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On a Cournot Dynamic Game with Differentiated Goods and Asymmetric Cost Functions

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Abstract. In this study we investigate the dynamics of a nonlinear Cournot – type duopoly game with differentiated goods, linear demand and different cost functions. The game is modelled with a system of two difference equations. Existence and stability of equilibrium of this system are studied. We show that the model gives more complex, chaotic and unpredictable trajectories as a consequence of change in the parameter of speed of adjustment of the bounded rational player and in the parameter of horizontal product differentiation. A higher (lower) degree of player's adjustment or a variation of the parameter of product differentiation (weaker or fiercer competition) destabilize (stabilize) the economy. The chaotic features are justified numerically via computing Lyapunov numbers and sensitive dependence on initial conditions. Also, we show that in the case of asymmetric costs there are stable trajectories and a higher (lower) degree of product differentiation does not tend to destabilize the economy.
Keywords: Cournot duopoly game, Product differentiation, Dynamical system, Heterogeneous expectations, Homogeneous expectations, Asymmetric costs, Stability, Chaotic behaviour.

1 Introduction

An oligopoly is a market structure between monopoly and perfect competition, where there are only a few numbers of firms in the market producing homogeneous products. The dynamic of an oligopoly game is more complex because firms must consider not only the behaviors of the consumers, but also the reactions of the competitors i.e. they form expectations concerning how their rivals will act. Cournot, in 1838 has introduced the first formal theory of oligopoly. He treated the case with naive expectations, so that in every step each player (firm) assumes the last values that were taken by the competitors without estimation of their future reactions.

Expectations play an important role in modeling economic phenomena. A producer can choose his expectations rules of many available techniques to adjust his production outputs. In this paper we study the dynamics of a duopoly model where each firm behaves with heterogeneous or homogeneous expectations strategies. We consider a duopoly model where each player forms

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a strategy in order to compute his expected output. Each player adjusts his outputs towards the profit maximizing amount as target by using his expectations rule. Some authors considered duopolies with homogeneous expectations and found a variety of complex dynamics in their games, such as appearance of strange attractors (Agiza [2], Agiza *et al*. [6], Agliari *et al*. [7],[8], Bischi and Kopel [12], Kopel [26], Puu [34], Sarafopoulos [38]). Also models with heterogeneous agents were studied (Agiza and Elsadany [4],[5], Agiza *et al*., [6], Den Haan [20], Fanti and Gori [22], Tramontana [41], Zhang [44]).
In the real market producers do not know the entire demand function, though it is possible that they have a perfect knowledge of technology, represented by the cost function. Hence, it is more likely that firms employ some local estimate of the demand. This issue has been previously analyzed by Baumol and Quandt [11], Puu [33], Naimzada and Ricchiuti [31], Askar [9], Askar [10]. Bounded rational players (firms) update their production strategies based on discrete time periods and by using a local estimate of the marginal profit. With such local adjustment mechanism, the players are not requested to have a complete knowledge of the demand and the cost functions (Agiza and Elsadany [4], Naimzada and Sbragia [32], Zhang *et al* [44], Askar, [10]). The paper is organized as follows: In sections 2 and 3 the dynamics of the duopoly game with heterogeneous and homogeneous expectations, linear demand and asymmetric cost functions are analyzed. The existence and local stability of the equilibrium points are also analyzed. Numerical simulations are used to show complex dynamics via bifurcations diagrams, computing Lyapunov numbers, and sensitive dependence on initial conditions. Also, we show that in the case of asymmetric costs there are stable trajectories and a higher (lower) degree of product differentiation does not tend to destabilize the economy.

2 Heterogeneous expectations

2.1 The game

There are two firms who offer their products at discrete-time periods (t = 0,1,2,…) on a common market. We consider a simple Cournot-type duopoly market where the two firms (players) produce differentiated goods and their production decisions are taken at discrete-time periods (t = 0,1,2,…). In this study we consider heterogeneous players and more specifically, we suppose that the Firm 1 decides in a rational way, following an adjustment mechanism (bounded rational player), while the Firm 2 chooses the production quantity by naïve way, selecting a quantity that maximizes its output (naïve player). At each period t, every firm must form an expectation of the rival's output in the next time period in order to determine the corresponding profit-maximizing quantities for period t+1. We suppose that q_1 , q_2 are the production quantities of each firm, then the inverse demand function (as a function of quantities) is given by the following equation:

$$p_i = a - q_i - dq_j \,, \quad \text{with } i \neq j$$

where p_i is the product price of firm i.
So, we have for each firm the following equations:

$$p_1 = a - q_1 - dq_2 \quad \text{and} \quad p_2 = a - q_2 - dq_1 \tag{1}$$

where α is a positive parameter which expresses the market size and $d \in (-1,1)$ is the parameter that reveals the differentiation degree between two products. For example, if $d = 0$ then both products are independent and each firm participates in a monopoly. But, if $d = 1$ then one product is a substitute for the other, since the products are homogeneous. It is understood that for positive values of the parameter d the larger the value, the less diversification we have in both products. On the other hand negative values of the parameter d are described that the two products are complementary and when $d = -1$ then we have the phenomenon of full competition between two companies.

