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Solution of Ancient Greek Problem of Trisection of Arbitrary Angle

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Abstract. A solution of the ancient Greek problem of trisection of an arbitrary angle employing only compass and straightedge that avoids the need for two marks on Archimedes marked ruler is presented. It is argued that although Wantzel [1-5] 1837 theory concerning non-existence of rational roots of Descartes-Wantzel cubic equation is correct it does not imply impossibility of trisection of 60^o angle. This is because according to the construction method introduced herein square of cosine of the trisected angle $\cos^2\alpha$ is related to cosine of its double $\cos 2\alpha$ thus requiring extraction of square root that is constructible rather than cubic root requiring rational solution of Descartes-Wantzel equation. In addition, the earlier formulation of the problem by Descartes the father of algebraic geometry is discussed. If one assumes that the ruler and compass employed in the geometric constructions are Platonic ideal instruments then the trisection solution proposed herein should be exact.

Keywords: The trisection problem, angle trisection, Wantzel theory, regular polygons, heptagon.

1 Introduction

The classical trisection problem requires trisecting an arbitrary angle employing only a compass and a straightedge or unmarked ruler. The general rules concerning the construction instruments and acceptable solution of the problem are most eloquently described by Dunham [4]

Indeed, Greek geometers performed trisection by introducing auxiliary curves like the quadratrix of Hippias or the spiral of Archimedes, but these curves were not themselves constructible with compass and straightedge and thus violated the rules of the game. It is rather like reaching the top of Everest by helicopter: It achieves the end by an unacceptable means. For a legitimate trisection, only compass and straightedge need apply.

The second rule is that the construction must require only a finite number of steps. There must be an end to it. An "infinite construction," even if it has trisection as a limiting outcome, is no good. Construction that

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goes on forever may be the norm for interstate highways, but it is impermissible in geometry.

Finally, we must devise a procedure to trisect *any* angle. Trisection a particular angle, or even a thousand particular angles, is insufficient. If our solution is not general, it is not a solution."

In this Note an unexpectedly simple solution of the ancient Greek trisection problem will be presented that results in Archimedes solution without the two marks on his ruler. Historically, it was proven by Wantzel [1] that the trisection of 60^o angle by only compass and straightedge is impossible if such a construction requires the existence of rational roots of the cubic equation

$$x^3 - 3x - 1 = 0 \tag{1}$$

as described by *Dunham* [4]:

(a) If we can trisect the general angle with compass and straightedge,

(b) Then we can surely trisect a 60^o angle,

(c) So, we can find a constructible solution of $x^3 - 3x - 1 = 0$,

(d) So, we can find a rational solution for $x^3 - 3x - 1 = 0$,

(e) And this rational solution must be either c/d = 1 or c/d = –1.

when x is a rational number denoted by the ratio x = c/d. Since by Wantzel's [1] proof Eq. (1) is irreducible and (d) is not true then one must conclude that (a) cannot be true. The algebraic equation (1) in Wantzel's theory [1] originates from the trigonometric equation

$$\cos\theta = 4\cos^3(\theta/3) - 3\cos(\theta/3) \tag{2}$$

that when applied to the angle $\theta = \pi/3$ with $x = 2\cos(\theta/3)$ results in Eq. (1).

A cubic equation of the form

$$x^3 + qx + r = 0 \tag{3}$$

was also employed by Descartes [6] in connection to the trisection problem [7]

"Descartes dealt with the problem of angle trisection by reducing the problem to a third-degree equation and constructing it via intersection of circle and parabola"

Descartes proposed a solution of trisection problem by employment of a parabola, a non-constructible hence transcendental curve, as shown in Fig. 1 reproduced from his book of geometry [6]

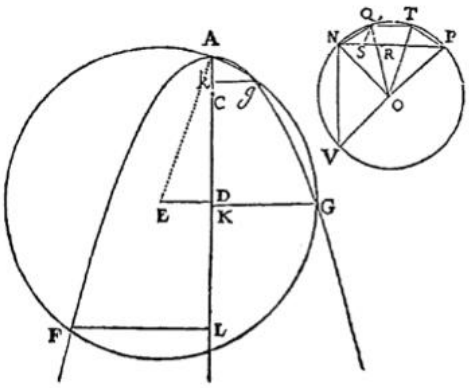


Fig. 1 Descartes solution of trisection of an arbitrary angle θ = NOP employing a parabola GAF [6].

