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Positive Solutions for Second Order Multi-Point Boundary Value Problems

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Abstract. By using double fixed point theorem, we study the existence of at least two positive solutions of a second order multi-point boundary value problem.

Keywords: Positive solutions, Fixed point theorem, Boundary value problems.

1 Introduction

In this paper we consider the second order multi-point boundary value problem (BVP)

$$\begin{cases} (\phi(u'(t)))' + q(t)f(t,u(t)) = 0, \quad t \in (0,1), \\ \phi(u'(0)) = \sum_{i=1}^{m-2} a_i \phi(u'(\xi_i)), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\eta_i), \end{cases} \tag{1}$$

where $\xi_i, \eta_i \in (0,1)(i = 1,2,...,m-2)$ with $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1$, $0 < \eta_1 < \eta_2 < ... < \eta_{m-2} < 1$, $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and homomorphism with $\phi(0) = 0$. A projection $\phi : \mathbb{R} \to \mathbb{R}$ is called an increasing homeomorphism and homomorphism if the following conditions are satisfied:
(i) If $x \le y$, then $\phi(x) \le \phi(y)$, for all $x, y \in \mathbb{R}$;
(ii) ϕ is continuous bijection and its inverse mapping is also continuous;
(iii) $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in \mathbb{R}$, where $\mathbb{R} = (-\infty, \infty)$.

We assume that the following conditions are satisfied:

(A1) $f \in \mathcal{C}([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$, $q \in \mathcal{C}[0,1]$ is nonnegative,

(A2) $a_i \in [0,\infty), b_i \in [0,\infty), i = 1,2,...,m-2$ with $0 < \sum_{i=1}^{m-2} a_i < 1$ and $0 < \sum_{i=1}^{m-2} b_i < 1$.

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The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1]. Since then, there has been a lot of recent attention focused on the study of nonlinear multi-point boundary value problems, see [2–5]. We cite some appropriate references here [6–9].

In [8], Ji *et al.* studied the existence of multiple positive solutions for one-dimensional p-Laplacian boundary value problem

$$\begin{cases} (\phi(u'(t)))' + q(t)f(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = \sum_{i=1}^{n} \alpha_i u(\xi_i), \quad u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i). \end{cases} \tag{2}$$

The authors established the existence of multiple positive solutions (2) by using fixed point theorem in a cone.

In [9], Ma *et al.* studied the existence of positive solutions for multi-point boundary value problem with p-Laplacian operator

$$\begin{cases} (\phi(u'(t)))' + q(t)f(t,u(t)) = 0, \quad t \in (0,1), \\ u'(0) = \sum_{i=1}^{n} \alpha_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i). \end{cases} \tag{3}$$

In this paper, motivated by the above research efforts on multi-point boundary value problems, criteria for the existence of at least two positive solutions of the BVP (1) are established by using the double fixed point theorem. Thus, our results are new for differential equations.

This paper is organized as follows. In Section 2, we give some preliminary lemmas which are key tools for our proof. The main result is given in Section 3.

2 Preliminaries

In this section, we give some lemmas which are useful for our main result.

We consider the Banach space $\mathbb{B} = \mathcal{C}^1[0,1]$ endowed with the norm

$$\|u\| = \max_{0 \le t \le 1} |u(t)|.$$

Define the cone $\mathcal{P} \subset \mathbb{B}$ by

$$\mathcal{P} = \{u \in \mathbb{B} : u \text{ is a concave, nonnegative and nonincreasing function,} \\ u(1) = \textstyle\sum_{i=1}^{m-2} b_i u(\eta_i)\}.$$

Lemma 1. *If $u \in \mathcal{P}$, then $\min_{0 \le t \le 1} u(t) \ge M\|u\|$, where* $M = \dfrac{\sum_{i=1}^{m-2} b_i(1-\eta_i)}{1 - \sum_{i=1}^{m-2} b_i \eta_i}$.