In this work we suppose that two players follow different cost functions.

$$C_1(q_1) = c \cdot q_1 \tag{2}$$

and

$$C_2(q_2) = c \cdot q_2^2 \tag{3}$$

and c > 0 is the marginal cost for player 1.
With these assumptions the profits of the firms are given by:

$$\mathrm{P}_1(q_1, q_2) = p_1 q_1 - C_1(q_1) = (\alpha - q_1 - dq_2)q_1 - cq_1 \tag{4}$$

and

$$\mathrm{P}_2(q_1, q_2) = p_2 q_2 - C_2(q_2) = (\alpha - q_2 - dq_1)q_2 - cq_2^2 \tag{5}$$

Then the marginal profits at the point of the strategy space are given by:

$$\frac{\partial \mathrm{P}_1}{\partial q_1} = a - c - 2q_1 - dq_2 \quad \text{and} \quad \frac{\partial \mathrm{P}_2}{\partial q_2} = a - dq_1 - 2(1+c)q_2 \tag{6}$$

We suppose that the first firm decides to increase its level of adaptation if it has a positive marginal profit, or decreases its level if the marginal profit is negative (bounded rational player). If k > 0 the dynamical equation of the first player is:

$$\frac{q_1(t+1) - q_1(t)}{q_1(t)} = k \frac{\partial \mathrm{P}_1}{\partial q_1} \tag{7}$$

where the parameter k∈ (0,1) expresses the speed of adjustment of player 1, it is a positive parameter which gives the extent of production variation of the firm following a given profit signal. Moreover, it captures the fact that relative effort variations are proportional to the marginal profit.

The second firm decides by naïve way, selecting a production that maximizes its profits (naïve player):

$$q_2(t+1) = \arg\max_{q_2} \mathrm{P}_2(q_1(t), q_2(t)) \tag{8}$$

The dynamical system of the players is described by:

$$\begin{cases} q_1(t+1) = q_1(t) + kq_1(t) \cdot \dfrac{\partial P_1}{\partial q_1} \\ q_2(t+1) = \dfrac{a - dq_1(t)}{2(1+c)} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \quad \begin{cases} q_1(t+1) = q_1(t) + kq_1(t)\left[a - c - 2q_1(t) - dq_2(t)\right] \\ q_2(t+1) = \dfrac{a - dq_1(t)}{2(1+c)} \end{cases} \tag{9}$$

We will focus on the dynamics of this system to the parameter k.

2.2 Dynamical analysis

2.2.1 The equilibriums of the game

The equilibriums of the dynamical system (9) are obtained as the nonnegative solutions of the algebraic system:

$$\begin{cases} k \cdot q_1^*\left[a - c - 2q_1^* - dq_2^*\right] = 0 \\ q_2^* = \dfrac{a - dq_1^*}{2(1+c)} \end{cases} \tag{10}$$

which obtained by setting $q_1(t+1) = q_1(t) = q_1^*$ and $q_2(t+1) = q_2(t) = q_2^*$.

- If $q_1^* = 0$, then $q_2^* = \dfrac{a}{2(1+c)}$ and we have the boundary equilibrium:

$$E_0 = \left(0 \, , \, \frac{a}{2(1+c)}\right) \tag{11}$$

- If $\dfrac{\partial P_1}{\partial q_1} = \dfrac{\partial P_2}{\partial q_2} = 0$, then we form the following system:

$$\begin{cases} q_1^* = \dfrac{a - c - dq_2^*}{2} \\ q_2^* = \dfrac{a - dq_1^*}{2(1+c)} \end{cases} \tag{12}$$

and the equilibrium is:

$$E_* = \left(q_1^*, q_2^*\right) = \left(\frac{2(1+c)(a-c) - da}{4(1+c) - d^2}, \frac{a(2-d) + dc}{4(1+c) - d^2}\right) \tag{13}$$

Since $a(2-d) + dc$ and $4(1+c) - d^2$ are always positive quantities, it means that for our game there is the following restriction:

$$2(1+c)(a-c) - da > 0 \tag{14}$$

2.2.2 Stability of equilibriums

The study of the local stability of the equilibrium is based on the localization on the complex plane of the eigenvalues of the Jacobian matrix of the dimensional map (Eq.(9)). In order to study the local stability of equilibrium points of the model (9), we consider the Jacobian matrix $J(q_1,q_2)$ along the variable strategy (q_1,q_2):

$$J(q_1,q_2)=\begin{bmatrix} f_{q_1} & f_{q_2} \\ g_{q_1} & g_{q_2} \end{bmatrix} \tag{15}$$