By geometric construction based on the trisected angle shown in Fig. 1 *Descartes* arrived at the cubic equation [6]

$$z^3 - 3z + q = 0 \tag{4}$$

where z = NQ and q = NP. It is now clear that for $\theta = 60 = \pi / 3$ and a circle of unity radius NO = 1 one has NP = q = NO = 1 and both Eq. (4) as well as the trigonometric relation

$$\sin(\theta / 2) = 3\sin(\theta / 6) - 4\sin^3(\theta / 6) \tag{5}$$

with the definition $z = 2\sin(\theta / 6)$ lead to

$$z^3 - 3z + 1 = 0 \tag{6}$$

that is identical to Eq. (1) with $z = -x$. One notes that the chord NQ in Fig. 1 of Descartes becomes the unknown $NQ = z = 2\sin(\theta / 6)$. In view of the equivalence of equations (1) and (6) the failure of Wantzel [1] to reference the work of Descartes is unfortunate. The important contributions of Gauss, Ruffini, and Abel have been discussed [2, 3].

If parallel to Eq. (5) instead of full angle in Eq. (2) one applies the trigonometric identity for half angle

$$\cos(\theta / 2) = 4\cos^3(\theta / 6) - 3\cos(\theta / 6) \tag{7}$$

and considers $\theta = 60 = \pi/3$ with $y = 2\cos(\theta / 6)$ one arrives at

$$y^3 - 3y - \sqrt{3} = 0 \tag{8}$$

that also does not possess any rational roots. It is interesting to note that cubic equation of the type

$$x^3 + b^2 x = b^2 c \tag{9}$$

that was first solved by Omar Khayyam using intersection of conics [8] also reduces to equations (1) and (6) when $(b = i\sqrt{3}, c = -1/3)$ and $(b = i\sqrt{3}, c = 1/3)$ respectively.

According to Wantzel's theory of 1837 [1] only rational numbers $x = c / d = \square$ that are roots of algebraic equations are acceptable solution to the trisection problem. This is because the criterion of geometric constructability based on *Descartes's* analytic geometry only admits rational operations of addition, subtraction, multiplication, division, and extraction of square roots thus requiring existence of rational roots of polynomials of various degrees [1-11]. Over three decades after Wantzel's work Hermite (1873) and then Lindemann (1882) respectively proved the existence of transcendental numbers e and π [4]. It is known that concerns about basing geometric constructability only on the application of geometrical (algebraic) curves and not mechanical (transcendental) curves were raised by ancient Greek mathematicians as well as Newton [7].

Clearly, in view of constructability of 17-sided polygon proved by Gauss, impossibility of trisection of 60º angle is counterintuitive. The main question regarding Wantzel's proof concerns rational numbers and their connection to geometric constructability. Since $\sqrt{2}$ that is not a rational number is nonetheless constructible, the proof rests on non-existence of constructible method for extraction of cubic root required by Eq. (1). However, when applying Descartes analytical geometry to translate geometric constructions into algebraic equations the question of uniqueness arises. In other words, the assumption that trisection of 60º angle only involves geometric constructions that lead to Eq. (1), hence steps (c)-(e) above, may not be valid. Therefore, Wantzel's proof may not rule out trisection of 60º angle because a construction method that does not require extraction of cubic root may exist. Such a situation will be somewhat similar to von Neumann's proof of impossibility of hidden variables in quantum mechanics [12] and the fact that later it was found to be inapplicable to quantum mechanics not because of an error in the theory but rather due to invalid assumptions made in its axiomatic foundation.