Proof. Since $u \in \mathcal{P}$, nonnegative and nonincreasing

$$\|u\| = u(0), \quad \min_{0\le t\le 1} u(t) = u(1).$$

On the other hand, $u(t)$ is concave on $[0,1]$. So, for every $t \in [0,1]$, we have

$$\frac{u(t)-u(1)}{1-t} \ge \frac{u(0)-u(1)}{1},$$

i.e., $u(t) \ge (1-t)u(0) + tu(1)$. Therefore,

$$\sum_{i=1}^{m-2} b_i u(\eta_i) \ge \sum_{i=1}^{m-2} b_i(1-\eta_i)u(0) + \sum_{i=1}^{m-2} b_i\eta_i u(1).$$

This together with $u(1) = \sum_{i=1}^{m-2} b_i u(\eta_i)$, implies that

$$u(1) \ge \frac{\sum_{i=1}^{m-2} b_i(1-\eta_i)}{1-\sum_{i=1}^{m-2} b_i\eta_i} u(0).$$

So, the proof of Lemma is completed. □

Lemma 2. *Assume that* $(A1),(A2)$ *hold. Then* $u \in \mathcal{C}^1[0,1]$ *is a solution to problem* (1) *if and only if* u *is a solution to the integral equation:*

$$u(t) = \int_t^1 \phi^{-1}\left(\int_0^s q(\tau)f(\tau,u(\tau))d\tau + A\right)ds$$

$$+\frac{1}{1-\sum_{i=1}^{m-2} b_i}\sum_{i=1}^{m-2} b_i \int_{\eta_i}^1 \phi^{-1}\left(\int_0^s q(\tau)f(\tau,u(\tau))d\tau + A\right)ds, \tag{4}$$

where

$$A = \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} q(s)f(s,u(s))ds. \tag{5}$$

Proof. First, suppose that $u \in \mathcal{C}^1[0,1]$ is a solution of problem (1). Integrating the equation (1) from 0 to t, one has

$$-\phi(u'(t)) + \phi(u'(0)) = \int_0^t f(s,u(s))ds. \tag{6}$$

and taking $t = \xi_i$ in (6), we have

$$\phi(u'(\xi)) = \phi(u'(0)) - \int_0^{\xi_i} q(s)f(s,u(s))ds.$$

So, we get

$$\sum_{i=1}^{m-2} a_i\phi(u'(\xi_i)) = \sum_{i=1}^{m-2} a_i\phi(u'(0)) - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} q(s)f(s,u(s))ds.$$

Since $\phi(u'(0)) = \sum_{i=1}^{m-2} \alpha_i\phi(u'(\xi_i))$, we have

$$\phi(u'(0)) = -\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} q(s)f(s,u(s))ds = -A. \tag{7}$$

Substituting (7) into (6), we get

$$u'(t) = -\phi^{-1}\left(\int_0^t q(s)f(s,u(s))ds + A\right). \tag{8}$$

Integrating the equation (8) from t to 1, one has

$$u(t) = u(1) + \int_t^1 \phi^{-1}\left(\int_0^s q(\tau)f(\tau,u(\tau))d\tau + A\right), \tag{9}$$

and taking $t = \eta_j$ in (9), we get

$$u(\eta_j) = u(1) + \int_{\eta_j}^1 \phi^{-1}\left(\int_0^s q(\tau)f(\tau,u(\tau))d\tau + A\right)ds.$$

So,

$$\sum_{i=1}^{m-2} b_i u(\eta_i) = u(1)\sum_{i=1}^{m-2} b_i + \sum_{i=1}^{m-2} b_i\int_{\eta_i}^1 \phi^{-1}\left(\int_0^s q(\tau)f(\tau,u(\tau))d\tau + A\right)ds.$$

Since $u(1) = \sum_{i=1}^{m-2} b_i u(\eta_i)$,

$$u(1) = \frac{1}{1-\sum_{i=1}^{m-2} b_i}[\sum_{i=1}^{m-2} b_i\int_{\eta_i}^1 \phi_{-1}\left(\int_0^s q(\tau)f(\tau,u(\tau))d\tau + A\right)ds. \tag{10}$$

Substituting (10) into (9), we get (4), which completes the proof of sufficiency.