Where

$$\begin{aligned} f(q_1,q_2) &= q_1 + kq_1[a-c-2q_1-dq_2] \\ g(q_1,q_2) &= \frac{a-dq_1}{2(1+c)} \end{aligned} \tag{16}$$

and we find the Jacobian matrix:

$$J(q_1,q_2)=\begin{bmatrix} 1+k[a-c-4q_1-dq_2] & -dkq_1 \\ -\frac{d}{2(1+c)} & 0 \end{bmatrix} \tag{17}$$

with

$$Tr[J]=1+k\left[a-c-4q_1^*-dq_2^*\right] \tag{18}$$

and

$$Det[J]=-\frac{kd^2q_1^*}{2(1+c)} \tag{19}$$

For E_0:

$$Tr[J(E_0)]=1+k\left[a-c-\frac{da}{2(1+c)}\right]=1+k\cdot\frac{2(1+c)(a-c)-da}{2(1+c)} \tag{20}$$

and

$$Det[J(E_0)]=0 \tag{21}$$

The characteristic equation of $J(E_0)$ is:

$$l^2-Tr\cdot l+Det=0 \tag{22}$$

the solutions of which are the following eigenvalues:

$$l_1=0 \quad \text{and} \quad l_2=1+k\cdot\frac{2(1+c)(a-c)-da}{2(1+c)} \tag{23}$$

Since $\frac{2(1+c)(a-c)-da}{2(1+c)}>0$ (because of Eq. (14)), it's clearly that $|l_2|>1$, and the point E_0 is unstable.

For E_*:

$$Tr[J(E_*)] = 1 - 2k \cdot \frac{2(1+c)(a-c) - da}{4(1+c) - d^2} \tag{24}$$

and

$$Det[J(E_*)] = -kd^2 \cdot \frac{2(1+c)(a-c) - da}{2 \cdot (1+c) \cdot \left[4(1+c) - d^2 \right]} \tag{25}$$

The equilibrium point is locally asymptotically stable if:

$$\begin{aligned} &i) \quad 1 - Det > 0 \\ &ii) \quad 1 - Tr + Det > 0 \\ &iii) \quad 1 + Tr + Det > 0 \end{aligned} \tag{26}$$

For (i), we have:

$$1 - Det = 1 + kd^2 \cdot \frac{2(1+c)(a-c) - da}{2 \cdot (1+c) \cdot \left[4(1+c) - d^2 \right]} > 0 \ , \tag{27}$$

an inequality which holds because of (14).
For (ii):

$$1 - Tr + Det = k \cdot \frac{2(1+c)(a-c) - da}{2(1+c)} > 0 \ , \tag{28}$$

which also holds because of (14).
The conditions (i) and (ii) of Eq.(26) are always satisfied and then the condition (iii) is the condition for the local stability of the Nash Equilibrium.
We suppose the third condition (iii):

$$1 + Tr + Det > 0 \ \Leftrightarrow k < \frac{4 \cdot (1+c) \cdot \left[4(1+c) - d^2 \right]}{\left[4(1+c) + d^2 \right] \cdot \left[2(1+c)(a-c) - da \right]} \tag{29}$$

So, the third condition (iii) is the stability condition for the Nash equilibrium of system Eq.(9), focusing to the parameter k.

Proposition: *The Nash equilibrium of the dynamical system Eq.(9) is locally asymptotically stable if:*

$$0 < k < \frac{4 \cdot (1+c) \cdot \left[4(1+c) - d^2 \right]}{\left[4(1+c) + d^2 \right] \cdot \left[2(1+c)(a-c) - da \right]} \tag{30}$$

2.3 Numerical simulations

To provide some numerical evidence for the chaotic behavior of the system Eq.(9), as a consequence of change in the parameter k of the speed of players' adjustment, we present various numerical results here to show the chaoticity, including its bifurcations diagrams, strange attractor, Lyapunov numbers and sensitive dependence on initial conditions (Kulenovic, M. and Merino, O. [27]). In order to study the local stability properties of the equilibrium points, it is convenient to take some specific values for the parameters a, c and d, for example: $a = 5$, $c = 0.5$, $d = 0.5$. And in this case the stability condition becomes as follows:

$$0 < k < 0.5018 \tag{31}$$

Numerical experiments are computed to show the bifurcation diagram with respect to k, strange attractors of the system Eq.(9) in the phase plane (q_1, q_2) and Lyapunov numbers. Figure 1 shows the bifurcation diagrams with respect to the parameter k. Also in this figure one observes complex dynamic behavior such as cycles of higher order and chaos. Figure 2 shows the Lyapunov numbers' diagram of the orbit of $(0.1,0.1)$ for $a = 5$, $c = 0.5$, $d = 0.5$, and for $k = 0.66$. Figure 3 shows the graphs of the same orbit (strange attractors) for $a = 5$, $c = 0.5$, $d = 0.5$, $k = 0.7$ (left) and $k = 0.76$ (right). From these results when all parameters are fixed and only k is varied the structure of the game becomes complicated through period doubling bifurcations, more complex bounded attractors are created which are aperiodic cycles of higher order or chaotic attractors.

