2 - v_1) + (v_0 - v_2) \cdot (v_3 - v_0)) \leq 0 \text{ by edgewise acuteness.}
\end{aligned}$$

Remark 46 *In the last lines of the previous proof we used the following trick: In an edgewise acute simplex*

$$\text{If } p = v_a - v_b, q = v_c - v_d \text{ then } p \cdot q - p^2 \leq 0$$

which follows from writing $q = v_c - v_b + v_b - v_d$*:*

$$\begin{aligned}
&(v_c - v_b) \cdot (v_a - v_b) + (v_b - v_d)(v_a - v_b) - (v_a - v_b) \cdot (v_a - v_b) \\
&= (v_c - v_a) \cdot (v_a - v_b) + (v_b - v_d)(v_a - v_b) \leq 0
\end{aligned}$$

Proposition 425 *In dimension* $d = 3$*:*

$$\mathcal{T} = \bigcap_{ijkl} \mathbf{M}(v_i + v_j - v_k - v_l)$$

In fact under OE, with $r = v_0 + v_1 - v_2 - v_3$*:*

$$\mathcal{T} = \mathcal{H} \cap \mathbf{M}(r) \cap \mathbf{M}(-r)$$

Proof. We have proven that all the vertices belong to the convex intersection $\cap \mathbf{M}(r)$ for appropriate subset of vectors r. Also we have proven that the faces of $\mathcal{T}$ belong to the boundaries $\partial \mathbf{M}(r)$ for appropriate r. In particular $r = v_i - v_j$ for the faces of $\mathcal{T}$ which are parts of the faces or $\mathcal{H}$ and the two additional faces are on the boundaries of $\mathbf{M}(r)$ for $\pm r = v_0 + v_1 - v_2 - v_3$.

Theorem 426 *For a face wise acute simplex in dimension* $d = 3$ *the set* $\mathcal{T}$ *is a closure with shared interior of a tile for the simplex lattice* $\mathbf{L}$.

Proof. It is enough to prove that the translations by lattice vectors of $\mathcal{T}$ fill the space around each vertex, with fitting edges and faces. We work under OE. There are six groups of four equivalent vertices:

1. Two groups of: four w_i and four m_i points.
2. Four groups of: two vertices with separated indices $w_i^k = m_j^l$ and $w_j^k = m_i^l$ together with two veritces with no separated indices w_l^k and m_k^l, were the choice is determined by the the direction of an edge (upper index k) of a point w, and the points m are determined by completion.

In general to a fixed (original) vertex of each group we translate three other equivalent vertices. Each of the translated points brings an adjacent face symmetric (with opposite external normals) to one of the three faces of the original vertex. Those paired faces share two equal (vector) edges. The remaining faces of translated points fit pairwise with each other around a "sticking out" edge which is common to all translated vertices but absent at the original one. In case of the points w and m all the edges are s_i edges of the same length λ^i with the "sticking out" edge being the s vector with the same index as the original point. In case of the doubly indexed points they share two s directions and a "horizontal" and a "vertical" one.

- Consider w_0 which is lattice equivalent to any w_i and translate each such vertex w_i to w_0 by $v_i - v_0$. Then each face $\mathcal{T}_{0i}$ adjacent to w_0 will be matched with the translated face $\mathcal{T}_{i0}$. Remark the change of order in the indices, which shows that the faces are matched with opposite external vectors. Each edge starting at w_0 in the direction s_j, $j \neq 0$ will be common with two such translated edges from the faces $\mathcal{T}_{i0}$, $i \neq j$, and they all share the length λ^j. There will be an additional edge, common to all three translated faces in the direction of s_0, which "sticks out" from $\mathcal{T}$. We recover the partition of A near $\mathcal{O}$ translated by $g - v_0$. Similar argument works for all other points w and by symmetry m.
- Consider now the point $w_0^3 = m_1^0$ with separated indices. It is adjacent to the faces $\mathcal{T}_{01}, \mathcal{T}_{31}, \mathcal{T}_{02}$, with three edges $\lambda^2 s_2$, $-\lambda_3 s_3$ and the additional "vertical" edge $(v_0 - v_3) + \lambda^0 s_0 - \lambda^3 s_3$. The points equivalent to $w_0^3 = w_0 + \lambda^3 s_3$ are all points w_i^3 and m_j^0. There are three such points in addition to w_0^3 itself. Each of such point is adjacent to a face symmetric to one of the faces adjacent to w_0^3 with the pair of adjacent edges equal (due to common upper index in case of s edges and an equal "vertical" edge). There is an additional edges "sticking out" common to all the other points which is "horizontal". Remark how the symmetric additional two faces stick together with a non s edge of $\mathcal{H}$ filling a wedge between them.

