Left invertibility of chaotic systems

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Abstract. When chaotic systems is used as support for secure data transmission, singularity of observability and left invertibility is an important problem. In this paper, data secure transmission is analyzed with respect to the observability and left invertibility concept. Moreover, in order to overcome observability and left invertibility singularities, a immersion technique is proposed. The use of high order sliding mode observer on a well known Qi circuit, allows to highlight the well founded of the proposed analysis and method.
Keywords: Chaotic systems, secure data transmission, Observability , Singularity, left invertibility , Immersion.

1 Introduction

There are lot of links between Chaos theory, synchronization and observation and many authors have studied for example, the synchronization of chaotic systems [1][18][22].... or the link between observation and synchronization [17][15], or the observability of chaotic system [16]. But, at the best of our knowledge, there are few papers about left invertibility of chaotic systems. This lacke of study is due to the fact, that at the beginning of chaotic cryptography, the method used was additional method [7][6] which based on the synchronization of two chaotic systems with an addition on the chaotic signal transmitted via the public channel the message. Obviously, this seminal method was not secure at all and some other methods was developed has for example the inclusion method [25][2] where the message is included directly in the dynamic

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of the chaotic system. In this context, the message recovering (estimation) by the receiver is a left inverse problem but this problem has not be studied formally. This is due to the fact that problems as preservation of chaotic behavior under the perturbation introduced by the message and lose of observability have received more attentions. Nevertheless, the study of left invertibility of chaotic system is of first importance and this not only for cryptography by inclusion. For example, the diagnosis of chaotic system can be treated by left invertibility approach when the fault can be considered as unknown input. But, if this problem was studied for classical dynamical system [20] for chaotic system some extra difficulties appear, as the observability singularity [4]. This singularity, under some specific conditions, this singularity can be overcome by immersion techniques [10][11][3]. From all of these considerations, in this paper we will first recall some important results with respect to left invertibility and immersion after that we will used their in the context of chaotic system. More precisely, we will consider a Qi circuit [23] in two configurations: firstly the case of one unknown input and two outputs is studied and secondly the case of two outputs and two unknown inputs is addressed . In the first case some immersion techniques can be use contrarily to the second one for which it is impossible to use such technique, and only some saturations are introduced in order than the estimated unknown input does not go fare a way went the system trajectory cross a set of observability or left invertibility singularity. In order to highlight the well-founded of our approach both cases are simulated on Mathlab. The paper is organized has followed: In the next section some observability concepts, left invertibility and observability singularity definitions are recalled. In section 3, left invertibility singularity is presented. After that, some recalls on HOSM differentiator are given in section 4. In section 5, two cases are simulated, with respect to Qi circuit. The paper ends with some conclusions.

2 Some recalls and definitions

We consider the following MIMO dynamical system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y_i &= h_i(x) \;\; i = 1, \ldots, m \end{aligned} \tag{1}$$

Where $x \in \mathbb{R}^n$ represents the state, $y \in \mathbb{R}^m$ represents the output and $u \in \mathbb{R}^p$ is the vector of inputs, perturbations or faults. $f(x), h(x)$ and $g(x)$ are supposed to be C^∞. Let us introduce some definitions for a MIMO dynamical system

Definition 1 [8] For the MIMO nonlinear system described by (1), r_1, the relative order of the output y_i, with respect to the manipulated input vector u is the smallest integer for which

$$L_g L_f^{r_i-1} h_i(x) = \left[L_{g_1} L_f^{r_i-1} h_i(x), L_{g_2} L_f^{r_i-1} h_i(x), \cdots, L_{g_m} L_f^{r_i-1} h_i(x) \right] \neq [0, 0, \cdots, 0] \tag{2}$$

Definition 2 [19] For the MIMO nonlinear system described by (1), with finite relative orders $r_1, \cdots, r_m$ the matrix

$$\Gamma(x) = \begin{pmatrix} L_{g_1} L_f^{r_1-1} h_1 & \cdots & L_{g_m} L_f^{r_1-1} h_1 \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m & \cdots & L_{g_m} L_f^{r_m-1} h_m \end{pmatrix}$$

is called the characteristic matrix of the system.