Conversely, if $u \in C^1[0,1]$ is a solution to (4), apparently

$$u'(t) = -\phi^{-1}\left(\int_0^t q(s)f(s,u(s))ds + A\right),$$

$$(\phi(u'(t)))' = -q(t)f(t,u(t)),$$

$$\phi(u'(0)) = \sum_{i=1}^{m-2} a_i\phi_p(u'(\xi_i)), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\eta_i).$$

The proof is complete. $\square$

Now define an operator $T : \mathcal{P} \longrightarrow \mathbb{B}$ by

$$Tu(t) = \int_t^1 \phi^{-1}\left(\int_0^s q(\tau)f(\tau,u(\tau))d\tau + A\right)ds$$

$$+\frac{1}{1-\sum_{i=1}^{m-2} b_i}\sum_{i=1}^{m-2} b_i \int_{\eta_i}^1 \phi^{-1}\left(\int_0^s q(\tau)f(\tau,u(\tau))d\tau + A\right)ds. \tag{11}$$

Lemma 3. *Assume that* $(A1)-(A2)$ *hold. Then* $T : \mathcal{P} \to \mathcal{P}$ *is a completely continuous operator.*

Proof. It is clear that $T\mathcal{P} \subset \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator by a standard application of the Arzela-Ascoli theorem.

3 Main Results

In this section we state and prove our main result. The following fixed point theorem is fundamental and important to the proof of main result.

For a nonnegative continuous functional γ on a cone $\mathcal{P}$ in a real Banach space $\mathbb{B}$, and each $d > 0$, we set

$$\mathcal{P}(\gamma, d) = \{x \in \mathcal{P} |\ \gamma(x) < d\}.$$

Lemma 4. *(Double Fixed Point Theorem) [10] Let* $\mathcal{P}$ *be a cone in a real Banach space* $\mathbb{B}$*. Let* α *and* γ *be increasing, nonnegative, continuous functionals on* $\mathcal{P}$*, and let* θ *be a nonnegative, continuous functional on* $\mathcal{P}$ *with* $\theta(0) = 0$ *such that, for some* $c > 0$ *and* $M > 0$*,*

$$\gamma(u) \le \theta(u) \le \alpha(u) \ \text{and}\ \|u\| \le M\gamma(u)$$

for all $u \in \overline{\mathcal{P}(\gamma, c)}$*. Suppose that there exist positive numbers* a *and* b *with* $a < b < c$ *such that*

$$\theta(\lambda u) \le \lambda\theta(u), \quad for \quad 0 \le \lambda \le 1 \quad and \quad u \in \partial\mathcal{P}(\theta, b)$$

and

$$T : \overline{\mathcal{P}(\gamma, c)} \to \mathcal{P}$$

is a completely continuous operator such that:

(i) $\gamma(Tu) > c$*, for all* $u \in \partial\mathcal{P}(\gamma, c)$*;*
(ii) $\theta(Tu) < b$*, for all* $x \in \partial\mathcal{P}(\theta, b)$*;*
(iii) $\mathcal{P}(\alpha, a) \ne \emptyset$*, and* $\alpha(Tu) > a$*, for all* $u \in \partial\mathcal{P}(\alpha, a)$*.*

Then T has at least two fixed points, u_1 and u_2 belonging to $\overline{\mathcal{P}(\gamma, c)}$ such that

$$a < \alpha(u_1), \quad \textit{with} \quad \theta(u_1) < b,$$

and

$$b < \theta(u_2), \quad \textit{with} \quad \gamma(u_2) < c.$$

Let us define the increasing, nonnegative, continuous functionals γ, β, and α on $\mathcal{P}$ by

$$\begin{aligned} \gamma(u) &= \min_{0 \le t \le \xi_1} u(t) = u(\xi_1), \\ \beta(u) &= \max_{\xi_1 \le t \le \xi_{n-2}} u(t) = u(\xi_1), \\ \alpha(u) &= \max_{0 \le t \le \xi_{n-2}} u(t) = u(0). \end{aligned}$$