To demonstrate the sensitivity to initial conditions of the system Eq.(9) we compute two orbits with initial points (0.1,0.1) and (0.101,0.1), respectively. Figure 4 shows sensitive dependence on initial conditions for x-coordinate of the two orbits, for the system Eq.(9), plotted against the time with the parameter values $a = 5$, $c = 0.5$, $d = 0.5$, $k = 0.7$. As in first case also, here at the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly.

3 Homogeneous expectations

Now we suppose that both firms decide to increase their level of adaptation if they have positive marginal profits, or decrease their level if the marginal profits are negative (bounded rational players). If k > 0 the dynamical equation of two players is:

$$\frac{q_i(t+1) - q_i(t)}{q_i(t)} = k\frac{\partial \mathrm{P}_i}{\partial q_i} , \qquad \forall i = 1,2 \tag{32}$$

k by the same way with the first case, expresses the same speed of adjustment of two players.

The dynamical system of the players is described by:

$$\begin{cases} q_1(t+1) = q_1(t) + kq_1(t)[a - c - 2q_1(t) - dq_2(t)] \\ q_2(t+1) = q_2(t) + kq_2(t)[a - dq_1(t) - 2(c+1)q_2(t)] \end{cases} \tag{33}$$

We will focus on the dynamics of this system to the parameter k.

3.1 Dynamical analysis

3.1.1 The equilibriums of the game

The equilibriums of the dynamical system (33) are obtained as nonnegative solutions of the algebraic system:

$$\begin{cases} q_1^* \cdot \dfrac{\partial P_1}{\partial q_1} = 0 \\ q_2^* \cdot \dfrac{\partial P_2}{\partial q_2} = 0 \end{cases} \tag{34}$$

which obtained by setting $q_1(t+1) = q_1(t) = q_1^*$ and $q_2(t+1) = q_2(t) = q_2^*$.

- If $q_1^* = q_2^* = 0$, then we have the boundary equilibrium:

$$E_0' = (0,0) \tag{35}$$

- If $q_1^* = 0$ and $\dfrac{\partial P_2}{\partial q_2} = 0$, the equilibrium point is:

$$E_1' = \left(0 \; , \; \frac{a}{2(1+c)} \right) \tag{36}$$

- If $q_2^* = 0$ and $\dfrac{\partial P_1}{\partial q_1} = 0$, the equilibrium point becomes as:

$$E_2' = \left(\frac{a-c}{2}, 0 \right) \tag{37}$$

- If $\dfrac{\partial P_1}{\partial q_1} = \dfrac{\partial P_2}{\partial q_2} = 0$, then we form the following system:

$$\begin{cases} q_1^* = \dfrac{a-c-dq_2^*}{2} \\ q_2^* = \dfrac{a-dq_1^*}{2(1+c)} \end{cases} \tag{38}$$

and the Nash equilibrium is the same as at the previous case (heterogeneous expectations):

$$E_*' = E_* = \left(q_1^*, q_2^* \right) = \left(\frac{2(1+c)(a-c)-da}{4(1+c)-d^2}, \frac{a(2-d)+dc}{4(1+c)-d^2} \right) \tag{39}$$

3.1.2 Stability of equilibriums

To study the local stability of the equilibrium positions we also need the Jacobian matrix of the dimensional map (Eq.(33)). Working by the same way as in the first case we find the following Jacobian matrix:

$$J(q_1,q_2)=\begin{bmatrix} 1+k\left(\frac{\partial P_1}{\partial q_1}+q_1^*\cdot\frac{\partial^2 P_1}{\partial q_1{}^2}\right) & k\cdot q_1^*\cdot\frac{\partial^2 P_1}{\partial q_1\partial q_2} \\ k\cdot q_2^*\cdot\frac{\partial^2 P_2}{\partial q_2\partial q_1} & 1+k\left(\frac{\partial P_2}{\partial q_2}+q_2^*\cdot\frac{\partial^2 P_2}{\partial q_2{}^2}\right) \end{bmatrix} \tag{40}$$

For E_0':

$$J(E_0')=\begin{bmatrix} 1+k(a-c) & 0 \\ 0 & 1+ka \end{bmatrix} \tag{41}$$

with

$$Tr[J(E_0')]=2+k(2a-c) \tag{42}$$

and

$$Det[J(E_0')]=0 \tag{43}$$

The characteristic equation of $J(E_0')$ is:

$$l^2-Tr\cdot l+Det=0 \tag{44}$$

the solutions of which are the following eigenvalues:

$$l_1=0 \quad \text{and} \quad l_2=Tr=2+k\cdot(2a-c)>1 \tag{45}$$

it's clearly that $|l_2|>1$, and the point E_0' is unstable.