Exemple 1 Let us consider the following MIMO dynamical system given in [19]:

$$f(x) = \begin{pmatrix} -k_1(x_3)x_1 - k_3(x_3)x_1^2 \\ k_1(x_3)x_1 - k_2(x_3)x_2 \\ \frac{1}{\rho c_p}\left[(-\Delta H_1)k_1(x_3)x_1 + (-\Delta H_2)k_2(x_3)x_2 + (-\Delta H_3)k_3(x_3)x_1^2\right] \end{pmatrix}$$

$$g_1(x) = \begin{pmatrix} C_a - x_1 \\ -x_2 \\ \frac{T_0 - x_3}{\rho c_p} \end{pmatrix} \quad g_2(x) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{V \rho c_p} \end{pmatrix}$$

Applying definition 1 for the calculation of the relative orders, we find:

$$\begin{aligned} L_{g_1} h_1 &= -x_2 \\ L_{g_1} h_2 &= \tfrac{T_0 - x_3}{\rho c_p} \\ L_{g_2} h_1 &= 0 \\ L_{g_2} h_2 &= \tfrac{1}{V \rho c_p} \end{aligned} \tag{3}$$

Consequently, the relative orders are $r_1 = 1$ and $r_2 = 1$

Definition 3 The system (1) is locally observable if the states x can be expressed as follows according to the outputs and of their derivative:

$$x(t) = \phi_x(y(t), ..., y^{(k)}(t)), \; k \in \mathbb{Z}^+. \tag{4}$$

where $x(t)$ is locally the only solution of (4). [24]

Now we speaks about observability singularity of a dynamical system for a certain order of derived of the outputs when its matrix of observation is not invertible at all point of original space.

Definition 4 In k order the matrix of observability is given by:

$$dO = \begin{pmatrix} dh(x) \\ dL_f h(x) \\ \vdots \\ dL_f^{n-1} h(x) \\ \vdots \\ dL_f^k h(x) \end{pmatrix}$$

with $dL_f^i h(x) = (\frac{\partial L_f^i h(x)}{\partial x_1}, \frac{\partial L_f^i h(x)}{\partial x_2}, \ldots, \frac{\partial L_f^i h(x)}{\partial x_n})$ The l-form associated with the usual Lie derivative. The singular observability manifold S_O for the order k is defined as [16]:

$$S_{O_{(k)}} = \{x \in \mathbb{R} | Rank\{dO\} < n\} \tag{5}$$

Exemple 2 Let us consider the following simple system:

$$\begin{aligned} \dot{x}_1 &= x_2 + x_2^2 \\ \dot{x}_2 &= -x_2^3 + 1 \\ \dot{x}_3 &= x_2 - x_2^3 \end{aligned} \tag{6}$$

with $y_1 = x_1$ and $y_2 = x_3$.
It can be seen that, if we choose the corresponding observability indices as $(2, 1)$, then

$$dO_{(2,1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + 2x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we obtain the following observability singularity set:

$$S_{(2,1)} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = -0.5\right\} \tag{7}$$

On the other hand, if the observability indices were chosen as $(1, 2)$, then

$$dO_{(1,2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 - 3x_2^2 & 0 \end{pmatrix}$$

and the observability singularity set is now given by:

$$S_{(1,2)} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2 = \frac{\mp 1}{\sqrt{3}}\right\} \tag{8}$$

It is clear that both observability matrices $dO(1, 2)$ and $dO(2, 1)$ contain singularities, but they are not the same. In order to overcome those singularities, one can compute further derivatives of the output. Indeed, consider the observability indices $(3, 1)$, then we obtain

$$dO_{(3,1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + 2x_2 & 0 \\ 0 & -8x_2^3 - 3x_2^2 + 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $S(3, 1) = \emptyset$, i.e. there is no longer any observability singularity. These highlight the fact that the choice of the observability indices is crucial in designing an observer for a nonlinear system.
Let us now recalls some definition for the invertibility of a dynamical system.

Definition 5 The system (1) is said to be locally invertible at x_0 if we can reconstruct in the neighborhood of x_0 its inputs from its outputs and their derivatives [20] [5].

