Central Configurations in a symmetric five-body problem

Muhammad Shoaib[1], Anoop Sivasankaran[2], and Yehia A. Abdel-Aziz[1,3]

[1] University of Hail, Department of Mathematical Sciences PO BOX 2440, Saudi Arabia
(E-mail: safridi@gmail.com)
[2] Khalifa University, Department of Applied Mathematics and Sciences, PO Box:573, Sharjah, UAE
(E-mail: anoop.sivasankaran@kustar.ac.ae)
[3] National Research Institute of Astronomy and Geophysics (NRIAG), Cairo, Egypt
(E-mail: yehia@nriag.sci.eg)

Abstract. A central configuration $q = (q_1, q_2, ..., q_n)$ is a particular configuration of the n-bodies where the acceleration vector of each body is proportional to its position vector and the constant of proportionality is the same for n-bodies. In the three-body problem, it is always possible to find three positive masses for any given three collinear positions given that they are central. This is not possible for more than four-body problems in general. In this paper we model a symmetric five-body problem with with position coordinates for the five bodies as $(-x, 0), (0, y), (x, 0), (0, -y)$ and (c_1, c_2). (c_1, c_2) is the centre of mass of the system. Regions of central configurations, where it is possible to choose positive masses, are derived using both analytical and numerical tools. We also identify regions in the phase space where no central configurations are possible. A certain relationship exists between the mass placed at the center of mass of the systems i.e (c_1, c_2) and the remaining four masses. This relationship is investigated both numerically and analytically. Similarly restrictions on the geometry and restrictions on the inter-body distances are investigated.
Keywords: Central Configurations, n-body problem, five-body problem, inverse problem of central configurations.

1 Introduction

The classical equation of motion for the n-body problem has the form

$$m_i \frac{d^2\mathbf{q}_i}{dt^2} = \frac{\partial U}{\partial q_i} = \sum_{j \neq i} \frac{m_i m_j \left(\mathbf{q}_j - \mathbf{q}_i\right)}{|\mathbf{q}_i - \mathbf{q}_j|^3} \qquad i = 1, 2, ..., n, \tag{1}$$

where the units are chosen so that the gravitational constant is equal to one, $\mathbf{q}_i$ is a vector in three space,

$$U = \sum_{1 \preceq i < j \preceq n} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|} \tag{2}$$

Received: 16 April 2013 / Accepted: 18 July 2013

ISSN 2241-0503

is the self-potential, $\boldsymbol{q}_i$ is the location vector of the ith body and m_i is the mass of the ith body.

A *central configuration* $q = (q_1, q_2, \cdots, q_n)$ is a particular configuration of the n-bodies where the acceleration vector of each body is proportional to its position vector, and the constant of proportionality is the same for the n-bodies, therefore

$$\sum_{j=1, j\neq i}^{n} \frac{m_j(\mathbf{q}_j - \mathbf{q}_k)}{|\mathbf{q}_j - \mathbf{q}_k|^3} = -\lambda(\mathbf{q}_k - \mathbf{c}) \qquad k = 1, 2, ..., n, \tag{3}$$

where

$$\lambda = \frac{U}{2I}, \qquad I = \sum_{i=1}^{n} m_i ||\mathbf{q}_i||^2, \text{ and } \mathbf{c} = \frac{\sum_{i=1}^{n} m_i \mathbf{q}_i}{\sum_{i=1}^{n} m_i}. \tag{4}$$

So far, in the non-collinear general four and five-body problems the main focus has been on the common question: For a given set of masses and a fixed arrangement of bodies does there exist a unique central configuration ([7],[6]). In this paper, we ask the inverse of the question i.e. given a four or five-body configuration, if possible, find positive masses for which it is a central configuration. Similar question has been answered by Ouyang and Xie (2005) for a collinear four body problem and by Mello and Fernades (2011) for a rhomboidal four and five-body problem. For other recent studies on the rhomboidal problem see [1],[2],[4], and [5]. In this paper we state and prove the following theorems.

Theorem 1. *Consider five bodies of masses* $(m_1, m_2, m_3, m_4, m_0)$ *located at* $(-x, 0), (y, 0), (x, 0), (0, -y)$ *and* $(0, 0)$ *respectively. The mass* m_0 *is taken to be stationary at the centre of mass of the system. Let* $m_1 = m_3 = 1, m_2 = m_4 = m$.