It is obvious that for each $u \in \mathcal{P}$,

$$\gamma(u) \le \beta(u) \le \alpha(u).$$

In addition, from by Lemma 1, for each $u \in \mathcal{P}$,

$$\|u\| \le \frac{1}{M} \min_{0 \le t \le 1} u(t) \le \frac{1}{M} \min_{0 \le t \le \xi_1} u(t) = \frac{1}{M} \gamma(u).$$

Thus,

$$\|u\| \le \frac{1}{M} \gamma(u), \quad \forall u \in \mathcal{P}.$$

For the convenience, we denote

$$\begin{aligned} K &= (1 - \xi_1) \phi^{-1} \left(\int_0^{\xi_1} q(\tau) d\tau \right), \\ L &= \frac{1}{1 - \displaystyle\sum_{i=1}^{m-2} b_i} \phi^{-1} \left(\frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\xi_1} q(\tau) d\tau \right). \end{aligned}$$

Theorem 1. *Suppose that assumptions $(A1), (A2)$ are satisfied. Let there exist positive numbers $a < b < c$ such that*

$$0 < a < \frac{K}{L} b < \frac{KM}{L} c,$$

and assume that f satisfies the following conditions

(A3) $f(t, u) > \phi\left(\frac{c}{K}\right)$, *for all* $(t, u) \in [0, \xi_1] \times [c, \frac{1}{M} c]$,
(A4) $f(t, u) < \phi\left(\frac{b}{L}\right)$, *for all* $(t, u) \in [0, 1] \times [0, \frac{1}{M} b]$,
(A5) $f(t, u) > \phi\left(\frac{a}{K}\right)$, *for all* $(t, u) \in [0, 1] \times [0, a]$.

Then the boundary value problem (1) *has at least two positive solutions* u_1 *and* u_2 *satisfying*

$$a < \alpha(u_1) \text{ with } \beta(u_1) < b, \quad b < \beta(u_2) \text{ with } \gamma(u_2) < c.$$

Proof. We define the completely continuous operator T by (11). So, it is easy to check that $T : \overline{\mathcal{P}(\gamma, c)} \to \mathcal{P}$. We now show that all the conditions of Lemma 4 are satisfied. In order to show that condition (i) of Lemma 4, we choose $u \in \partial\mathcal{P}(\gamma, c)$. Then $\gamma(u) = \min\limits_{0\le t\le \xi_1} u(t) = u(\xi_1) = c$, this implies that $c \le u(t)$ for $t \in [0, \xi_1]$. Recalling that $\|u\| \le \frac{1}{M}\gamma(u) = \frac{1}{M}c$, we get

$$c \le u(t) \le \frac{1}{M}c, \quad t \in [0, \xi_1].$$

Then assumption $(A3)$ implies $f(t, u) > \phi\left(\frac{c}{A}\right)$, for all $(t, u) \in [0, \xi_1] \times [c, \frac{1}{M}c]$. Therefore,

$$\begin{aligned}
\gamma(Tu) &= \min_{t\in[0,\xi_1]} (Tu)(t) = (Tu)(\xi_1) \\
&\ge \int_{\xi_1}^{1} \phi^{-1}\left(\int_0^s q(\tau)f(\tau, u(\tau))d\tau\right) ds \\
&\ge \int_{\xi_1}^{1} \phi^{-1}\left(\int_0^{\xi_1} q(\tau)f(\tau, u(\tau))d\tau\right) ds \\
&= (1-\xi_1)\phi^{-1}\left(\int_0^{\xi_1} q(\tau)f(\tau, u(\tau))d\tau\right) \\
&> \frac{c}{K}(1-\xi_1)\phi^{-1}\left(\int_0^{\xi_1} q(\tau)d\tau\right) \\
&= c.
\end{aligned}$$

Hence, condition (i) is satisfied.