Definition 6 The unknown input $u(t)$ of the system (1) can be estimated if it can be expressed locally as [10][11]:

$$u(t) = \phi_u(y(t), ..., y^{(k)}(t)), \ k \in \mathbb{Z}^+. \tag{9}$$

Exemple 3 Let us consider the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + x_4^2 + u \\ \dot{x}_3 &= x_1 x_4 + u \\ \dot{x}_4 &= -x_3 \end{aligned} \tag{10}$$

With $y_1 = x_1$ and $y_2 = x_3$. We first analyze observability and after that study the left inversibility and finally determine the unknown input when the system is invertible. The successive derivatives of the output y_1 and y_2 of the system (10) give us:

$$\begin{aligned} y_1 &= x_1 \\ \dot{y_1} &= x_2 \\ y_2 &= x_3 \\ \dot{y_2} &= x_1 x_4 \end{aligned}$$

the matrix of observability limited here to $n = 4$ of the system gives us:

$$dO_{(2,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_4 & 0 & 0 & x_1 \end{pmatrix}$$

It is easy to see that the determinant is equals to zero for $x_1 = 0$. The system (4) with the indices of observability $(2, 2)$ is locally weakly regularly observable excepted on the set of singularity

$$S_{O_0} = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1 = 0\right\} \tag{11}$$

Then the directly accessible states are:

$$\begin{aligned} y_1 &= x_1 \\ \dot{y_1} &= x_2 \\ y_2 &= x_3 \end{aligned}$$

The variables to be estimated are x_4 and u, x_4 is accessible only if $x_1 \neq 0$, now consider the matrix Ω combining the information linked to the state and the input with the characteristic number of y_1 and y_2.

$$\Omega = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} & \frac{\partial y_1}{\partial x_4} & \frac{\partial y_1}{\partial u} \\ \frac{\partial \dot{y}_1}{\partial x_1} & \frac{\partial \dot{y}_1}{\partial x_2} & \frac{\partial \dot{y}_1}{\partial x_3} & \frac{\partial \dot{y}_1}{\partial x_4} & \frac{\partial \dot{y}_1}{\partial u} \\ \frac{\partial \ddot{y}_1}{\partial x_1} & \frac{\partial \ddot{y}_1}{\partial x_2} & \frac{\partial \ddot{y}_1}{\partial x_3} & \frac{\partial \ddot{y}_1}{\partial x_4} & \frac{\partial \ddot{y}_1}{\partial u} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} & \frac{\partial y_2}{\partial x_4} & \frac{\partial y_2}{\partial u} \\ \frac{\partial \dot{y}_2}{\partial x_1} & \frac{\partial \dot{y}_2}{\partial x_2} & \frac{\partial \dot{y}_2}{\partial x_3} & \frac{\partial \dot{y}_2}{\partial x_4} & \frac{\partial \dot{y}_2}{\partial u} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2x_4 & 1 \\ x_4 & 0 & 0 & x_1 & 1 \end{pmatrix}$$

The left inversion matrix Ω_0 is given by:

$$\Omega_0 = \begin{pmatrix} \frac{\partial \ddot{y}_1}{\partial x_4} & \frac{\partial \ddot{y}_1}{\partial u} \\ \frac{\partial \dot{y}_2}{\partial x_4} & \frac{\partial \dot{y}_2}{\partial u} \end{pmatrix} = \begin{pmatrix} 2x_4 & 1 \\ x_1 & 1 \end{pmatrix}$$

Then this system admits a set of singularities S_{01} such that

$$S_{O_1} = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1 = 2x_4\right\} \tag{12}$$

To reduce the size of this set of singularities we can combine the outputs as follows:
Let $\tilde{y}$ be the new output such that $\tilde{y} = \ddot{y}_1 - \dot{y}_2$, because u is not differentiable in:

$$\tilde{y} = -x_2 + x_4^2 - x_1 x_4$$

and

$$\dot{\tilde{y}} = x_2 - x_4^2 - 2x_4 x_3 - x_2 x_4 + x_1 x_3 - u$$

Then we have the left inversion matrix Ω_1

$$\Omega_1 = \begin{pmatrix} \frac{\partial \ddot{y}_1}{\partial x_4} & \frac{\partial \ddot{y}_1}{\partial u} \\ \frac{\partial \dot{y}_2}{\partial x_4} & \frac{\partial \dot{y}_2}{\partial u} \\ \frac{\partial \dot{\tilde{y}}}{\partial x_4} & \frac{\partial \dot{\tilde{y}}}{\partial u} \end{pmatrix} = \begin{pmatrix} 2x_4 & 1 \\ x_1 & 1 \\ -2x_4 - 2x_3 - x_2 & -1 \end{pmatrix}$$

The determinants are:

- 1. $2x_4 - x_1 = 0$
- 2. $2x_3 + x_2 = 0$
- 3. $-x_1 + 2x_4 + 2x_3 + x_2 = 0$

If we put (1) and (2) into (3) we get $x_1 = x_1$ so (3) did not provide any additional information about the singularity set. Which gives us the intersection of two hyperplans the set of singularities becomes

$$S_{02} = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1 = 2x_4 \ , x_2 = -2x_3\right\}$$

We reduced the singularity set of dimension 3 ($dimS_{01}$) to a set of singularity of dimension 2 ($dimS_{02}$). In the literature, this technique is called immersion and several authors have used this immersion technique in order to obtain a specific normal form [21]. Most often, the immersion is obtained by output integration.

