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Chaos in hydrodynamic models of pulsating BL Her-type stars

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Abstract. We present hydrodynamic models of pulsating BL Her-type stars that show a wealth of dynamical behaviours characteristic for deterministic chaos. Interesting phenomena detected in our models include period doubling and intermittent routes to chaos, periodic windows within chaotic domain, type I and type III intermittency, interior crisis bifurcation and others. Before we describe the models, we briefly review the current knowledge about type II Cepheids, a group of radially pulsating stars to which BL Her class belongs, and the methods used to model such stars.
Keywords: astrophysics, pulsating stars, type II Cepheids, chaos, intermittency.

1 Introduction

2 Type II Cepheids

Type II Cepheids are low-mass ($M \approx 0.5-0.7\mathrm{M}_{\odot}$), giant stars pulsating radially with periods from one to several tenths of days (see e.g. Wallerstein[1] or Soszynski et al.[2]). In the H-R diagram, a plot of absolute luminosity (L) vs. the effective temperature ($T_{\rm eff}$), these stars are located in the cool and luminous part, within the instability strip (IS), in which pulsations are driven with the opacity (kappa) mechanism (e.g. Cox[3]). Type II Cepheids are divided into three classes: BL Her stars, with periods between 1 and 4 days, W Vir stars with periods between 4 and 20 days and RV Tau stars with periods above 20 days. The borderline between BL Her and W Vir stars is somewhat arbitrary (see Soszynski et al.[2]). RV Tau stars, on the other hand, are distinguished by period-doubled pulsation which starts to appear at periods above 20 days. Recent studies show however, that effect can appear also at shorter periods, in particular the W Vir star, a prototype of the W Vir class, shows the effect (Templeton & Henden[4]). In addition, the period doubling effect was discovered in one BL Her star with period ≈ 2.4 days (Smolec et al[5]). The possible existence of period doubled BL Her stars was predicted by Buchler & Moskalik[6] 20 years earlier, based on hydrodynamic models.

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Type II Cepheids are at advanced evolutionary stages (see Wallerstein[1], Gingold[7]). The division into three classes is believed to reflect different evolutionary stages of the stars. After hydrogen is depleted in the core, the star climbs up the Red Giant Branch (RGB) increasing its luminosity at nearly constant effective temperature. After the helium is ignited in the degenerate core, the progenitors of type II Cepheids arrive at the blue side of the Zero-Age Horizontal Branch (ZAHB), steadily burning helium in the center. They evolve redward, towards the Asymptotic Giant Branch (AGB) and, as they cross the instability strip, they pulsate as BL Her variables. As helium is depleted in the core, its burning continues, along with the hydrogen burning, in the shells surrounding the carbon/oxygen core. The star, now climbing up the AGB, may loop back into the IS due to instabilities in the shell burning, becoming a W Vir-type variable. Finally, as the star leaves the AGB on the way to the white dwarf sequence, it crosses the IS for the last time, pulsating as RV Tau-type variable.

In majority of cases, the least luminous, shortest period BL Her stars are very regular pulsators, with repeatedly stable cycle-to-cycle variation. As luminosity increases the light variation becomes less regular. Irregular amplitude and period variation is frequently observed in W Vir stars. Strong irregularities on top of period-doubled pulsations are common in RV Tau stars. The behaviour is more pronounced in longer period stars. Closely related to RV Tau stars, even more luminous and longer period semi-regular and Mira-type pulsators, show very strong irregular cycle-to-cycle variation, without evident period doubling. Deterministic chaos was detected in two RV Tau-type stars and in a few semi-regular and one Mira-type variable, for which long (at least 30 years) and good quality observations allowed a rigorous analysis (Buchler et al.,[8], Kolláth et al.,[9], Buchler, Kolláth & Cadmus,[10], Kiss & Szatmáry[11]). Hydrodynamic models of type II Cepheids indicate that indeed, as pulsation period increases a period-doubling route leads to deterministic chaos (Buchler & Kovács[12], Kovács & Buchler[13]). We note however, that no period-4 (or other than period-2) pulsating star is known to date.

3 Hydrodynamic models of BL Her stars

For more than 50 years now, large amplitude radially pulsating stars are investigated with the help of one dimension pulsation hydrocodes. The first calculations were purely radiative, neglecting the energy transfer by convection. Nowadays, simple 1D recipes for time-dependent turbulent convection are used. In our study of BL Her models we used our nonlinear code (Smolec & Moskalik[14]) implementing the Kuhfuß[15] one-equation, turbulent convection recipe. Equations we solve are momentum, internal and turbulent energy equations:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial}{\partial r}\big(p + p_{\mathrm{t}}\big) + U_{\mathrm{q}} - \frac{GM_r}{r^2}, \tag{1}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} + p\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial\big[r^2\big(F_{\mathrm{r}} + F_{\mathrm{c}}\big)\big]}{r^2\partial r} - C, \tag{2}$$

