Chaotic Modeling and Simulation (CMSIM) 3: 519–527, 2012

Linear Least Squares Estimate of Noise Level in Chaotic Time Series via L_∞ Norm Correlation Sum

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Abstract. In this work we extend the details of a linear least squares method to estimate the noise level in chaotic time series which has been previously proposed in [1]. For this purpose we analyze a non iterative algorithm based on the functional form obtained by Schreiber in 1993 where the effects of noise on L_∞ norm correlation sums can be quantified via the nonlinear functional. The modified version of the functional leads to a linear approach that gives satisfactory results for simulated continuous flow data even for high level of noise contamination (up to 80%). The approach is especially useful to determine the effective fitting range of data. The range is limited by the curvature effects of the attractor and fluctuations in small scales. We also seek for a phenomenological model for the curvature effect depending on the empirical distribution of estimation errors.
Keywords: linearization, noise level, chaos, curvature effect, data analysis.

1 Introduction

For the last three decades, analysis of chaotic time series has seen a great many numbers of improvements and has became one of the most demanded approach while investigating the systems with unpredictable complex behavior. Chaotic analysis of complex systems usually takes its form through determination of global structural properties (invariants) and the concept of nonlinear prediction. The system under investigation may be perceived as a random fluctuation whereas its behavior is controlled by a system of nonlinear deterministic equations, sometimes disturbed by observational or dynamic noise source. Although it is possible to separate the noise and signal via conventional spectral techniques, it is not the case for chaotic systems which show broadband spectra.

Received: 30 July 2011 / Accepted: 31 March 2012

ISSN 2241-0503

For such a situation, it is crucial to comprehend the main source of fluctuation whether it is generated by pure random noise or any nonlinear dynamical system. If we are interested in the analysis of real world observations such as financial time series, atmospherical measurements or trajectory of planetary motions, then the observations are mostly a combination of the two. However the weights in this combination may be related to the nature of the phenomena.

For noisy observations of a deterministic system, random noise contaminant is evaluated as a negative effect that give rise to serious bias in the estimations of statistical quantities related to the dynamics. This effects the reliability of the information obtained from the system which is supposed to define its overall behavior. Especially for the algorithms to calculate the invariants of the chaotic dynamics, noise narrows the effective scaling ranges for computations, since most of them has been derived under noise-free assumptions. Taking into account of the effects of random noise on the analytical form of the invariants is also advantageous to describe the exact amplitude of noise corruption which can be extracted from the usual invariant statistics. Many of the algorithms have been proposed using the mentioned framework.

In this work, we give a linear least square algorithm for the noise level determination approach used in [2]. We also seek for a phenomenological model for the curvature effects depending on the empirical distribution of estimation errors. The sections are arranged as follows: in Section 2 we give the brief results of the literature dealing with the noise estimation algorithms depending on correlation sum. We also describe a new linear algorithm for noise estimation. In Section 3 we discuss about the *curvature effects* which has strong bias for the map data while using Schreiber's approach. In this section we also propose an empirical model for the characterization of this effect depending on our high-resolution simulations using synthetic chaotic data. Our conclusions and future perspectives about some open problems are given in Section 4.

2 A Linear Algorithm for Noise Estimation

In chaotic systems, the spatial distribution of phase space vectors follows the power law for relatively small length scales compared to the attractor size. Noise shows its disturbing effects on the distribution of nearby vectors that are closer than ϵ distance in phase space. General approach to determine the noise amplitude is to append the analytical form of the disturbance effects on to the mathematical form of invariant descriptions. For example Liu et.al [11] proposed an analytic technique where noise level could be estimated from the geometrical form of the exponential divergence curves [12]. On the other hand studies adopting correlation sum approaches exploit the effects of noise disturbance on the point density over the attractor. Due to the self-similarity, the point distribution follows the very basic power law. If we consider the definition of correlation sum approach [13], the point distribution depends on the fractal dimension of (D_2) the system. For the length scales $\epsilon \to 0$ the distribution is given by,

$$C(\epsilon) \propto \epsilon^{D_2} \tag{1}$$

where $C(\epsilon)$ is defined as,

$$C(\epsilon) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=1}^{N} \Theta(\epsilon - \parallel \mathbf{s}_i - \mathbf{s}_j \parallel) \tag{2}$$