3 Recalls on high-order sliding-mode

In what follows we will use a real-time exact robust differentiator with higher-order sliding mode (HOSM)[12] . Consider a signal $y(t) \in C^k$ (at least k times differentiable), let us suppose $(y, \ldots, y^{(k)}) = (z_1, \ldots, z_{k+1})$. The HOSM robust differentiator proposed in [14] takes the following form:

$$\begin{aligned} \dot{\hat{z}}_0 &= \hat{z}_1 - \lambda_k M^{1/k+1} \lceil z_0 - y \rfloor^{\frac{k}{k+1}} \\ \dot{\hat{z}}_1 &= \hat{z}_2 - \lambda_{k-1} M^{2/k+1} \lceil z_0 - y \rfloor^{\frac{k-1}{k+1}} \\ &\vdots \\ \dot{\hat{z}}_{k-1} &= \hat{z}_k - \lambda_1 M^{k/k+1} \lceil z_0 - y \rfloor^{\frac{1}{k+1}} \\ \dot{\hat{z}}_k &= -\lambda_0 M sign(z_0 - y). \end{aligned} \tag{13}$$

where $\lceil x \rfloor^\alpha = sgn(x).|x|^\alpha, \;\; \alpha > 0$ M is chosen to be larger than the k-th derivative of $y(t)$, λ_i are positive design parameters, and the adjustment or tuning of those parameters is described in detail in [14] and [13].
. Defining the observation errors as: $e_i = z_i - \hat{z}_i$, then the observation errors dynamics is given by:

$$\begin{aligned} e_1 &= \hat{z}_1 - y \\ e_2 &= \dot{e}_1 = \lambda_0 M^{1/k} \lceil e_1 \rfloor^{\frac{k}{k+1}} \\ &\vdots \\ e_k &= \dot{e}_{k-1} = \lambda_{k-1} M^{1/2} \lceil e_{k-1} \rfloor^{\frac{1}{2}} \\ e_{k-1} &= -\lambda_k M sign(e_k). \end{aligned} \tag{14}$$

It has been proven in [14] that there exists t_0 such that $\forall t > t_0$ we have

$$e_i = z_i - \hat{z}_i = 0 \; pour \; 1 \leqslant i \leqslant k+1 \tag{15}$$

In the next section, an example, representing a type of observability singularity that appears for example in the Qi circuit [23] is presented.

4 Application to secure communication

Let us consider the following chaotic system given in [23]:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_2 x_3 \\ \dot{x}_2 &= b(x_1 + x_2) - x_1 x_3 \\ \dot{x}_3 &= -c x_3 - e x_4 + x_1 x_2 \\ \dot{x}_4 &= -d x_4 + f x_3 + x_1 x_3 \end{aligned} \tag{16}$$

a, b, c, d, e, f are the system parameters such as $a = 42.5, b = 24, c = 13, d = 20, e = 50, f = 40$.
Figure 1 illustrates the chaotic behaviour of the system in the phase plot on x_1, x_2 and x_3 .

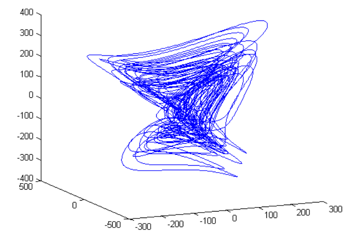

Fig. 1. Phase plot of x_1, x_2 and x_3.

case 1(one unknown input and two outputs) In order to send the confidential messages u, the following transmitter is designed:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_2 x_3 \\ \dot{x}_2 &= b(x_1 + x_2) - x_1 x_3 + u \\ \dot{x}_3 &= -cx_3 - ex_4 + x_1 x_2 \\ \dot{x}_4 &= -dx_4 + fx_3 + x_1 x_3 + u \end{aligned} \tag{17}$$

The chosen outputs are $y_1 = x_1$, $y_2 = x_3$ and u, should be seen as unknown input $x \in [x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4$, $y \in \mathbb{R}^2$. It can be seen that, if we choose the corresponding observability indices respectively as (2.2), we obtain the following observability matrix:

$$dO = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a & a + x_3 & x_2 & 0 \\ 0 & 0 & 1 & 0 \\ x_2 & x_1 & -c & -e \end{pmatrix}$$