5 Background

Knuth [16] gives a formal definition of an algorithm as a computational method which involves the iteration of a function (a dynamical system) which takes an input and after a finite number of steps produces a fixed point which determines an output. This notion can be expanded to include algorithms which we call *streaming algorithms* which transform sequences of inputs into sequences of outputs by means a family of parameterized maps acting on a space internal states, the inputs supplying the parameter, the outputs being evaluations of some function of internal states. Such dynamical systems need not settle down to a fix point or even a periodic orbit. The action on internal states with nonconstant inputs is a time dependent dynamical system and with constant inputs a time independent one. All this suggests that one might find interesting dynamical systems and associated concepts to analyze algorithms which occur in *grey scale and color printing*, *resource management*, *game theory*, *signal* *processing* (*e.g. sigma-delta modulators*), *scheduling*, *et al.* Dynamical systems that have often turned up are ones employing piecewise isometries of Euclidean space, actually piecewise translations and rotations.

The problem studied in the present work is one derived from an algorithm used in digital printing. This algorithm goes under the generic name of *error diffusion* and is described in [10], [14]Here the usefulness of applying dynamical system concepts is apparent. EDA is an approximation algorithm, the outputs being discrete versions of continuous inputs and the differences forming errors which can accumulate in time. The internal state space is one dimensional for grey scale printing with the inputs coming from an interval. It is three dimensional for color printing with the inputs coming from a polytope with five or more vertices [1], [3].The family of maps are piecewise translations of Euclidean space. The bounding of orbits is important because it leads to the result that the time averages of inputs and outputs become arbitrary close. This is enough to allow the use of a few printable output colors to be a good approximations to the continuous color input palette. The proof of boundedness of orbits, which is easy in one dimension but difficult in three and higher, was achieved in [2].

After boundedness is guaranteed, the quality of the algorithm can be investigated. For instance, EDA is known to be the best algorithm in the class of online, greedy, algorithms on the standard simplex [8]. Of course, its outputs are not optimal (as can be obtained by non-algorithmic methods (that require unbounded look ahead): see [21] and references therein).

In the study of dynamical systems an important consideration is the description of invariant sets. With respect to invariant sets for piecewise translations we cite [12], [13], [18] for constant input and [19] for nonconstant input. Further works on the constant input case are [5] and in preparation [7].

EDA on an interval is reduced to rotation on a circle which is connected to Diophantine approximations. This was one of the motivating factors to [5] and the present work. In [5] there is a stronger two dimensional result than in the present work in that the assumption of ergodicity is not required. However [7] will contain a stronger two dimensional result than [5]: namely the invariant set is a polygonal tile as are its intersection with any union of Voronoï domains, and the dynamic translates each such subtile providing a different partition of the invariant set with isometric pieces. This in turn connects EDA with the interval exchange maps.

In scheduling algorthims such as in the so-called *Carpool problem* the polytope may vary among a finite collection. It leads one to study the dynamics in the *error space*, the vector space that model the affine space for the family of polytopes under consideration. This change of viewpoint is necessitated by the fact that there is no invariant region in the affine space where the polytopes are defined but there is one in the vector space (see [8], [22], [23]).

One finds algorithms whose dynamics use piecewise translations in other settings like game theory [20]. In *sigma-delta modulators* boundedness results in [9] correspond to boundedness results for EDA in [2]. Similarly tiling results for sigma-delta modulators with ergodic inputs proven in [13] correspond to the tiling properties of EDA proven in the present work. In other works in *digital filters* [17] piecewise rotations were studied. Piecewise rotations were investigated in their own right in [4], [11], [12].

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