1. *In this particular set up, using polar coordinates, of the rhomboidal five body problem where* $m(\theta) > 0$, $m_0(\theta) > 0$ *and* $r = 1$ *will form central configuration when* $\theta \in (-1.94, -1.04) \cup (0.74, 1.04)$. *For all other values of* θ *at least one of the masses will become negative.*
2. *For* $r \neq 1$, *the central configuration region is given in figure (1).*

Theorem 2. *Let five bodies of masses* $m_1 = m_3 = M, m_2 = m_4 = m$ *be placed at the vertices* $m_1(-1, 0), m_2(y, 0), m_3(1, 0), m_4(0, -y)$ *and* $m_0(0, 0)$ *of a rhombus. The mass* m_0 *is taken to be stationary at the centre of mass of the system. There exist a region*

$$R_1 = (R_{1m} \cup R_{1m}^*) \cap (R_{1M} \cup R_{1M}^*). \tag{5}$$

in the ym_0-*plane where it is possible to choose positive masses which will make the configuration central, where*

$$\begin{aligned} R_{1m} &= \{(y, m_0) | m_0 > \frac{y^3\left(8 - \left(1+y^2\right)^{3/2}\right)}{-8y^3 + \left(1+y^2\right)^{3/2}} \\ &\quad \textit{and} \;\; y \in (0, 2-\sqrt{3}) \cup (2+\sqrt{3}, \infty)\}, \end{aligned} \tag{6}$$

$$
\begin{aligned}
R^*_{1m} &= \{(y,m_0)|m_0 < \frac{y^3\left(8-\left(1+y^2\right)^{3/2}\right)}{-8y^3+(1+y^2)^{3/2}} \textit{ and } y \in (2-\sqrt{3}, 2+\sqrt{3})\}, && (7)\\
R_{1M} &= \{(y,m_0)|m_0 > \frac{8y^3-\left(1+y^2\right)^{3/2}}{-8+(1+y^2)^{3/2}} && (8)\\
&\quad \textit{and } \; y \in (0, 2-\sqrt{3}) \cup (2+\sqrt{3}, \infty)\},\\
R^*_{1M} &= \{(y,m_0)|m_0 < \frac{8y^3-\left(1+y^2\right)^{3/2}}{-8+(1+y^2)^{3/2}} \textit{ and } y \in (2-\sqrt{3}, 2+\sqrt{3})\}. && (9)
\end{aligned}
$$

In the complement of this region no central configurations exist for $m, m_0 > 0$.

Theorem 3. *Consider five bodies of masses $(m_1, m_2, m_3, m_4, m_0)$ located at $(-x, 0), (y, 0), (x, 0), (0, -y)$ and $(0, 0)$ respectively. The mass m_0 is taken to be stationary at the centre of mass of the system. Let $m_1 = m_3 = M, m_2 = m_4 = m$. There exist a region*

$$R_3 = ((R_d \cap R_{3m}) \cup (R^c_d \cap R^c_{3m})) \cap (R_d \cap R_{3M}) \cup (R^c_d \cap R^c_{3M}), \qquad (10)$$

in the $xy-$plane where it is possible to choose positive masses which will make the configuration central. Here

$$R_{3m} = \{(x,y)|r(x,y) > 2y\sqrt[3]{\frac{m_0+x^3}{m_0+y^3}}, x>0, y>0, m_0>0\}, \qquad (11)$$

$$R_{3M} = \{(x,y)|r(x,y) > 2x\sqrt[3]{\frac{m_0+y^3}{m_0+x^3}}, x>0, y>0, m_0>0\}. \qquad (12)$$

In the complement of this region no central configurations exist for $M, m, m_0 > 0$.