Whose determinant is equal to zero for $x_3 = -42.5$. The system (17) is locally weakly observable except on the singularity set

$$S_O = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_3 = -42.5\right\} \tag{18}$$

The successive derivatives of the output y of the system (28) give us:

$$\begin{aligned} &y_1 = x_1 \\ &\dot{y}_1 = \dot{x_1} = a(x_2 - x_1) + x_2 x_3 \\ &\ddot{y}_1 = bu + (x_2 x_3 - a(x_1 - x_2))(b - x_3) + b(b(x_1 + x_2) - x_1 x_3) + x_1(cx_3 + ex_4 - x_1 x_2) \\ &y_2 = x_3 \\ &\dot{y}_2 = \dot{x_3} = -cx_3 - ex_4 + x_1 x_2 \end{aligned}$$

$$\begin{aligned} \ddot{y}_2 = (x_1 - e)u + x_1(b(x_1 + x_2) - x_1 x_3) + c(cx_3 + ex_4 - x_1 x_2) \\ - e(fx_3 - dx_4 + x_1 x_3) + x_2(x_2 x_3 - a(x_1 - x_2)) \end{aligned} \tag{19}$$

The variables to be estimated are x_2, x_4 and u, now consider the matrix Ω combining the information linked to the state and the input with the characteristic number of y_1 and y_2

$$\Omega = \begin{pmatrix} \frac{\partial \dot{y}_1}{\partial x_2} & \frac{\partial \dot{y}_1}{\partial x_4} & \frac{\partial \dot{y}_1}{\partial u} \\ \frac{\partial \ddot{y}_1}{\partial x_2} & \frac{\partial \ddot{y}_1}{\partial x_4} & \frac{\partial \ddot{y}_1}{\partial u} \\ \frac{\partial \dot{y}_2}{\partial x_2} & \frac{\partial \dot{y}_2}{\partial x_4} & \frac{\partial \dot{y}_2}{\partial u} \end{pmatrix} = \begin{pmatrix} a+x_3 & 0 & 0 \\ B & ex_1 & b \\ x_1 & -e & 0 \end{pmatrix}$$

With

$$B = (a+x_3)(b-x_3) + b^2 - x_1^2 \tag{20}$$

Then this system admits a set of singularities S_{01} such that

$$S_{01} = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_3 = -42.5\right\} \tag{21}$$

To reduce the size of this set of singularities we can calculate $\ddot{y}_2$, the matrix Ω becomes as follows:

$$\Omega = \begin{pmatrix} \frac{\partial \dot{y}_1}{\partial x_2} & \frac{\partial \dot{y}_1}{\partial x_4} & \frac{\partial \dot{y}_1}{\partial u} \\ \frac{\partial \ddot{y}_1}{\partial x_2} & \frac{\partial \ddot{y}_1}{\partial x_4} & \frac{\partial \ddot{y}_1}{\partial u} \\ \frac{\partial \dot{y}_2}{\partial x_2} & \frac{\partial \dot{y}_2}{\partial x_4} & \frac{\partial \dot{y}_2}{\partial u} \\ \frac{\partial \ddot{y}_2}{\partial x_2} & \frac{\partial \ddot{y}_2}{\partial x_4} & \frac{\partial \ddot{y}_2}{\partial u} \end{pmatrix} = \begin{pmatrix} a+x_3 & 0 & 0 \\ B & ex_1 & b \\ x_1 & -e & 0 \\ E & ce+de & x_1-e \end{pmatrix}$$

With

$$E = 2x_2(a+x_3) + x_1(b-a-c) \tag{22}$$

The determinants are:

- 1. $(a+x_3)eb = 0$
- 2. $(a+x_3)(ex_1(x_1-e) - (ce+de)b) = 0$
- 3. $-x_1(ex_1(x_1-e) - (ce+de)b) - e(B(x_1-e) - Eb) = 0$

Then we finds the new set of singularities S_{02} such that

$$S_{02} = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_3 = -42.5, x_1 = 3.622, x_1 = 46.377, x_1 = 0 \ and \ x_2 = 0\right\}$$

$\ddot{y}_2$ gave us more information about singularity , in this case it is a pseudo immersion we can move to a total immersion in computations of $\tilde{y}$ and its derivative such that

$$\tilde{y} = \ddot{y}_1(x_1 - e) - \ddot{y}_2 b \tag{23}$$

To look for more information about the singularities in order to be able to circumvent them, but in this case the computation becomes more complicate. We obtained the estimates of the states and the estimation of the unknown input of the system (28) as a function of the output y and its derivatives.