Finally, the impossibility of trisection of an arbitrary angle poses a fundamental paradox concerning reversibility of mathematical operations in analytic geometry. In other words, if ruler and compass are capable of tripling an arbitrary angle as shown in Fig. 2, why these same instruments could not trisect a given arbitrary angle? Since mathematical operations are considered to be non-dissipative and do not generate entropy, one expects that ruler and compass should be capable of undergoing a reversible process by trisecting an arbitrary angle.

2 Archimedes Marked Ruler Solution

Because the solution of trisection problem described in the following Section is closely related to Archimedes' classical contribution known as *Marked Ruler Solution* [4, 5] schematically shown in Fig. 2, the latter solution will be discussed first. Given the arbitrary angle $\theta = 3\alpha = \angle \mathrm{BOA}$ and circle with arbitrary radius $\mathrm{R}' = \mathrm{OB}$, one places a ruler with two marks at points C and D separated by distance $\mathrm{R}' = \mathrm{CD}$ at point B and moves the ruler to obtain Fig. 2 thus leading to the trisected angle $\alpha = \angle \mathrm{CDO} = \theta / 3$. Archimedes recognized that this solution violated the construction rule since it involved a marked ruler and the process known as verging [4, 5].

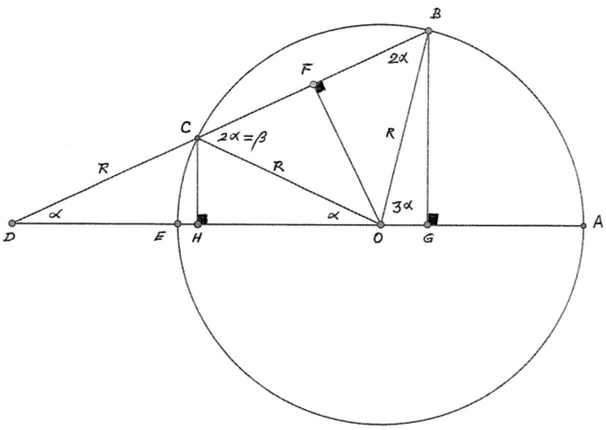


Fig. 2 Archimedes Marked Ruler solution (with ruler marks at points C and D) for trisecting an arbitrary angle $\theta = 3\alpha = \angle \mathrm{BOA}$ into $\alpha = \angle \mathrm{BDA} = \theta / 3$ constructed on Geogebra.

However, the geometric construction shown in Fig. 2 could be viewed from an entirely different perspective namely how to solve the inverse problem of constructing the angle $\theta = 3\alpha = \angle \mathrm{BOA}$ given an arbitrary angle $\alpha = \angle \mathrm{BDA}$. Hence, given the angle $\alpha = \angle \mathrm{BDA}$, one places the compass at point D with arbitrary radius $\mathrm{R}' = \mathrm{CD}$ to find point C and similarly from point C at

$R' = OC$ one finds point O and with compass at O one finds the point B at $R' = OB$. The External Angle Theorem applied to triangle $\triangle BDO$ gives $\theta = 3\alpha = \angle BOA$. With unity radius, $DC = CO = OB = R' = 1$ the $\triangle DBG$ in Fig. 2 leads to

$$\cos\alpha = \frac{2\cos\alpha + \cos 3\alpha}{2\cos 2\alpha + 1} = \frac{2\cos\alpha + \cos 3\alpha}{4\cos^2\alpha - 1} \tag{10}$$

The second equality in Eq. (10) leads to Descartes-Wantzel cubic Eq. (1) whereas the first equality results in

$$\cos\alpha = \frac{\cos 3\alpha}{2\cos 2\alpha - 1} \tag{11}$$

that for $\alpha = 20^o$ gives

$$2\cos 20 = \frac{1}{2\cos 40 - 1} \tag{12}$$

with the unknown x = 2cos20 now inversely related to cos40.