Let's consider five bodies of masses m_i, $i = 0, 1, 2, 3, 4$. Four of the masses are placed at the vertices of a rhombus and the fifth mass m_0 is stationary at the centre of mass of the system. The coordinates for the five bodies are chosen as below:

$$\mathbf{q}_0 = (c_1, c_2), \mathbf{q}_1 = (-x, 0), \mathbf{q}_2 = (0, y), \qquad (13)$$

$$\mathbf{q}_3 = (x, 0), \mathbf{q}_4 = (0, -y), \qquad (14)$$

Using (3) and (13) we obtain the following equation for central configurations.

$$\frac{m_0\mathbf{q}_1}{x^3} + \frac{m_2\mathbf{q}_{12}}{\left(\sqrt{x^2+y^2}\right)^3} + \frac{m_3\mathbf{q}_{13}}{8x^3} + \frac{m_4\mathbf{q}_{14}}{\left(\sqrt{x^2+y^2}\right)^3} = -\lambda(\mathbf{q}_1 - \mathbf{c}), \qquad (15)$$

$$\frac{m_0\mathbf{q}_2}{y^3} + \frac{m_1\mathbf{q}_{21}}{\left(\sqrt{x^2+y^2}\right)^3} + \frac{m_3\mathbf{q}_{23}}{\left(\sqrt{x^2+y^2}\right)^3} + \frac{m_4\mathbf{q}_{24}}{8y^3} = -\lambda(\mathbf{q}_2 - \mathbf{c}), \qquad (16)$$

$$\frac{m_0\mathbf{q}_3}{x^3}+\frac{m_1\mathbf{q}_{31}}{8x^3}+\frac{m_2\mathbf{q}_{32}}{\left(\sqrt{x^2+y^2}\right)^3}+\frac{m_4\mathbf{r}_{34}}{\left(\sqrt{x^2+y^2}\right)^3}=-\lambda(\mathbf{q}_3-\mathbf{c}), \tag{17}$$

$$\frac{m_0\mathbf{q}_4}{y^3}+\frac{m_1\mathbf{q}_{41}}{\left(\sqrt{x^2+y^2}\right)^3}+\frac{m_2\mathbf{q}_{42}}{8y^3}+\frac{m_3\mathbf{q}_{43}}{\left(\sqrt{x^2+y^2}\right)^3}=-\lambda(\mathbf{q}_4-\mathbf{c}). \tag{18}$$

2 Proof of Theorem 1.

Let $m_1 = m_3 = 1, m_2 = m_4 = m$. As CC's are invariant up to translation and re-scaling therefore we assume that the centre of mass is at the origin. This assumption leads to some simplifications in the CC equations. Therefore from the four CC equations ((15 to 18) the following two linearly independent equations are obtained.

$$-\frac{1}{4x^2}+\frac{m_0}{x^2}-\frac{2mx}{(x^2+y^2)^{3/2}}=-x\lambda, \tag{19}$$

$$\frac{m}{4y^2}-\frac{m_0}{y^2}+\frac{2y}{(x^2+y^2)^{3/2}}=y\lambda. \tag{20}$$

Let $\lambda = 1$. Equations (19 and 20) are solved to obtain m and m_0 as functions of $x > 0$ and $y > 0$.

$$m(x,y)=\frac{8y^3-\left(x^2+y^2\right)^{3/2}(1-4x^3+4y^3)}{8x^3-(x^2+y^2)^{3/2}} \tag{21}$$

$$m_0(x,y)=\frac{32x^3y^3(2-\left(x^2+y^2\right)^{3/2})-\left(x^2+y^2\right)^3(1-4x^3)}{4\,(x^2+y^2)^{3/2}\left(8x^3-(x^2+y^2)^{3/2}\right)}. \tag{22}$$

It is not possible to explicitly solve for x and y therefore we use polar coordinates to re-write $m(x,y)$ and $m_0(x,y)$ as $m(r,\theta)$ and $m_0(r,\theta)$, where $x = r\cos\theta$ and $y = r\sin\theta$.

$$m(r,\theta)=\frac{1+4r^3\cos^3\theta-4\left(2+r^3\right)\sin^3\theta}{1-6\cos\theta-2\cos 3\theta}. \tag{23}$$

$$m_0(r,\theta)=\frac{\left(1-6\sin 2\theta+2\sin 6\theta-r^3(3\cos\theta-3\sin 2\theta+\cos 3\theta+\sin 6\theta)\right)}{4\,(1-6\cos\theta-2\cos 3\theta)}. \tag{24}$$

Let $r = 1$. The denominator of both $m(\theta)$ and $m_0(\theta)$ becomes zero at $\theta = -\frac{\pi}{3}, \frac{\pi}{3}$. The denominator is negative when $\theta \in (-\frac{\pi}{3}, \frac{\pi}{3})$ and is positive elsewhere. The numerator of $m(\theta)$ when $r = 1$ is given by $1 + \cos^3\theta - 12\sin^3\theta$. This has real zeros at $\theta = -2.61$ and $\theta = 0.673$. The numerator is positive