For E_1':

$$J(E_1')=\begin{pmatrix} \mathrm{A} & 0 \\ -dkq_2^* & \mathrm{B} \end{pmatrix} \tag{46}$$

where

$$\mathrm{A}=1+k\cdot\frac{2(1+c)(a-c)-da}{2(1+c)} \quad \text{and} \quad \mathrm{B}=1-ka \tag{47}$$

with

$$Tr[J(E_1')]=\mathrm{A}+\mathrm{B} \tag{48}$$

and

$$Det[J(E_1')]=\mathrm{A}\cdot\mathrm{B} \tag{49}$$

The characteristic equation of $J(E_1')$ is:

$$l^2-Tr\cdot l+Det=0 \Leftrightarrow (l-\mathrm{A})\cdot(l-\mathrm{B})=0 \tag{50}$$

the solutions of which are the following eigenvalues:

$$l_1=\mathrm{A} \quad \text{and} \quad l_2=\mathrm{B} \tag{51}$$

Since $2(1+c)(a-c)-da>0$ from Eq.(14), it's clearly that $|l_1|>1$, and the point E_1' is unstable.

For E_2':

$$J(E_2') = \begin{pmatrix} C & -dkq_1^* \\ 0 & D \end{pmatrix} \quad (52)$$

where

$$C = 1 - k \cdot \frac{a-c}{2} \quad \text{and} \quad D = 1 + k \cdot \frac{a(2-d)+dc}{2} \quad (53)$$

with

$$Tr[J(E_2')] = C + D \quad (54)$$

and

$$Det[J(E_2')] = C \cdot D \quad (55)$$

Also, the characteristic equation of $J(E_2')$ is:

$$(l - C)\cdot(l - D) = 0 \quad (56)$$

the solutions of which are the following eigenvalues:

$$l_1 = C \quad \text{and} \quad l_2 = D \quad (57)$$

Since $a(2-d)+dc > 0$, it's clearly that $|l_2| > 1$, and the point E_2' is unstable.

For E_*':

$$J(E_*') = \begin{pmatrix} 1-2kq_1^* & -dkq_1^* \\ -dkq_2^* & 1-2(1+c)kq_2^* \end{pmatrix} \quad (58)$$

with

$$Tr[J(E_*')] = 2 - 2kq_1^* - 2(1+c)q_2^* \quad (59)$$

and

$$Det[J(E_*')] = 1 + k^2 q_1^* q_2^* \left[4(1+c) - d^2\right] - 2(1+c)kq_2^* - 2kq_1^* \quad (60)$$

The equilibrium point is locally asymptotically stable if :

$$\begin{aligned} &i)\ \ 1 - Det > 0 \\ &ii)\ \ 1 - Tr + Det > 0 \\ &iii)\ \ 1 + Tr + Det > 0 \end{aligned} \quad (61)$$

For (ii):

$$1 - Tr + Det = k^2 q_1^* q_2^* \left[4(1+c) - d^2\right] > 0\ , \quad (62)$$

an inequality which always holds.

Condition (i) becomes as:

$$1 - Det > 0 \Leftrightarrow k < \frac{2(1+c)q_2^* + 2q_1^*}{q_1^* q_2^* \left[4(1+c) - d^2\right]}\ , \quad (63)$$

and for the inequality (iii):

$$1+Tr+Det>0 \Leftrightarrow q_1^* q_2^* \left[4(1+c)-d^2\right]\cdot k^2 - 4\left[q_1^* + (1+c)q_2^*\right]\cdot k + 4 > 0 \quad (64)$$

Since, the condition (ii) of Eq.(62) is always satisfied, the conditions (i) and (iii) are the conditions for the local stability of the Nash Equilibrium focusing to the parameters k or d.

Proposition : *The Nash equilibrium of the dynamical system Eq.(33) is locally asymptotically stable if:*

$$0 < k < \frac{2(1+c)q_2^* + 2q_1^*}{q_1^* q_2^* \left[4(1+c)-d^2\right]} \quad \text{and} \quad q_1^* q_2^* \left[4(1+c)-d^2\right]\cdot k^2 - 4\left[q_1^* + (1+c)q_2^*\right]\cdot k + 4 > 0 \quad (65)$$

where: $d \in (-1,1)$,

$$q_1^* = \frac{2(1+c)(a-c)-da}{4(1+c)-d^2} \quad \textit{and} \quad q_2^* = \frac{a(2-d)+dc}{4(1+c)-d^2}$$

3.2 Numerical simulations

3.2.1 Stability space (k, d)

In our game there are two important parameters, the parameter k, the speed of adjustment of players to their mechanisms and the parameter d, which is the differentiation degree between two products. At first, it is needed to export the stability space allowing both parameters k and d. Setting the specific values of the parameters α=5, c=0.5 in two stability conditions, from Figure 5 it seems that there is a closed stability space and that for small values of the parameter k there are stable trajectories for all the values of the parameter d.