To show that Archimedes angle trisection avoids Descartes-Wantzel cubic equation (1) one notes that application of External Angle Theorem to $\triangle DCO$ of Fig. 2 results in $\beta = 2\alpha$. Also, in $\triangle CFO$ of Fig. 2 equalities $CO = CD = b'$ and $CD = CO = R'$ result in $\cos\beta = b' / R' = b$. Similarly, defining $DH = HO = a'$ in $\triangle DCH$ leads to $\cos\alpha = a' / R' = a$. Finally, since lines CB and CD are aligned, $\triangle DFO$ gives

$$\cos\alpha = a = \frac{1+b}{2a} = \frac{1+\cos\beta}{2\cos\alpha} \tag{13}$$

resulting in the trigonometric identity

$$\cos\beta = \cos 2\alpha = 2\cos^2\alpha - 1 \tag{14}$$

Hence, according to equation (14), the square of cosine of trisected angle $\cos^2\alpha$ is related to cosine of its double $\cos 2\alpha$ thus requiring extraction of square root that is constructible rather than cubic root associated with rational solution of Descartes-Wantzel equation. Therefore, Archimedes solution shown in Fig. 2 avoids the constraint imposed by existence of rational roots of Descartes-Wantzel equation (1). As a result, even though Wantzel's proof concerning non-existence of rational roots of Eq. (1) is valid, it does not rule out trisection of 60^o

angle because according to Eq. (14) admissible construction algorithm not involving extraction of cubic root exists.

Archimedes solution shown in Fig. 2 is better illustrated in Fig. 3

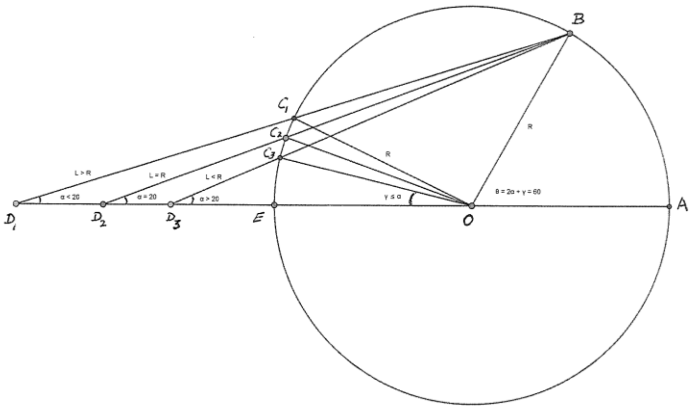


Fig. 3 Division of angle $\theta = \angle BOA$ into two parts $\theta = 2\alpha + \gamma$ constructed on Geogebra.

that helps to identify two distinguishable types of partition of an angle θ as

$$\theta = 2\alpha + \gamma \qquad \gamma \neq \alpha \tag{15a}$$

$$\theta = 2\alpha + \gamma = 3\alpha \qquad \gamma = \alpha \tag{15b}$$

Hence, for the problem of dividing angle θ into two parts, one has the general case in Eq. (15a) and the special case corresponding to angle trisection in Eq. (15b). Depending on the relative size of the angles (α, γ) one has three distinguishable cases associated with relative lengths L = (C_1D_1, C_2D_2, C_3D_3) versus radius R'

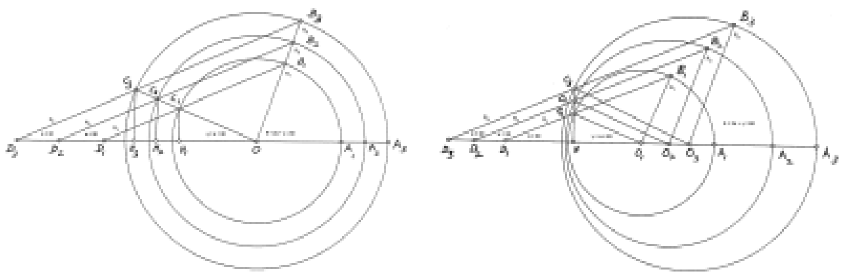
$$L > R' \qquad \gamma > \alpha \tag{16a}$$