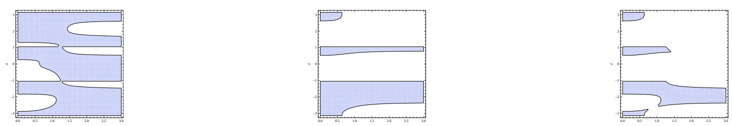

Fig. 1. left: $m_0(r,\theta) > 0$. **Centre:** Region, when $m(r,\theta) > 0$ **Right:** Region, when $m_0(r,\theta) > 0$ and $m(r,\theta) > 0$

when $\theta \in (-2.61, 0.673)$. Therefore $m(\theta)$ is positive when $\theta \in (-2.61, -1.04) \cup (0.673, 1.04)$.

The numerator of $m_0(\theta)$ when $r = 1$ is given by $-1 + 3\cos\theta + \cos 3\theta + 3\sin 2\theta - \sin 6\theta$. This has real zeros at $\theta = -2.541$, $\theta = -1.935$, $\theta = -0.449$, and $\theta = 1.248$. The numerator of $m_0(\theta)$ is positive when $\theta \in (-\pi, -2.54) \cup (-1.935, -0.449) \cup (1.248, \pi)$. Therefore $m_0(\theta)$ is positive when $\theta \in (-\pi, -2.54) \cup (-1.935, -1.04) \cup (-0.449, 1.04) \cup (1.248, \pi)$.

Hence, this particular set up of the rhomboidal five body problem where $m(\theta) > 0$, $m_0(\theta) > 0$ and $r = 1$ will form central configuration when $\theta \in (-1.94, -1.04) \cup (0.74, 1.04)$. For all other values of θ at least one of the masses will become negative.

In the case when $r \neq 1$,The central configuration region is given in figure (1)

3 Proof of Theorem 2.

Let $\lambda = x = 1$. Solve equations (19 and 20) to obtain m and M as functions of m_0 and y .

$$m(y, m_0) = \frac{4\left(1+y^2\right)^{3/2} N_m(y, m_0)}{(1-4y+y^2)\left(1+4y+18y^2+4y^3+y^4\right)}, \tag{25}$$

$$M(y, m_0) = \frac{4\left(1+y^2\right)^{3/2} N_M(y, m_0)}{(1-4y+y^2)\left(1+4y+18y^2+4y^3+y^4\right)}, \tag{26}$$

where

$$\begin{aligned} N_m(y, m_0) = {} & y^3\left(-2+\sqrt{1+y^2}\right)\left(5+y^2+2\sqrt{1+y^2}\right) \\ & +m_0\left(\left(-2y+\sqrt{1+y^2}\right)\left(1+5y^2+2y\sqrt{1+y^2}\right)\right), \end{aligned} \tag{27}$$

$$
\begin{aligned}
N_M(y,m_0) = {} & \left(-2y+\sqrt{1+y^2}\right)\left(1+5y^2+2y\sqrt{1+y^2}\right) \\
& +m_0\left(\left(-2+\sqrt{1+y^2}\right)\left(5+y^2+2\sqrt{1+y^2}\right)\right). \qquad (28)
\end{aligned}
$$

The factor $1-4y+y^2$ of the denominator of $m(y,m_0)$ and $M(y,m_0)$ is positive when $y\in(0,2-\sqrt{3})\cup(2+\sqrt{3},\infty)$ and is negative when $y\in(2-\sqrt{3},2+\sqrt{3})$. Therefore to find the sign of $m(y,m_0)$ and $m(y,m_0)$ we need to analyze $N_m(y,m_0)$ and $N_M(y,m_0)$. The component of the numerator of $m(y,m_0)$, $N_m(y,m_0)$, has two factors i.e. $-2+\sqrt{1+y^2}$ and $-2y+\sqrt{1+y^2}$ which can become negative and hence can make $N_m(y,m_0)$ negative. The factor $-2+\sqrt{1+y^2}>0$ when $y\in(\sqrt{3},\infty)$ and $-2y+\sqrt{1+y^2}>0$ when $y\in(0,\frac{1}{\sqrt{3}})$. As both the intervals have empty intersection therefore we must have the following bound on m_0 for $N_m(y,m_0)$ to be positive.