Secondly, we show that (ii) of Lemma 4 is satisfied. For this, we select $u \in \partial\mathcal{P}(\beta, b)$. Then, $\beta(u) = \max\limits_{t\in[\xi_1,\xi_{n-2}]} u(t) = u(\xi_1) = b$, this means $0 \le u(t) \le b$, for all $t \in [\xi_1, 1]$. Noticing that $\|u\| \le \frac{1}{M}\gamma(u) = \frac{1}{M}\beta(u) = \frac{1}{M}b$, we get

$$0 \le u(t) \le \frac{1}{M}b, \text{ for } 0 \le t \le 1.$$

Then, assumption $(A4)$ implies $f(t, u) < \phi\left(\frac{b}{L}\right)$. Therefore

$$\begin{aligned}
\beta(Tu) &= \max_{t\in[\xi_1,\xi_{m-2}]} (Tu)(t) = (Tu)(\xi_1) \\
&\le \frac{1}{1-\sum\limits_{i=1}^{m-2} b_i}\phi^{-1}\left(\frac{1}{1-\sum\limits_{i=1}^{m-2} a_i}\int_0^1 q(\tau)f(\tau, u(\tau))d\tau\right)
\end{aligned}$$

$$
\begin{aligned}
&< \frac{b}{L}\frac{1}{1-\sum_{i=1}^{m-2} b_i}\phi^{-1}\left(\frac{1}{1-\sum_{i=1}^{m-2} a_i}\int_0^1 q(\tau)d\tau\right)\\
&= b.
\end{aligned}
$$

So, we get $\beta(Tu) < b$. Hence, condition (ii) is satisfied.

Finally, we show that the condition (iii) of Lemma 4 is satisfied. We note that $u(t) = a\left(\frac{\sum_{i=1}^{m-2} b_i - 1}{1-\sum_{i=1}^{m-2} b_i\eta_i}t+1\right)$, $0 \le t \le 1$ is a member of $\mathcal{P}(\alpha, a)$, and so $\mathcal{P}(\alpha, a) \neq \emptyset$. Now, let $u \in \partial\mathcal{P}(\alpha, a)$. Then $\alpha(u) = \max_{t\in[0,\xi_{n-2}]} u(t) = u(0) = a$. This implies

$$0 \le u(t) \le a,\ t \in [0, 1].$$

By assumption $(A5)$, $f(t, u) > \phi\left(\frac{a}{A}\right)$. Then,

$$
\begin{aligned}
\alpha(Tu) &= \max_{t\in[0,\xi_{n-2}]}(Tu)(t) = (Tu)(0)\\
&\ge \int_0^1 \phi^{-1}\left(\int_0^s q(\tau)f(\tau, u(\tau))d\tau + A\right)ds\\
&\ge \int_{\xi_1}^1 \phi^{-1}\left(\int_0^s q(\tau)f(\tau, u(\tau))d\tau\right)ds\\
&\ge (1-\xi_1)\phi^{-1}\left(\int_0^{\xi_1} q(\tau)f(\tau, u(\tau))d\tau\right)\\
&> (1-\xi_1)\frac{a}{A}\phi^{-1}\left(\int_0^{\xi_1} q(\tau)d\tau\right) = a.
\end{aligned}
$$

So, we get $\alpha(Tu) > a$. Thus, (iii) of Lemma 4 is satisfied. Hence, the boundary value problem (1) has at least two positive solutions u_1 and u_2 satisfying

$$a < \alpha(u_1) \text{ with } \beta(u_1) < b, \text{ and } b < \beta(u_2) \text{ with } \gamma(u_2) < c.$$

□

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