$$L = R' \qquad \gamma = \alpha \tag{16b}$$

$$L < R' \qquad \gamma < \alpha \tag{16c}$$

In view of the continuity of case $L > R'$ across $L = R'$ to $L < R'$ clearly shown in Fig. 3, it is reasonable to anticipate that constructible trisection of all angles and not just angles such as 45°, 90°, 180°, … should be possible. In other words, amongst the numbers associated with cosines of angles cos20 = 0.9396926208, cos20.01 = 0.9396329127, and cos19.99 = 0.9397523002, there is nothing unique or especial about the number cos20 = 0.9396926208. Therefore, one expects that given a desired degree of accuracy, cosines of all angles should be constructible as ratios of various real numbers that themselves are either irrational or transcendental. Amongst infinite number of possible partitions $\theta = 2\alpha + \gamma$, the trisection solution corresponds to the unique case $\theta = 2\alpha + \gamma = 3\alpha$.

A hierarchy of solutions similar to that shown in Fig. 2 but corresponding to different values of (a', R') for a given θ are shown in Fig. 4. As (a', R') changes, either the position H is fixed with the origin O moving to the right as in Fig. 4(a) or the location of origin O is fixed with position H moving to the left as in Fig. 4(b). Hence, the solution in Fig. 2 corresponds to a pair (a', R') values amongst the hierarchy of solutions shown in Fig. 4.



(a) (b)

Fig. 4 Hierarchy of solutions of trisection of $\theta = \angle BOA$ with (a) fixed-H and (b) fixed-O coordinates.

In view of Figs. 2-4, in the following Section we do not consider $\triangle DBG$ that directly relates angles (α, 3α) leading to Eq. (10) and hence Descartes-Wantzel equation (1). Instead, one considers $\triangle DFO$ that relates angles (α, 2α) and leads to Eq. (14).

3 Solution of Trisection Problem

Following Archimedes, one looks for the solution of trisection problem by attempting to arrive at geometric construction shown in Fig. 2. For clarity of presentation, the solution is first applied to the case $\theta = 3\alpha = 75^o$ and the historically more important case $\theta = 3\alpha = 60^o$ is considered next. The geometric construction made on Geogebra is shown in Fig. 5

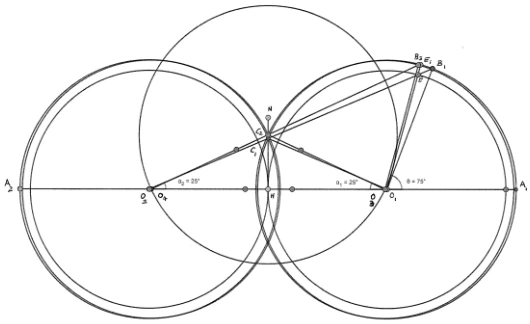


Fig. 5 Trisection of angle $\theta = 3\alpha = \angle EO_1A_1 = 75^o$ into $\alpha = \theta / 3 = \angle B_3O_4A_1 = 25^o$ constructed on Geogebra

The solution of trisection problem shown in Fig. 5 that resembles the *Olympic* sign, most appropriately for this ancient Greek problem, involves the following steps:

(1) On line A_1A_2 make two circles at points (O_1, O_2) with arbitrary radius $O_2H = O_1H = R'$ and draw the line of symmetry NH perpendicular to line O_1O_2 at midpoint H. Construct the given angle $\theta = 3\alpha = \angle EO_1A_1$ to be trisected at point O_1 to get point E with $O_1E = R'$.

(2) Draw line from point E to point O_2 to cross line of symmetry NH at point C_1.