$$
m_0 > \frac{y^3\left(8-\left(1+y^2\right)^{3/2}\right)}{-8y^3+\left(1+y^2\right)^{3/2}}. \qquad (29)
$$

Hence $m(y,m_0)$ will be positive in the following two regions.

$$
\begin{aligned}
R_{1m} &= \{(y,m_0)|m_0 > \frac{y^3\left(8-\left(1+y^2\right)^{3/2}\right)}{-8y^3+\left(1+y^2\right)^{3/2}} \\
& \quad \text{and } \ y\in(0,2-\sqrt{3})\cup(2+\sqrt{3},\infty)\}, \qquad (30)\\
R^*_{1m} &= \{(y,m_0)|m_0 < \frac{y^3\left(8-\left(1+y^2\right)^{3/2}\right)}{-8y^3+\left(1+y^2\right)^{3/2}} \\
& \quad \text{and } \ y\in(2-\sqrt{3},2+\sqrt{3})\}. \qquad (31)
\end{aligned}
$$

Similarly $M(y,m_0)$ is positive in the following two regions

$$
\begin{aligned}
R_{1M} &= \{(y,m_0)|m_0 > \frac{8y^3-\left(1+y^2\right)^{3/2}}{-8+\left(1+y^2\right)^{3/2}} \\
& \quad \text{and } \ y\in(0,2-\sqrt{3})\cup(2+\sqrt{3},\infty)\}, \qquad (32)\\
R^*_{1M} &= \{(y,m_0)|m_0 < \frac{8y^3-\left(1+y^2\right)^{3/2}}{-8+\left(1+y^2\right)^{3/2}} \text{ and } y\in(2-\sqrt{3},2+\sqrt{3})\} \qquad (33)
\end{aligned}
$$

Hence, the central configuration region for this particular set up of the rhomboidal five body problem where both $m(x,y,m_0)$ and $M(x,y,m_0)$ are positive is given by

$$
R_1 = (R_{1m}\cup R^*_{1m})\cap(R_{1M}\cup R^*_{1M}). \qquad (34)
$$

This completes the proof of theorem 2. This central configuration region is given in figure (2)

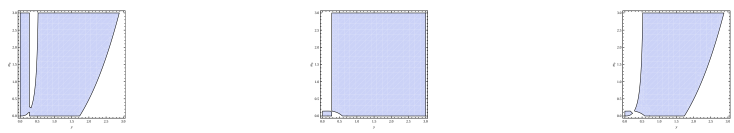

Fig. 2. left: $m(y, m_0) > 0$. **Centre:** $M(y, m_0) > 0$ **Right:** $m(y, m_0) > 0$ and $M(y, m_0) > 0$

4 Proof of Theorem 3.

Let $\lambda = 1$. Solve equations (19 and 20) to obtain m and M as functions of x, y and m_0.

$$m(x, y, m_0) = \frac{4\left(x^2+y^2\right)^{3/2}\left(\begin{array}{c} y^3\left(-8x^3+\left(x^2+y^2\right)^{3/2}\right) \\ +m_0\left(-8y^3+\left(x^2+y^2\right)^{3/2}\right)\end{array}\right)}{\left(x^2-4xy+y^2\right)\left(x^4+4x^3y+18x^2y^2+4xy^3+y^4\right)} \tag{35}$$

$$M(x, y, m_0) = \frac{4\left(x^2+y^2\right)^{3/2}\left(\begin{array}{c} x^3\left(-8y^3+\left(x^2+y^2\right)^{3/2}\right) \\ +m_0\left(-8x^3+\left(x^2+y^2\right)^{3/2}\right)\end{array}\right)}{\left(x^2-4xy+y^2\right)\left(x^4+4x^3y+18x^2y^2+4xy^3+y^4\right)} \tag{36}$$

It can be immediately seen that the denominator of both $m(x, y, m_0)$ and $M(x, y, m_0)$ becomes singular at $y = (2 \pm \sqrt{3})x$. Therefore $y = (2 \pm \sqrt{3})x$ will form two singular curves for the two masses m and M. Therefore the denominator will be positive in region R_d given below and will be negative in its complement.

$$R_d = \{(x, y)|0 < y < (2-\sqrt{3})x \text{ or } y > (2+\sqrt{3})x, x > 0\}. \tag{37}$$

