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Comparison of non-relativistic and relativistic Lyapunov exponents for a low-speed system

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Abstract. The Newtonian and special-relativistic Lyapunov exponents are compared for a low speed system – the periodically-delta-kicked particle. We show that although the agreement between the Newtonian and special-relativistic transient Lyapunov exponents rapidly breaks down initially, they converge to values which are very close to each other.
Keywords: kicked particle, Lyapunov exponent, special relativity, Newtonian approximation

1 Introduction

It is conventionally believed [1-3] that if the speed *v* of a dynamical system is *low* compared to the speed of light *c*, that is, $v \ll c$, then the special-relativistic dynamical predictions for the system will be well-approximated by the Newtonian predictions. However, it was shown in recent numerical studies [4-9] that, contrary to the conventional belief, the agreement between the Newtonian and special-relativistic dynamical predictions for a single trajectory [4-7] and for an ensemble of trajectories [8,9] can break down completely although the speed of the system is low. Here, we extend the previous studies [4-9] to a comparison of the Newtonian and special-relativistic predictions for the Lyapunov exponent of a prototypical chaotic Hamiltonian system – the periodically-delta-kicked particle – at low speed. Details of the system and calculations will be given next, followed by the results and discussion.

2 Method

In the Newtonian framework, the equations of motion for the periodically-delta-kicked particle are reducible to an exact mapping, which is called the standard map [10,11]:

$$P_n = P_{n-1} - \frac{K}{2\pi}\sin\left(2\pi X_{n-1}\right) \tag{1}$$

$$X_n = \left(X_{n-1} + P_n\right) \bmod 1 \tag{2}$$

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where X_n and P_n are, respectively, the dimensionless scaled position and momentum of the particle just before the nth kick (n = 1, 2, …), and K is a dimensionless positive parameter.

In the special-relativistic framework, the equations of motion for the periodically-delta-kicked particle are also reducible to a mapping, which is called the relativistic standard map [12,13]:

$$P_n = P_{n-1} - \frac{K}{2\pi}\sin(2\pi X_{n-1}) \quad (3)$$

$$X_n = \left(X_{n-1} + \frac{P_n}{\sqrt{1+\beta^2 P_n^2}} \right) \bmod 1 \quad (4)$$

where β, like K, is also a dimensionless positive parameter.

The transient Lyapunov exponent for a map is generally defined [14] as

$$\lambda_n = \frac{1}{n}\ln[\text{abs}(\text{trace}\, M_n)] \quad (5)$$

where $M_n = J_n J_{n-1} \ldots J_2 J_1$ and J_n is the Jacobi matrix. In the limit $n \to \infty$, λ_n yields [14] the largest Lyapunov exponent. A hallmark of chaos is the existence of a positive Lyapunov exponent. For the standard map in Eqs. (1) and (2), the Jacobi matrix is

$$J_n = \begin{bmatrix} 1 & -K\cos(2\pi X_n) \\ 1 & 1 - K\cos(2\pi X_n) \end{bmatrix}. \quad (6)$$

For the relativistic standard map in Eqs. (3) and (4), the Jacobi matrix is

$$J_n = \begin{bmatrix} 1 & -K\cos(2\pi X_n) \\ \left(1+\beta^2 P_{n+1}^2\right)^{-3/2} & 1 - \left(1+\beta^2 P_{n+1}^2\right)^{-3/2}\left[K\cos(2\pi X_n)\right] \end{bmatrix}. \quad (7)$$

In each theory, the transient Lyapunov exponent [Eq. (5)] is calculated twice to determine its accuracy. The calculation for the transient Lyapunov exponent is first performed in 32-significant-figure precision and then repeated in quadruple (35 significant figures) precision. The accuracy of the transient Lyapunov exponent is determined by the common digits of the 32-significant-figure-precision and quadruple-precision calculations. For example, if the former calculation yields 1.234… and the latter calculation yields 1.235…, the transient Lyapunov exponent is accurate to 1.23.