3.2.2 Focusing to the parameter k

Numerical experiments are computed again to show the bifurcation diagram with respect to k, strange attractors of the system Eq.(33) in the phase plane (q_1, q_2) and Lyapunov numbers. Figure 6 shows the bifurcation diagrams by the same way with the previous case with respect to the parameter k. Also in this figure one observes complex dynamic behavior such as cycles of higher order and chaos. Figure 7 shows the graphs of the same orbit (strange attractors) and Lyapunov numbers' diagram of the orbit of (0.1,0.1) for $a=5$, $c=0.5$, $d=0.5$, and $k=0.57$. From these results when all parameters are fixed and only k is varied the structure of the game becomes complicated through period doubling bifurcations, more complex bounded attractors are created which are aperiodic cycles of higher order or chaotic attractors.
To demonstrate the sensitivity on initial conditions of the system Eq.(33) we compute two orbits with initial points (0.1,0.1) and (0.101,0.1), respectively.

Figure 8 shows sensitive dependence on initial conditions for x-coordinate of the two orbits, for the system Eq.(33), plotted against the time with the parameter values $a = 5$, $c = 0.5$, $d = 0.5$, $k = 0.57$. As in first case also, here at the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly.

3.2.3 Focusing to the parameter d

Now some numerical experiments are computed, with respect to differentiation parameter d. Bifurcation diagrams, strange attractors of the system Eq.(33) in the phase plane (q_1, q_2) and Lyapunov numbers are presented. Figure 9 shows the bifurcation diagrams with respect to the parameter d. Also in this figure one observes complex dynamic behavior such as cycles of higher order and chaos. Figure 10 shows the graphs of the same orbit (strange attractors) and Lyapunov numbers' diagram of the orbit of (0.1,0.1) for $a = 5$, $c = 0.5$, $k = 0.3$, and $d = -0.79$. From these results when all parameters are fixed and only d is varied the structure of the game becomes complicated through period doubling bifurcations, more complex bounded attractors are created which are aperiodic cycles of higher order or chaotic attractors.
To demonstrate the sensitivity on initial conditions of the system Eq.(33) we compute two orbits with initial points (0.1,0.1) and (0.101,0.1), respectively. Figure 11 shows sensitive dependence on initial conditions for x-coordinate of the two orbits, for the system Eq.(33), plotted against the time with the parameter values $a = 5$, $c = 0.5$, $k = 0.3$, $d = -0.79$. As in first case also, here at the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly.

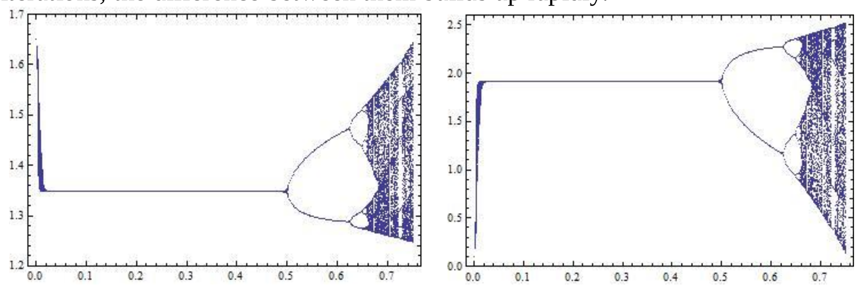

Fig. 1: Bifurcation diagrams with respect to the parameter d against variable q_1 (left) and q_2 (right), with 400 iterations of the map Eq. (9) for $a = 5$, $c = 0.5$, $d = 0.5$.

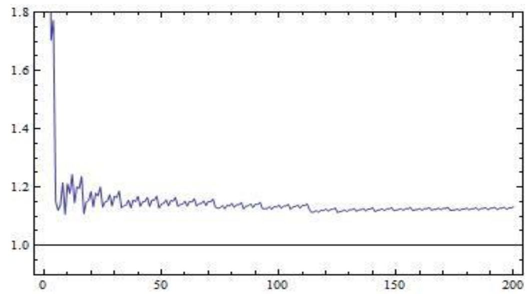

Fig. 2: Lyapunov numbers of the orbit of the point A(0.1,0.1) versus the number of iterations for $a = 5, c = 0.5, d = 0.5$ and for $k = 0.66$.