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= \frac{\dot{y}_1 + ay_1}{a + y_2} \\ x_3 &= y_2 \\ x_4 &= \frac{-cy_2 - \dot{y}_2 + \frac{\dot{y}_1 + ay_1}{a+y_2} y_1}{e} \\ u &= \frac{\ddot{y}_1 - X}{b} \ or \ u = \frac{\ddot{y}_2 - Y}{(y_1 - e)} \end{aligned}$$

with

$$X = (\frac{\dot{y}_1 + ay_1}{a + y_2} y_2 - a(y_1 - \frac{\dot{y}_1 + ay_1}{a + y_2}))(b - y_2) + b(b(y_1 + \frac{\dot{y}_1 + ay_1}{a + y_2}) - y_1 y_2)$$
$$+ y_1(cy_2 + e\frac{-cy_2 - \dot{y}_2 + \frac{\dot{y}_1 + ay_1}{a + y_2} y_1}{e} - y_1 \frac{\dot{y}_1 + ay_1}{a + y_2}) \quad (24)$$

and

$$Y = y_1(b(y_1 + \frac{\dot{y}_1 + ay_1}{a + y_2}) - y_1 y_2 - c\dot{y}_2 - e(fy_2 - d\frac{-cy_2 - \dot{y}_2 + \frac{\dot{y}_1 + ay_1}{a + y_2} y_1}{e} + y_1 y_2)$$
$$+ \frac{\dot{y}_1 + ay_1}{a + y_2}(\frac{\dot{y}_1 + ay_1}{a + y_2} y_2 - a(y_1 - \frac{\dot{y}_1 + ay_1}{a + y_2})) \quad (25)$$

Note that $(y, \ldots, y(k)) = (z_1, \ldots, z_{k+1})$. We obtained real-time exact robust differentiator with higher-order sliding mode (HOSM) [13] [9] :

observer 1 : differentiator for y_1

$$\begin{array}{l} \dot{\hat{z}}_1 = \hat{z}_2 - 1.5 M_1^{1/2} |\hat{z}_1 - y_1|^{1/2} sign(\hat{z}_1 - y_1) \\ \dot{\hat{z}}_2 = -1.1 M_1 sign(\hat{z}_1 - y_2). \end{array} \quad (26)$$

observer 2 : differentiator for y_2

$$\begin{array}{l} \dot{\hat{z}}_4 = \hat{z}_5 - 1.5 M_2^{1/2} |\hat{z}_4 - y_2|^{1/2} sign(\hat{z}_4 - y_2) \\ \dot{\hat{z}}_5 = -1.1 M_2 sign(\hat{z}_4 - y_2). \end{array} \quad (27)$$

Simulation results The simulation results were obtained with the following parameters. Simulation time $T = 0.3s$ with a fixed step $T_s = 10^{-7}s$, The used solver is the Runge Kutta (ODE 4) with the initial conditions $x_0 = (10, 5, -10, -3)^T$, $z_0 = (3, 3, 3, 3)^T$ and we set $M_1 = 10^8$ and $M_2 = 10^7$

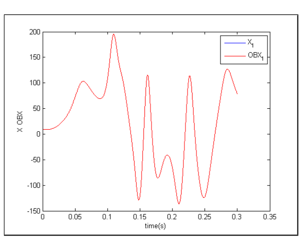

Fig. 2. x_1 in blue and $\hat{x}_1$ in red

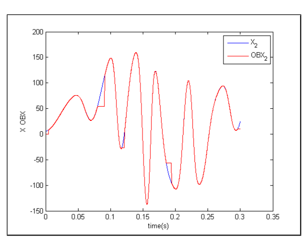

Fig. 3. x_2 in blue and $\hat{x}_2$ in red

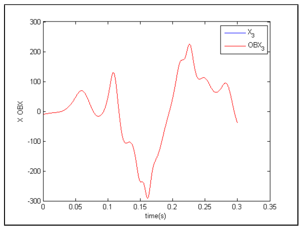

Fig. 4. x_3 in blue and $\hat{x}_3$ in red

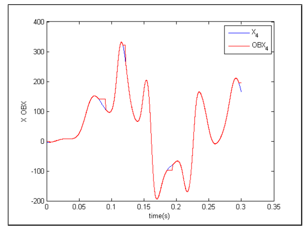

Fig. 5. x_4 in blue and $\hat{x}_4$ in red

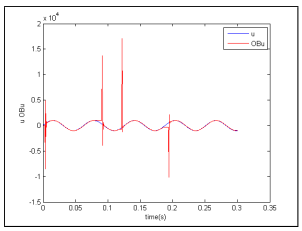

Fig. 6. The unknown input u and it estimate $\hat{u}$ in red before adding saturation