3 Results and discussion

Here we will present an example to illustrate the typical result. In this example, X_0 = 0.5, P_0 = 99.9, K = 7.0 and $\beta = 10^{-7}$. For these initial conditions and parameters, both the Newtonian and special-relativistic trajectories are chaotic. In this case, the speed of the particle is low, about $10^{-5}c$, up to 8800 kicks.

Fig. 1, which plots the Newtonian and special-relativistic transient

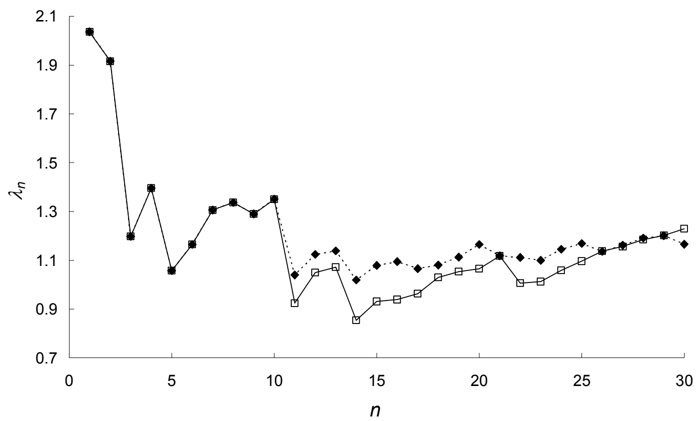

Fig. 1. Newtonian (squares) and special-relativistic (diamonds) transient Lyapunov exponents versus kick.

Lyapunov exponents for the first 30 kicks, shows that the two transient Lyapunov exponents agree with each other for the first 10 kicks but the agreement breaks down from kick 11 onwards. The agreement between the Newtonian and special-relativistic transient Lyapunov exponents breaks down rapidly because the difference between the two grows, on average, exponentially – see Fig. 2. The exponential growth constant of the difference

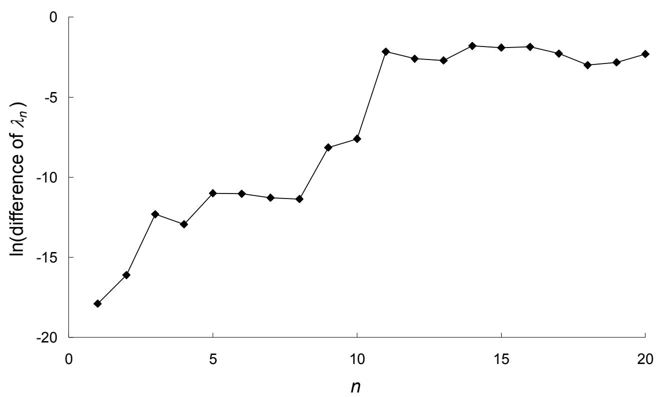

Fig. 2. Difference between the Newtonian and special-relativistic transient Lyapunov exponents versus kick.

between the two transient Lyapunov exponents, measured from kick 1 to kick 10, is 0.96.

However, asymptotically, the Newtonian and special-relativistic transient Lyapunov exponents converge to values which are very close to one another. In particular, at kick 8800, the Newtonian and special-relativistic transient Lyapunov exponents are both accurate to 1.27, which is quite close to the analytical estimate [10] of the asymptotic Newtonian Lyapunov exponent given by $\ln(K/2) = 1.253$. This result is surprising since the chaotic trajectories predicted by the two theories agree only for the first 16 kicks, which suggests that the two asymptotic Lyapunov exponents should not agree.

Conclusions

We have shown that although the agreement between the Newtonian and special-relativistic transient Lyapunov exponents rapidly breaks down initially, the asymptotic special-relativistic Lyapunov exponent is well-approximated by the asymptotic Newtonian value. The same result should hold for other low-speed chaotic Hamiltonian systems since the periodically-delta-kicked particle is a prototype.

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