In Eq.(2), $\parallel \cdot \parallel = \parallel \cdot \parallel_{L_\infty}$ and $\mathbf{s}_{i,j}$ are embedding vectors satisfying $\mathbf{s}_i = (y_i, y_{i+\tau}, \ldots, y_{i+(m-1)\tau})$ that are generated by the time-delay reconstruction of the time series sequence $\{y_i\}_{i=0}^{N}$ with delay time (τ) and embedding dimension (m). Then the Heaviside step function $\Theta(\cdot)$ is used to estimate the probability of a nearby trajectory vector to fall inside the selected hypercube of side length 2ϵ.

If the measurements are noisy, then the functional form in Eq.(2) for L_2 can be represented by the complicated form given by Smith [3] and a modified version discussed in [6]. By using the correlation sum definition in [6], Jayawardena et.al [4] gave a linear least squares approach to detect the noise level in chaotic time series. They have shown that the correlation function in [6] satisfies an ordinary linear differential equation where it is possible to extract both dimension and noise level information by a least squares fitting of calculated correlation sum data.

Our approach here is different from [4] in terms of the norm definition. Schreiber [2] has shown that the L_∞ norm definition of correlation sum can be used to estimate the noise amplitude σ via the nonlinear functional $g(\cdot)$. In this case the effects of noise on the spatial distribution is characterized by (the usual form),

$$d_m(\epsilon) = d_r(\epsilon) + (m-r) g(\frac{\epsilon}{2\sigma}), \quad g(z) = \frac{2}{\sqrt{\pi}} \frac{z\, e^{-z^2}}{\mathrm{erf}(z)} \tag{3}$$

The correlation dimension estimates obtained from n dimensional embedding space d_n is defined by,

$$d_n = \lim_{\epsilon\to 0} \lim_{N\to\infty} d_n(\epsilon), \quad d_n(\epsilon) = \frac{d \ln(C_n(\epsilon))}{d \ln(\epsilon)} \tag{4}$$

In Eq.(3) embedding space of m dimensions should theoretically satisfy $m > r > 2d$, whereas $m > r > d$ give better estimates. It is possible to show that the noise functional $g(z)$ can be represented in terms of the confluent type hypergeometric function of the first kind ${}_1F_1$, such that,

$$g(z) = \left({}_1F_1(1, \frac{3}{2}, z^2) \right)^{-1} \tag{5}$$

where ${}_1F_1(a, c, x)$ can be defined by the integral representation in Eq.(6).

$${}_1F_1(a, c, x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xt}\, t^{a-1} (1-t)^{c-a-1}\, dt \quad c > a > 0 \tag{6}$$

We approximate the gaussian noise functional by a *stretched exponential decay* function (Eq.(7)), where longer derivations including the asymptotic expansion of the form in Eq.(5) was explained in [1]. The final form is,

$$g(z) = \frac{2}{\sqrt{\pi}} \frac{z\, e^{-z^2}}{\mathrm{erf}(z)} \approx e^{-\alpha\, z^{\lambda}} \tag{7}$$