$$\frac{\mathrm{d}e_{\mathrm{t}}}{\mathrm{d}t}+p_{\mathrm{t}}\frac{\mathrm{d}V}{\mathrm{d}t}=-\frac{1}{\rho}\frac{\partial\left(r^2F_{\mathrm{t}}\right)}{r^2\partial r}+E_{\mathrm{q}}+C. \tag{3}$$

with

$$u=\frac{\mathrm{d}r}{\mathrm{d}t} \tag{4}$$

Above, u is fluid velocity, M_r is mass enclosed in radius r, V is specific volume (inverse of specific density, $V=1/\rho$), p and E are pressure and energy of the gas. F_{r}, F_{c} and F_{t} are radiative, convective and turbulent fluxes, respectively. Radiative flux is computed assuming diffusion approximation and radiation pressure and radiation energy are included in p and E. Turbulent energy, e_{t}, is computed according to model of Kuhfuß. p_{t} is turbulent pressure and U_{q} and E_{q} are viscous momentum and energy transfer rates. The internal and turbulent energy equations are coupled through the term C:

$$C=S-D-D_{\mathrm{r}}, \tag{5}$$

with source (or driving) function, S, describing the rate of turbulent energy generation/damping through the buoyant forces, D modelling the decay of turbulent energy through the turbulent cascade and D_{r} describing the rate at which turbulent energy is transformed to the internal energy, through the radiative cooling of the convective eddies. The model contains eight parameters, values of which are calibrated using observational constraints. The reader is referred to Smolec & Moskalik[14] for further details.

To construct a model of pulsating star we first solve the static version of equations (1)–(3). The model is divided typically into 150–200 lagrangian mass shells extending down to a fixed temperature of a few million Kelvin. It is not necessary to model the deeper stellar interior as pulsation amplitudes are negligibly small there. The equilibrium model is subject to linear stability analysis, which yields periods and linear eigenvectors of the pulsation modes. All known type II Cepheids pulsate in the lowest frequency fundamental mode. The static model is perturbed with the scaled velocity eigenvector and equations (1)–(4) are integrated in time till steady pulsation state is reached. In majority of the studies focused on classical pulsators, RR Lyrae stars or classical Cepheids, the model converges to a limit cycle – full amplitude, single-periodic pulsation. In our recent studies of type-II Cepheids of BL Her type a much more interesting solutions were found, including period doubled pulsation, nicely reproducing the observations of the only BL Her star showing the effect (Smolec et al.[5]) and periodic and quasiperiodic modulation of pulsation (Smolec & Moskalik[16]). In this contribution we discuss an even more complex behaviour we found in BL Her type models with decreased eddy-viscous dissipation – deterministic chaos.

We discuss a single sequence of BL Her-type models, with the same mass ($M=0.55\mathrm{M}_{\odot}$), the same luminosity ($L=136\mathrm{L}_{\odot}$), the same chemical composition and varying effective temperature, which is a control parameter in the following. The models cover a 170 K stripe in the H-R diagram and were computed with the maximum step in effective temperature of 1 K, decreased to 0.1 K in the most interesting domains. The models were integrated typically for 10 000 pulsation cycles (up to 50 000 for few cases) and radius variation, in particular the values of maximum radius, were analysed in detail. Smolec & Moskalik[17] present a detailed description and analysis of these models.

4 Chaotic phenomena in BL Her models – a showcase

Figure 1 presents a bifurcation diagram for the computed BL Her models. It is a stack a grey-scaled histograms. For each effective temperature we computed the probability with which the maximum radii fall into 120 bins into which the range of maximum radius variation in the models was divided. In the bottom part of Fig. 1 values of the largest Lyapunov exponents, computed using the algorithm of Rosenstein, Collins & De Luca[18], are plotted. They are positive, with typical values between $0.15\,\mathrm{d}^{-1}$ and $0.20\,\mathrm{d}^{-1}$, dropping significantly at the edges of the chaotic bands. Period doubling route to chaos is evident both from the cool and the hot side of the computation domain. Period doubling cascade up to period-16 (on the hot side) is detected in our model grid. The length of period-$2k$ domain, d_{2k}, decreases as k increases. The ratios d_k/d_{2k} are estimated to $d_2/d_4 = (3.6 \pm 0.4)\,\mathrm{K}$ and $d_4/d_8 = (5 \pm 2.5)\,\mathrm{K}$ (on the hot side), and $d_2/d_4 = (3.5 \pm 0.9)\,\mathrm{K}$ (on the cool side), and do not differ significantly from the Feigenbaum constant. The chaotic band is split into parts by several windows with periodic variation. The largest, period-3 window, is the most interesting. At its cool side, as effective temperature decreases, an intermittent route to chaos is detected (Pomeau and Mannevile[19]). On its hot side, the period doubling route leads to three chaotic bands which merge into one in an interior crisis bifurcation (Grebogi, Ott and Yorke[20]). Below we highlight these and other interesting phenomena we detect in our models.