It is not possible to explicitly solve the numerator of either $m(x, y, m_0)$ or $M(x, y, m_0)$ for x or y therefore we choose the inter body distance $x^2 + y^2$ to find regions of central configuration where both m and M are positive. In the numerator of $m(x, y, m_0)$ the factor

$$y^3\left(-8x^3+\left(x^2+y^2\right)^{3/2}\right)+m_0\left(-8y^3+\left(x^2+y^2\right)^{3/2}\right) = N_{3m}$$

can be become negative. By taking $r = \sqrt{x^2 + y^2}$, the factor N_{3m} is simplified as below.

$$N_{3m} = y^3 \left(-8x^3 + r^3\right) + m_0 \left(-8y^3 + r^3\right) \tag{38}$$

After some algebraic manipulation it can be shown that N_{3m} is positive in the following region.

$$R_{3m} = \{(x, y)|r(x, y) > 2y \sqrt[3]{\frac{m_0 + x^3}{m_0 + y^3}}, x > 0, y > 0, m_0 > 0\}. \tag{39}$$

N_{3m} is negative in the complement of R_{3m}. Therefore, in this particular set up, the central configuration region where m is positive is given by

$$(R_d \cap R_{3m}) \cup (R_d^c \cap R_{3m}^c). \tag{40}$$

Similarly $N_{3M} = x^3 \left(-8y^3 + \left(x^2 + y^2\right)^{3/2}\right) + m_0 \left(-8x^3 + \left(x^2 + y^2\right)^{3/2}\right)$ is positive in the following region.

$$R_{3M} = \{(x, y)|r(x, y) > 2x \sqrt[3]{\frac{m_0 + y^3}{m_0 + x^3}}, x > 0, y > 0, m_0 > 0\}. \tag{41}$$

N_{3M} is negative in the complement of R_{3M}. Therefore, in this particular set up, the central configuration region where M is positive is given by

$$(R_d \cap R_{3M}) \cup (R_d^c \cap R_{3M}^c). \tag{42}$$

Hence, the central configuration region for this particular set up of the rhomboidal five body problem where both $m(x, y, m_0)$ and $M(x, y, m_0)$ are positive is given by

$$R_3 = ((R_d \cap R_{3m}) \cup (R_d^c \cap R_{3m}^c)) \cap (R_d \cap R_{3M}) \cup (R_d^c \cap R_{3M}^c). \tag{43}$$

In the complement of this region no central configurations are possible as at least one of the masses will become negative. This completes the proof of theorem 3.

5 Acknowledgments

The authors, Muhammad Shoaib and Yehia A Abdel-Aziz thank the Deanship of Scientific research at the University of Hail, Saudi Arabia for funding this project (reference number SM3) for the year 1433 Hijri (2012-2013).

References

1.B. Lennard, and S. Simmons. Stability of the Rhomboidal Symmetric-Mass Orbit.*arXiv preprint arXiv:1208.3183* (2012).

2.D. Diarova and N. I. Zemtsova. The instability of the rhombus-like central configurations in newton 9-body problem. *Computer Algebra in Scientific Computing.* Springer Berlin Heidelberg, 2006.

3.L. F. Mello and A. C. Fernandes. Co-circular and co-spherical kite central configurations. *Qualitative theory of dynamical systems.* 10 (2011) 29-41.

4.M. Kulesza, M. Marchesin, and C. Vidal. Restricted rhomboidal five-body problem. *Journal of Physica A: Mathematical and Theoritical.* 44 (2011) 485204.

5.M. Marcelo, and C. Vidal. Spatial restricted rhomboidal five-body problem and horizontal stability of its periodic solutions. *Celestial Mechanics and Dynamical Astronomy* (2013): 1-19.

6.M. Shoaib, B.A. Steves, and A. Széll. Stability analysis of quintuple stellar and planetary systems using a symmetric five-body model. *New Astronomy.* 13 (2008) 639-645.

7.M. Shoaib and I. Faye. Collinear equilibrium solutions of four-body problem. *Journal of Astrophysics and Astronomy.* 32 (2011) 411-423.

8.T Ouyang and Z. Xie. Collinear central configuration in four-body problem, *Celestial Mechanics and Dynamical Astronomy.* 93 (2005) 147-166.