in which (α, λ) are optimized parameters for $g(z)$ (see [1]). Although there has been successful attempts to directly fitting (global optimum) the exponentials, it is noted that there is still not a direct technique that linearize the exponentially scattered data in to a linear one except ordinary *log* transformation (see [7]). In the present work we follow a relatively practical way to linearize the functional form that was obtained in Eq.(7) by converting the original exponential fitting problem in to an initial value problem (IVP) which is linear in its parameters [14]. Since $g(z) = \exp(-\alpha z^{\lambda})$, then $g(z)' = -\alpha\, \lambda z^{\lambda-1} g(z)$ with initial condition $g(0) = 1$. Despite being a basic property of dimension estimates for small length scales, statistical fluctuations of the data accumulated in $g(z)$ can be efficiently smoothed via the integration based solution. Finally, the linear least squares algorithm for the solution of the mentioned differential equation for $z_i = \epsilon_i / 2\sigma$ yields with,

$$\min \left(\sum_{i=1}^{N} w(z_i) \left\{ g(z_i) + \alpha\, \lambda \int_0^{z_i} z^{\lambda-1} g(z)\, dz - g(0) \right\}^2 \right) \tag{8}$$

including the multiplier term $w(z_i)$ as the statistical weighting factors.

System $N = 20000$	NR	NR *(real)*	$\hat{NR}$ *(estimated)*	σ *(real)*	$\hat{\sigma}$ *(estimated)*	Linear Region
Henon	0.05	0.0502	0.0518	0.0363	0.0374	0.12-0.77
$\sigma_s = 0.72210$	0.20	0.1995	0.2285	0.1444	0.1653	0.19-0.71
	0.50	0.5015	0.6094	0.3629	0.4410	0.17-0.47
Rössler	0.10	0.1004	0.1122	0.8020	0.8957	0.07-0.71
$\sigma_s = 7.9328$	0.40	0.3999	0.4451	3.1933	3.5534	0.16-0.77
	0.80	0.8001	0.8698	6.3876	6.9442	0.23-0.88
Lorenz	0.10	0.1001	0.1090	0.7923	0.8628	0.11-0.734
$\sigma_s = 7.9193$	0.40	0.3989	0.4452	3.1575	3.5251	0.15-0.77
	0.80	0.7963	0.8548	6.3047	6.7675	0.23-0.70

Table 1. Estimated noise amplitudes ($\hat{\sigma}$) by the proposed linear algorithm in Eq.(8). Number of observations for all series is $N = 20000$ where transients are discarded.

We test the algorithm on Lorenz, Rössler flow systems and Henon map to see the efficiency (Table.1). Here and in the following, NR ratio is defined as the standart deviation of the noise contaminant σ_n divided by the standard deviation of the original noise free signal σ_s, $NR = \sigma_n/\sigma_s$. In Table.1 it can be observed that the linear algorithm works relatively well for the flow data whereas it has strong positive bias for maps. It is clear that the noise estimations for flow systems give reasonably acceptable estimates for extreme noise levels up to 80% $(NR = 0.8)$. In Section 3 we discuss the dominant effect of curvature of the attractor geometry which cause bias for map data.

3 Empirical Modeling of Curvature Effects: The Peak Function Approach

For some statistics obtained from a chaotic time series, the *curvature effect* is dominant. From a technical point of view, the curvature effect is highly related to the limit assumption of $\epsilon \to 0$ made on the point distribution over the attractor. In the literature the bias of density estimates sourced from geometric effects is related to various concepts such as *edge* or *boundary* effect. The sparsity pattern in relatively small scales also cause measurement bias and related to the *lacunarity* of the attractor. However the relationship between noise level and the bias of estimations caused by macroscopic geometric effects has not been clearly described. In this section our aim is to represent the results of our simulations and describe the effects of attractor geometry when the noise level is extremely high.

The point density measurements obey the power law for very small scales. However for large distances ($\epsilon >> 0$) the macro-scale geometrical characteristics of the attractor is dominant for the density measurements which may violate the power law assumption. This problem comes into prominence especially for the correlation dimension estimates where they suffer from positive bias.