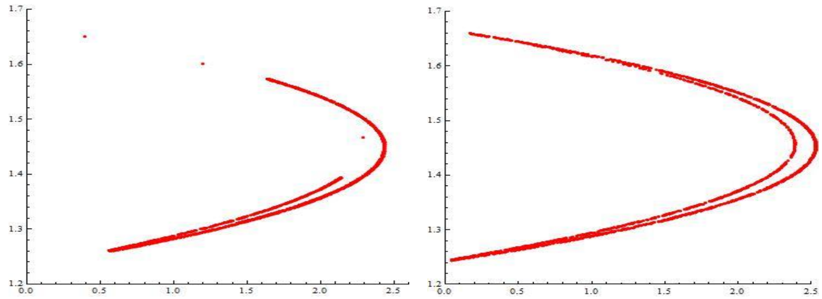

Fig. 3: Phase portrait (strange attractors). The orbit of (0.1,0.1) with 2000 iterations of the map Eq.(9) for $a = 5, c = 0.5, d = 0.5$ and for $k = 0.7$ (left) and $k = 0.76$ (right).

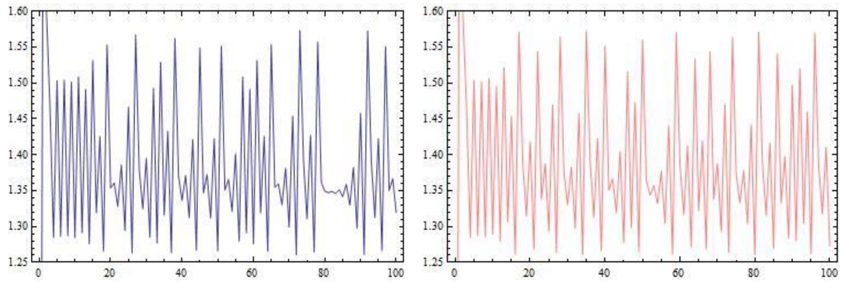

Fig. 4: Sensitive dependence on initial conditions for x-coordinate plotted against the time: the two orbits: the orbit of (0.1,0.1) (left) and the orbit of (0.101,0.1) (right), for the system Eq.(9), with the parameters values $a = 5, c = 0.5, d = 0.5$ and $k = 0.7$.

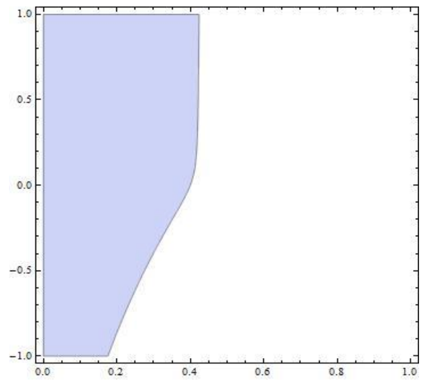

Fig. 5: Stability space for $a = 5$ and $c = 0.5$, (horizontal axis for k and vertical axis for d).

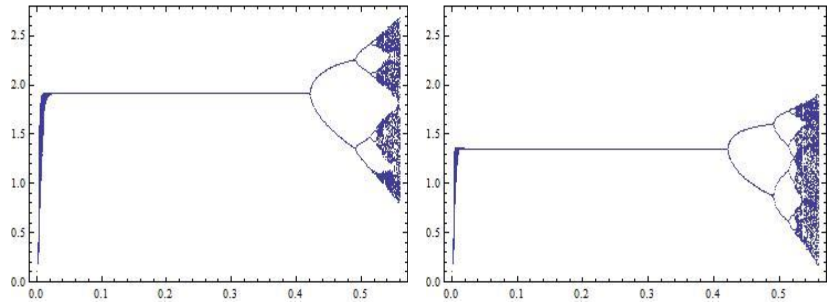

Fig. 6: Bifurcation diagrams with respect to the parameter k against variable q_1 (left) and q_2 (right), with 400 iterations of the map Eq. (33) for $a = 5, c = 0.5, d = 0.5$.

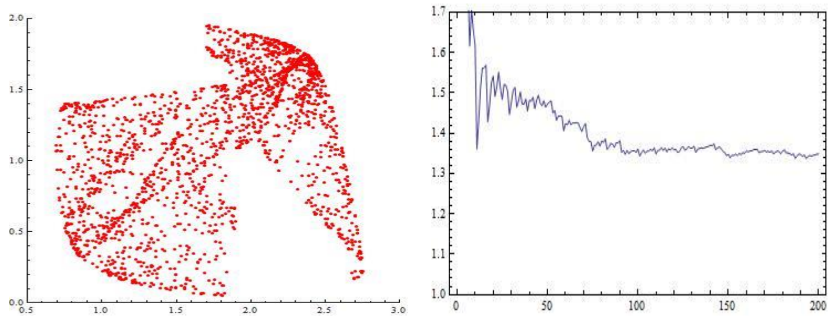

Fig. 7: Phase portrait (strange attractor) and Lyapunov numbers of the orbit of the point A(0.1,0.1) versus the number of iterations for $a = 5, c = 0.5, d = 0.5, k = 0.57$.