(3) Draw two circles at points (O_1, O_2) with radius $C_1O_2 = C_1O_1 = R_1$. Extend line O_1E to get point E_1 and line O_2E to get point B_1 on circle. Note that by Archimedes solution (Fig. 2) with $\angle B_1O_2A_1 = \beta$ one has $\angle B_1O_1A_1 = 3\beta$.

(4) Connect point E_1 to O_2 to get point C_2 on line of symmetry NH. Placing compass at point C_2 make arcs at radius $C_2O_3 = C_2O_4 = R_1$ to find new origins (O_4, O_3) on line A_1A_2. Make two circles at origins (O_4, O_3) with radius R_1 and extend line O_4C_2 to get point B_3 on circle.

(5) Connect point B_3 to point O_3. Since $O_4C_2 = O_3C_2 = O_3B_3 = R_1$ by construction, $B_3O_3 = E_1O_1$, and $B_3O_3 \parallel O_1E_1$, in accordance with Archimedes solution (Fig. 2) one gets $\angle C_2O_4H = \angle C_2O_3H = \alpha = 25^o$ and External Angle Theorem applied to triangles $\triangle C_2O_4O_3$ and $\triangle C_2O_3B_3$ give $\angle B_3C_2O_3 = \angle C_2B_3O_3 = 2\alpha = 50^o$ and $\angle B_3O_3A_1 = 3\alpha = 75^o$.

Figure 6 shows the application of the solution method following steps (1)-(5) above to trisect angle $\theta = \angle EO_1A_1 = 60^o$.

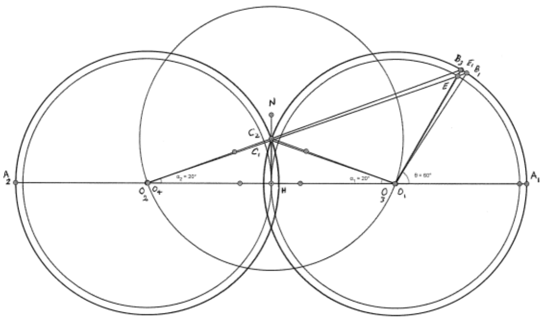


Fig. 6 Trisection of angle $\theta = 3\alpha = \angle EO_1A_1 = 60^o$ into $\alpha = \theta / 3 = \angle B_3O_4A_1 = 20^o$ constructed on Geogebra.

An equivalent but simpler solution of trisection problem applied to trisect angle $\theta = \angle EO_1A_1 = 60^o$ is shown in Fig. 7. After steps (1)-(2) above, in reference to Fig.7, one places compass at point C making arcs at radius $CO_3 = CO_4 = O_1E = R$ to find points $(O_3,\ O_4)$ on line A_1A_2. From points (O_3, O_4) make two lines parallel to (O_1E, O_2E') that cross extensions of the lines (O_4C, O_3C) at points (B, B') , respectively. Draw a line perpendicular to line A_1A_2 at point O_3 to cross line O_4B at point D and draw line $CF \parallel O_3H$ to get point F. One can show that $\angle CO_4H = \angle CO_3H = \angle FCO_3 = \angle DCF = \alpha$ and $\angle CBO_3 = \angle BCO_3 = 2\alpha$. Application of External Angle Theorem to $\triangle O_4BO_3$ results in $\angle BO_4H + \angle O_4BO_3 = \alpha + 2\alpha = \angle BO_3A_1 = 3\alpha = 60^o$. Hence, angle $\angle BO_4H = \alpha = 20^o$ is the trisected angle.