Noise level determination algorithms that use information coming from for both micro-macro geometrical features are highly effected from the curvature effects. For instance the ones that use the point density measurements. For low level of noise ratios (NR) the estimated amplitudes are very accurate, whereas estimated noise levels that are comparable to the size of the original attractor show strong positive bias up to a 50% relative error rate. Here and in the following, we define the relative error rate (RE) as the deviation of estimated noise amplitude $\hat{\sigma}$ from the real measurement σ_n, $RE = (\hat{\sigma} - \sigma_n)/\sigma_n$. It is known that Schreiber's algorithm gives overestimated results for $0.2 < NR < 0.8$ which was investigated by Leontitsis et.al in [8]. They have shown that the maximum norm estimation of noise function can be used as a practical way to eliminate negative effects curvature. The prediction algorithm has been also implemented to an adaptive locally projective noise reduction technique in [9]. The algorithm is also useful to describe the overall noise measurements through non-adaptive techniques while analyzing financial time series [10].

In this section we investigate the empirical properties of curvature effects and give a phenomenological model for the distribution of error rates. For a computational description in noise-free systems we refer the reader to the work in [15]. Analytic description of the effect is possible, however we do not consider such a mathematical model here. Here we will investigate another situation: Destruction of the attractor geometry will eliminate any kind of geometrical effect, since the chaotic attractor is converted in to a '*geometrically formless*' situation resembling the multivariate normal distribution.

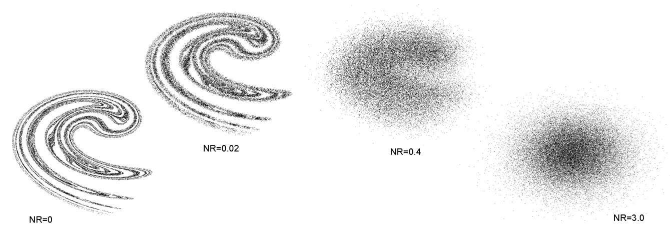

Fig. 1. Destruction of curvature effects on the chaotic Ikeda Map by adding gaussian noise to data ($NR = 0\%,\ 2\%, 40\%, 300\%$).

One way to eliminate all the curvature effects from the attractor is to add appropriate amount of white noise to the original system. In this case, the macro scale geometric form of the system is destroyed and becomes formless (Figure(1)). To support the idea, we test the situation on well known chaotic systems including Henon, Ikeda, Predator-Prey and Lorenz systems. All series have $N = 1500$ observations where transients are excluded. We continuously add normally distributed noise to the systems and estimate the noise level via the nonlinear version of Schreiber's algorithm on $\sigma < \epsilon < 4\sigma$ interval. When the noise ratio exceeds $3\sigma_s$ ($NR > 3$), then any curvature effect is eliminated. The relative error rates for all chaotic systems has a peak position in the interval $0.5 < NR < 1.5$ where the peak level and position seems to depend on the characteristics of the macroscopic geometry. The common features of error distributions is concordant with the behavior of a peak function: 0 level error rates for low noises, a peak point for some noise level and an asymptotic descent of the error rates until convergence to a minimum, while $\sigma_n/\sigma_s \to \infty$. Since we do not make any prior assumptions, we have selected a log-normal type peak function to describe the situation. The relative error rates of a chaotic system arisen from the curvature effect is then modeled by h function in Eq.(9).

$$h(\sigma) = re_0 \left(\frac{\sigma}{\sigma + 1} \right) + \frac{\kappa}{\sqrt{2\pi}\, w\, \sigma}\, e^{-\frac{(\ln(\sigma/x_c))^2}{2\,w^2}} \tag{9}$$

If the original series is corrupted by gaussian noise with standard deviation of ε, it is possible to model the curvature effects via adding increasing amounts of noise with standard deviation of σ_{add}. The final noise amplitude is estimated by $\sigma \approx (\sigma_{add}^2 + \varepsilon^2)^{1/2}$. Since we do not have any information about the relative error rates, observed standard deviations should be used. By using the form used in Eq.(9), the peak model yields the final form in Eq.(10),