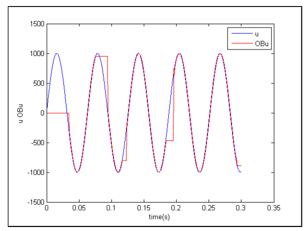

Fig. 7. The unknown input u and it estimate $\hat{u}$ in red after adding saturation

case 2(two outputs and two unknown inputs) In order to send the confidential messages u_1 and u_2, the following transmitter is designed:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_2x_3 + u_1 \\ \dot{x}_2 &= b(x_1 + x_2) - x_1x_3 \\ \dot{x}_3 &= -cx_3 - ex_4 + x_1x_2 \\ \dot{x}_4 &= -dx_4 + fx_3 + x_1x_3 + u_1 + u_2 \end{aligned} \tag{28}$$

The chosen outputs are $y_1 = x_2$, $y_2 = x_3$ and u_1, u_2 should be seen as two unknown inputs $x \in [x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4$, $y \in \mathbb{R}^2$.
It can be seen that, if we choose the corresponding observability indices respectively as (2.2), we obtain the following observability matrix:

$$dO = \begin{pmatrix} 0 & 1 & 0 & 0 \\ b - x_3 & b & -x_1 & 0 \\ 0 & 0 & 1 & 0 \\ x_2 & x_1 & -c & -e \end{pmatrix}$$

Whose determinant is equal to zero for $x_3 = 24$. The system (28) is locally weakly observable except on the singularity set

$$S_O = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_3 = 24\right\} \tag{29}$$

The successive derivatives of the output y of the system (28) give us:

$y_1 = x_2$
$\dot{y}_1 = \dot{x_2} = b(x_1 + x_2) - x_1 x_3$
$\ddot{y}_1 = (b - x_3)u + (x_2 x_3 - a(x_1 - x_2))(b - x_3) + b(b(x_1 + x_2) - x_1 x_3) + x_1(cx_3 + ex_4 - x_1 x_2)$
$y_2 = x_3$
$\dot{y}_2 = \dot{x_3} = -cx_3 - ex_4 + x_1 x_2$

$$\ddot{y}_2 = -eu_2 + x_2(u_1 + x_2 x_3 - a(x_1 - x_2)) - e(u_1 - dx_4 + fx_3 + x_1 x_3) + x_1(b(x_1 + x_2) - x_1 x_3) + c(cx_3 + ex_4 - x_1 x_2) \quad (30)$$

We obtained the estimates of the states and the estimation of the unknown input of the system (28) as a function of the output y and its derivatives.

$$x_1 = \frac{\dot{y}_1 - by_1}{b - y_2}$$
$$x_2 = y_1$$
$$x_3 = y_2$$
$$x_4 = \frac{cy_2 - \dot{y}_2 + \frac{\dot{y}_1 - by_1}{b - y_2} y_1}{e}$$
$$u_1 = \frac{\ddot{y}_1 - A}{(b - y_2)}$$
$$u_2 = \frac{-\ddot{y}_2 + B}{e}$$

with

$$A = (y_1 y_2 - a(\frac{\dot{y}_1 - by_1}{b - y_2} - y_1))(b - y_2) + b(b(\frac{\dot{y}_1 - by_1}{b - y_2} + y_1) - \frac{\dot{y}_1 - by_1}{b - y_2} y_2) + \frac{\dot{y}_1 - by_1}{b - y_2}(cy_2 + e\frac{cy_2 - \dot{y}_2 + \frac{\dot{y}_1 - by_1}{b - y_2} y1}{e} - \frac{\dot{y}_1 - by_1}{b - y_2} y_1) \quad (31)$$

$$B = y_1(\frac{\ddot{y}_1 - A}{(b - y_2)} + y_1 y_2 - a(\frac{\dot{y}_1 - by_1}{b - y_2} - y_1)) - e(\frac{\ddot{y}_1 - A}{(b - y_2)} - d(\frac{cy_2 - \dot{y}_2 + \frac{\dot{y}_1 - by_1}{b - y_2} y_1}{e}) + fy_2 + \frac{\dot{y}_1 - by_1}{b - y_2} y_2) + \frac{\dot{y}_1 - by_1}{b - y_2}(b(\frac{\dot{y}_1 - by_1}{b - y_2} + y_1) - \frac{\dot{y}_1 - by_1}{b - y_2} y_2) + c(cy_2 + e\frac{cy_2 - \dot{y}_2 + \frac{\dot{y}_1 - by_1}{b - y_2} y_1}{e} - \frac{\dot{y}_1 - by_1}{b - y_2} y_1) \quad (32)$$