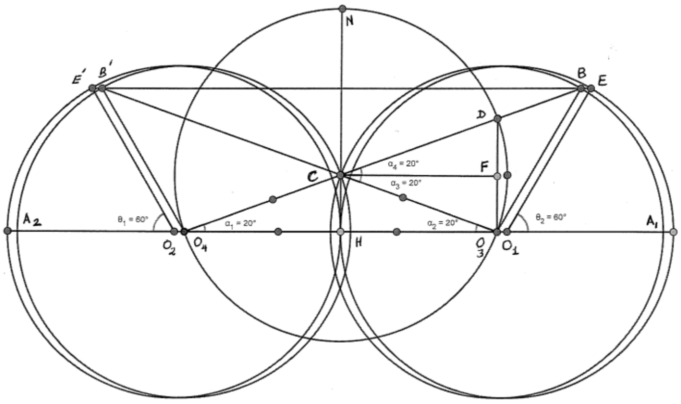


Fig. 7 Application of the equivalent but simpler solution for trisection of $\theta = 3\alpha = \angle EO_1A_1 = 60^\circ$ into $\alpha = \theta / 3 = \angle BO_4H = 20^\circ$ constructed on Geogebra.

Although trisection of 90º angle by ruler and compass is trivial, to prove applicability of the solution method to the special case $\theta = 3\alpha = \angle B_1O_1A_1 = 90^o$ construction on Geogebra is shown in Fig. 8.

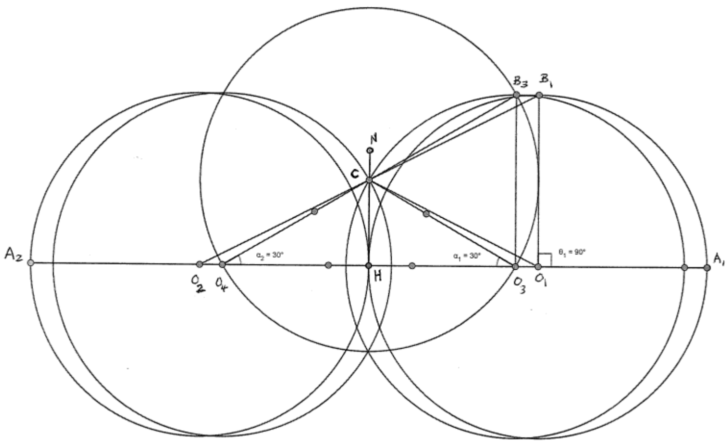


Fig. 8 Trisection of angle $\theta = 3\alpha = \angle B_1O_1A_1 = 90^\circ$ into $\alpha = \theta / 3 = \angle B_3O_4A_1 = 30^\circ$ constructed on Geogebra.

The solution also applies to angles larger than $\theta > 90^o$ and $\theta > 180^o$ since they are expressible as $90^o + \theta$ and $180^o + \theta$ prior to trisection**.** The construction procedure (1)-(5) above gives Archimedes solution that is known to be exact but removes the need for the two marks on his ruler.

A simple argument for non-existence of rational root $\cos\alpha = \cos 20 \neq c/d$ of Eq. (1) could be that this equation is based on a trisection method involving ratios of lengths (hence arithmetic numbers) that are associated with circles of different radii hence *measures*. For example, as shown in Fig. 2, line BE defines the smallest angle $\angle BEA = \theta/2 = 1.5\alpha$ associated with the arc (AB) with $EA = R_{max} = 2R_1$. For smaller angles such as $\angle BDA = \theta/3 = \alpha < 1.5\alpha$ line BD will have point D lie outside of the circle of radius R_1 as shown in Figs. 2 and 3. Accordingly, equation (1) of Wantzel that simultaneously relates angles $\angle BDA = \alpha = 20^o$ and $\angle BOA = 3\alpha = 60^o$ (Fig. 2) involves $R_2 > R_1$ thus a larger circle with radius $R_2 = DB = 2R_1 + R_1 \cos\theta / \cos\alpha$. Inside circle of radius R_1 dimensionless lengths will involve numbers in the range $r_\beta = (0 \to r'/R_1 = 1)_\beta$ with subscript β referring to scale. According to a scale-invariant definition of hierarchies of embedded coordinates shown in Fig. 9 [16]