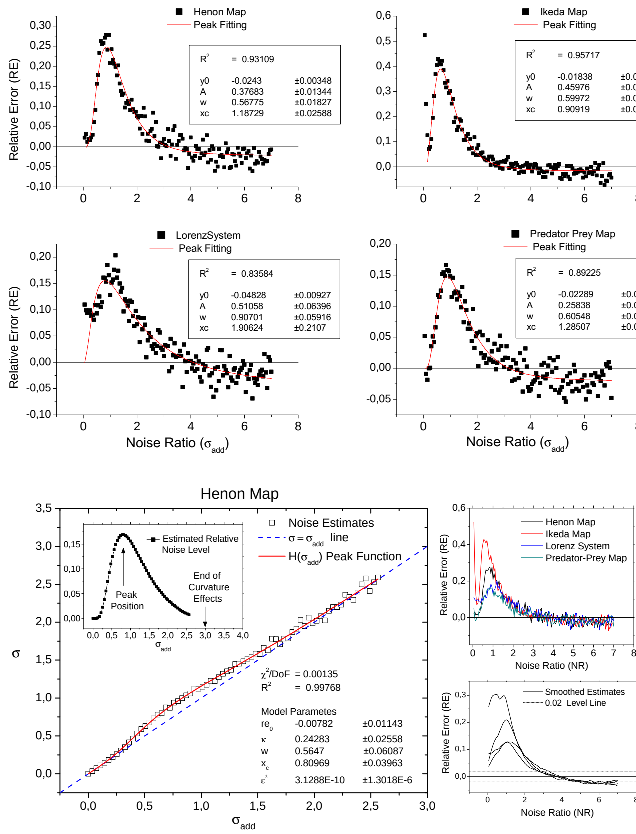

Fig. 2. Upper Panel: The right skewed distribution of relative error rate scores with respect to the noise level for various chaotic systems. The distribution is characterized by a log-normal peak function of type $y = y_0 + \frac{A}{\sqrt{(2\pi)}\, w\, xc} \exp(-(\ln(x) - xc)^2/2w^2)$. Lower Panel: Modeling curvature effects via peak function approach described in the text. Left panel: The noise estimates represent significant deviation from the 45^0 degree line (blue) which can be efficiently modeled by the phenomenological approach. Right panel: The simulation results for four different chaotic systems and moving average smoothed curves.

$$H(\sigma_{add}) = (\sigma_{add}^2 + \varepsilon^2)^{1/2}\,(re_0(\frac{(\sigma_{add}^2 + \varepsilon^2)^{1/2}}{(\sigma_{add}^2 + \varepsilon^2)^{1/2} + 1}) + \frac{\kappa}{\sqrt{2\pi}\, w\, (\sigma_{add}^2 + \varepsilon^2)^{1/2}}\, e^{-\frac{(\ln((\sigma_{add}^2+\varepsilon^2)^{1/2}/x_c))^2}{2\,w^2}} + 1) \qquad (10)$$

which is obviously extremely nonlinear, but still useful to describe the effects. From Figure.(2), it can be seen that the fitting of error function H accommodates well with the simulated data (significance of model and parameters). If the noise level let one to describe the position of peak error rate, it is possible to make better estimates for the exact level of noise via the peak function approach. On the other hand, for noise levels $NR >> 1$ where it goes far beyond the peak position, it would be difficult to make reasonable estimations for exact noise amplitude due to the nonlinearity of $H(\cdot)$ suffering from local minima.

4 Results and Discussion

In this work we have discussed a linear least squares method to estimate the noise level in chaotic time series. The efficiency of the method on map and flow data are discussed. Although the proposed linear approach is used for to estimate the noise level of a chaotic time series, it could also be used to determine the initial feasible estimates for nonlinear algorithms. Positive bias still exists for linear approach where we have proposed a novel approach to model the curvature effects depending on the distribution of estimation errors. Our future investigations will be based on the analytical description of curvature effect.

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