Simulation results The simulation results were obtained with the following parameters. Simulation time $T = 0.4s$ with a fixed step $T_s = 10^{-7}$, The

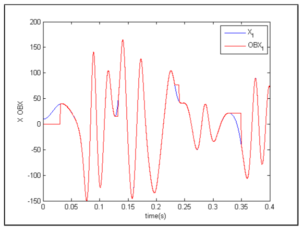

Fig. 8. x_1 in blue and $\hat{x}_1$ in red

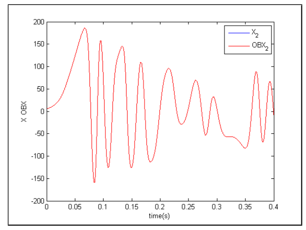

Fig. 9. x_2 in blue and $\hat{x}_2$ in red

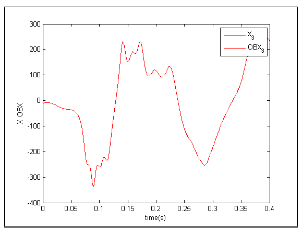

Fig. 10. x_3 in blue and $\hat{x}_3$ in red

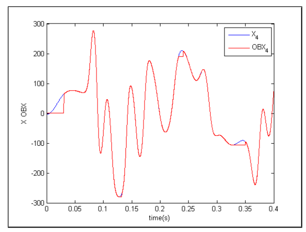

Fig. 11. x_4 in blue and $\hat{x}_4$ in red

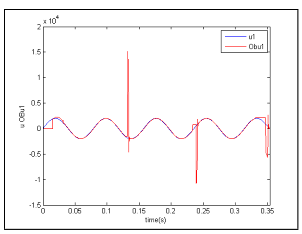

Fig. 12. The unknown input u_1 and it estimate $\hat{u}_1$ in red before adding saturation

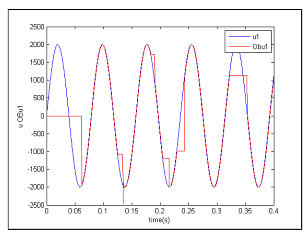

Fig. 13. The unknown input u_1 and it estimate $\hat{u}_1$ in red after adding saturation

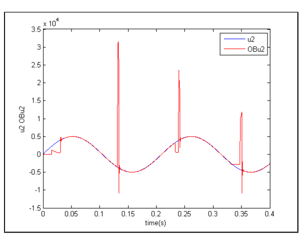

Fig. 14. The unknown input u_2 and it estimate $\hat{u}_2$ in red before adding saturation

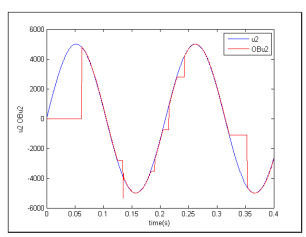

Fig. 15. The unknown input u_2 and it estimate $\hat{u}_2$ in red after adding saturation

used solver is the Runge Kutta (ODE 4) with the initial conditions $x_0 = (10, 5, -10, -3)^T$, $z_0 = (3, 3, 3, 3)^T$ and we set $M_1 = 10^8$ and $M_2 = 10^7$

In the figures (8) and (11) we note that the exact estimates of x_1 and x_4 are obtained after a time $t < 0.05$. The blocking of the estimates is due to a saturation that has been introduced on simulink to avoid the peaks of inversion singularity ($S_I = \{x \in \mathbb{R}^4 | x_3 = 24\}$) and of observation ($S_O = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_3 = 24\}$) .

5 Conclusion

In this document an analysis of the observability with respect to dynamic MIMO systems, was done. After that we studied the left inversion and immersion technique an academic example was treated The practical interest of the method was also demonstrated With an application for secure communication the unknown inputs were restored in both cases Using high-order sliding mode observers (HOSM).

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