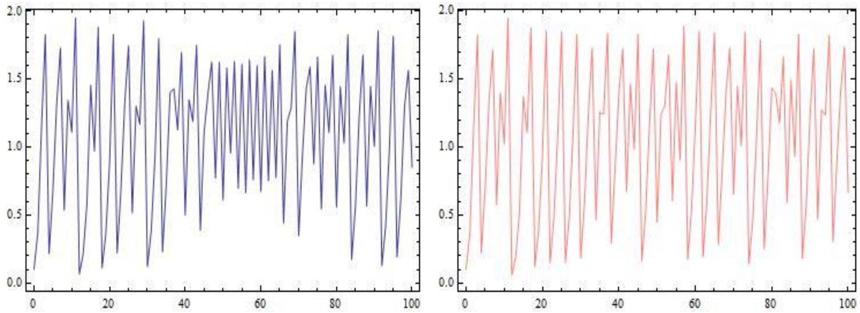

Fig. 8: Sensitive dependence on initial conditions for x-coordinate plotted against the time: the two orbits: the orbit of (0.1,0.1)(left) and the orbit of (0.101,0.1)(right), for the system Eq.(33), for $a = 5, c = 0.5, d = 0.5$ and $k = 0.57$.

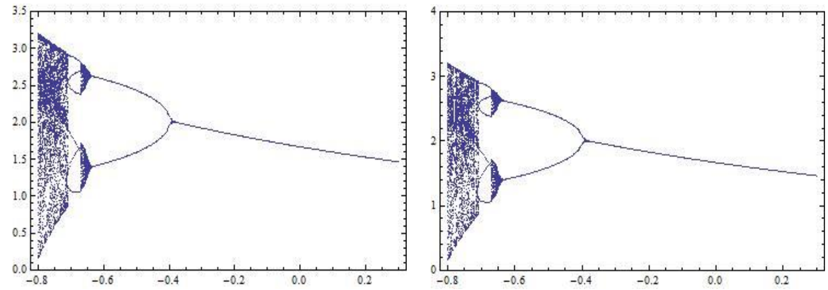

Fig. 9: Bifurcation diagrams with respect to the parameter d against variable q_1 (left) and q_2 (right), with 400 iterations of the map Eq. (33) for $a = 5, c = 0.5, k = 0.3$.

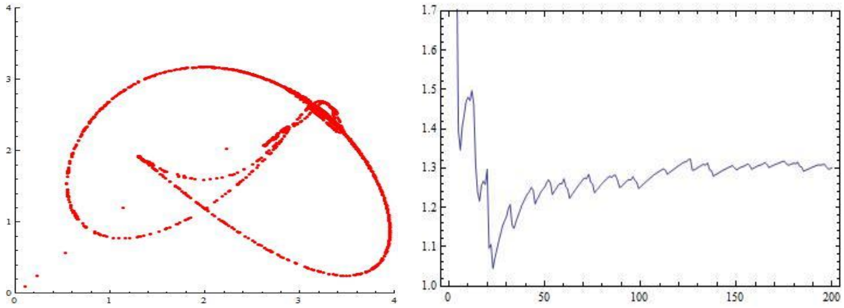

Fig. 10: Phase portrait (strange attractor) and Lyapunov numbers of the orbit of the point A(0.1,0.1) versus the number of iterations for $a = 5$, $c = 0.5$, $k = 0.3$, $d = -0.79$.

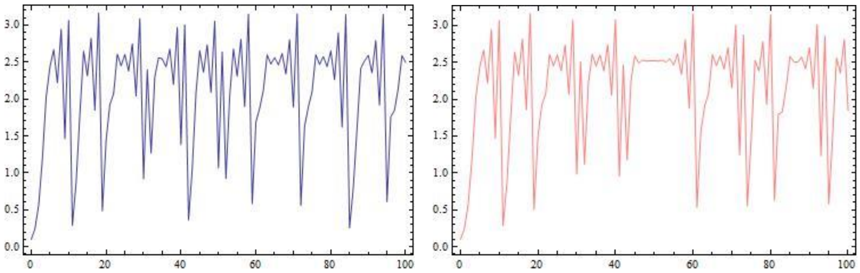

Fig. 11: Sensitive dependence on initial conditions for x-coordinate plotted against the time: the two orbits: the orbit of (0.1,0.1) (left) and the orbit of (0.101,0.1) (right), for the system Eq.(33), with the parameter values $a = 5$, $c = 0.5$, $k = 0.3$ and $d = -0.79$.

Conclusions

In this paper we analyzed the dynamics of two nonlinear Cournot – type dynamic game with heterogeneous or homogeneous expectations, differentiated goods, linear demand and different cost functions for each player. Existence and stability of equilibriums are studied. We proved that the parameter of horizontal product differentiation and the parameter of the speed of adjustment may change the stability of the Nash equilibrium and cause a structure to behave chaotically through period – doubling bifurcation. The chaotic features are justified numerically via bifurcation diagrams, computing Lyapunov numbers and sensitive dependence on initial conditions. Also, we proved that at the case of asymmetric costs there are stable trajectories and a higher (lower) degree of product differentiation does not tend to destabilize the economy.

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