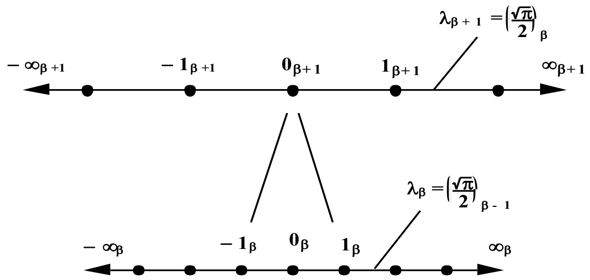


Fig. 9 Hierarchy of normalized coordinates for cascades of embedded statistical fields [16].

the normalized coordinate

$$r_\beta = \frac{2}{\sqrt{\pi_{\beta-1}}} \int_0^{r'_{\beta-1}} e^{-y^2} dy = \mathrm{erf}\,(r'_{\beta-1}) \tag{17}$$

relates the number-range of adjacent scales as $(0,1)_\beta \Leftrightarrow (0,\infty)_{\beta-1}$. Therefore, Archimedes solution gives $\cos\alpha = (c/d)_{R_1}$ because it relates $(\cos\alpha)_{R_1}$ to $(\cos 2\alpha)_{R_1}$ *inside* the circle of radius R_1 involving rational numbers defined in terms of a common measure. However, this same angle $\angle BDA = \alpha$ based on the outer radius R_2 cannot be expressed as $(\cos\alpha)_{R_1}$ because it lies outside of the number field of R_1 circle and requires number field of R_2 with a different measure for its normalization. The incommensurability of the number fields of R_1 versus R_2 circles, due to their different measures and transcendental nature of the number π, leads to $(\cos 3\alpha)_{R_1}$ and $(\cos\alpha)_{R_2}$ in equation (10) thereby accounting for non-existence of rational root of Descartes-Wantzel equation (1).

If one assumes that the ruler and compass employed in geometric construction of Archimedes solution shown in Figs. 2 and 5 are Platonic ideal instruments, the trisection solution presented herein is exact. Clearly, the precise meaning of "exactness" just mentioned is intimately connected to the *continuum* problem [13] thus requiring elaborate mathematics of re-normalization [14], Internal Set Theory [15], and scale-invariant definition of hierarchies of infinitesimals [16]. An ideal Platonic ruler is defined as an instrument that is capable of resolving spatial coordinates at all infinite scales [16]. A Platonic compass is an ideal compass that is capable of following all points of a Platonic ruler. Therefore, a Platonic ruler can resolve all real numbers on line, including irrational and transcendental ones, to any desired accuracy by choosing ever-smaller measures ad infinitum since each point on the real line contains an infinite number of Aristotle's potential infinite [16].

The procedures outlined in steps (1)-(5) can be applied to construct regular polygons of various sizes such as nonagon (9-sides) having 40° angles per sector shown in Fig. 10.

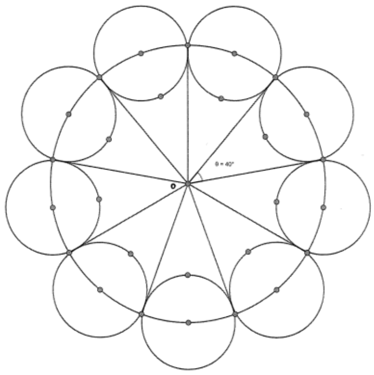


Fig. 10 Inner and outer circles of regular nonagon with 40° sectors constructed on Geogebra.

The result shown in Fig. 10 also leads to regular 18-sided polygon with 20° per sector or its conjugate a regular 20-sided polygon with 18° per sector. Finally, the trisection solution described above could be employed to construct a heptagon following the algorithm discovered by Gleason [17].

4 Concluding Remarks

The unexpectedly simple solution shown in Fig. 5 fully justifies the intuition of all Trisectors [18, 19] amongst both professional and amateur mathematicians since Wantzel's 1837 paper who believed that solution of the trisection problem might indeed be possible.

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