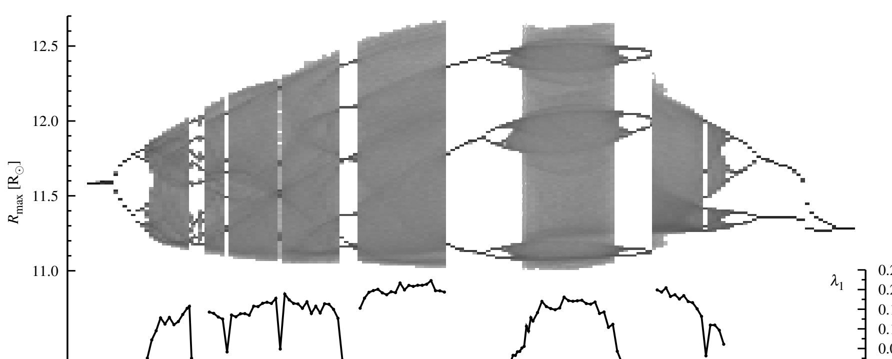

Fig. 1. Bifurcation diagram for the computed hydrodynamic models (top) and variation of the largest Lyapunov exponent (bottom).

- **Chaotic models.** In Fig. 2 we display first return maps for two hydrodynamic models followed for 50 000 pulsation cycles. Complex and likely fractal structure of the attractor is well visible.
- **Periodic windows.** We detect seven windows with periodic behaviour. Three of the windows, with period-6 (at $T_{\rm eff} = 6371\,$K), period-5 (at $T_{\rm eff} = 6383\,$K) and period-6 (at $T_{\rm eff} = 6479\,$K) behaviour, are less than 2 K wide. Return map for the model located in first of these windows is displayed in Fig. 3 (left). In a window extending between 6397 K and 6400 K period-7 and, after a period doubling bifurcation, period-14 behaviours are detected. In a window extending between 6363 K and 6366 K complex scenario is observed – see return map in Fig. 3 (right), including type-III intermittency discussed below. The two largest windows, period-3 (6421 K–6438 K) and period-6 (6459 K–6468 K) windows, show a rich internal structure, with period-doubling and intermittent routes to chaos. These are also discussed in more detail below.

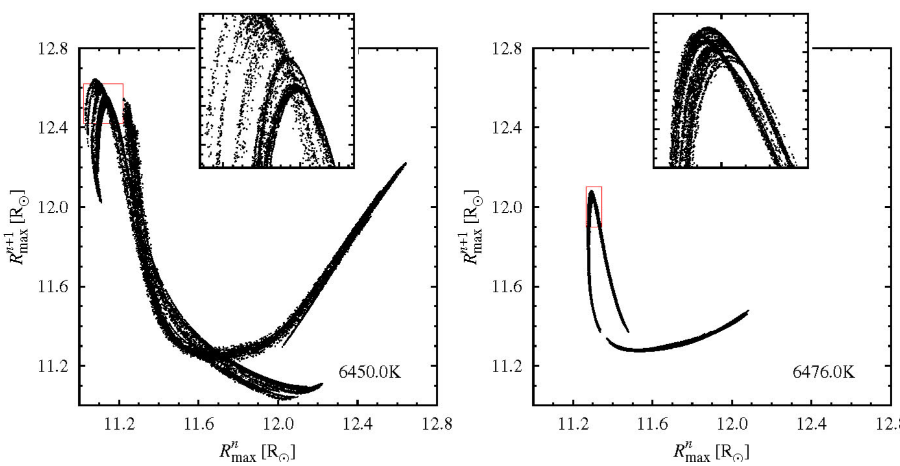

Fig. 2. First return maps for two models showing chaotic variability.

- **Type III intermittency.** The effect is clearly observed in one model from period-9 window ($T_{\rm eff} = 6365\,$K) In a return map (Fig. 3, right) 9 bands are clearly visible, while inspection of maximum radii (Fig. 4) clearly reveals type-III intermittency: switching between period-9 and period-18 behaviour (see Pomeau and Mannevile[19]).
- **Type I intermittency.** The effect is best visible at the cool edge of the largest, period-3 window, at which, as effective temperature of the models is decreased, the intermittent route to chaos is evident. In Fig. 5 the maximum radii are plotted vs. the pulsation cycle number for two models with 6420.9 K and 6420.7 K. We note that a slightly hotter model (6421 K) displays a strictly periodic, period-3 behaviour. As effective temperature is decreased, period-3 cycle losses its stability (tangent bifurcation) and type

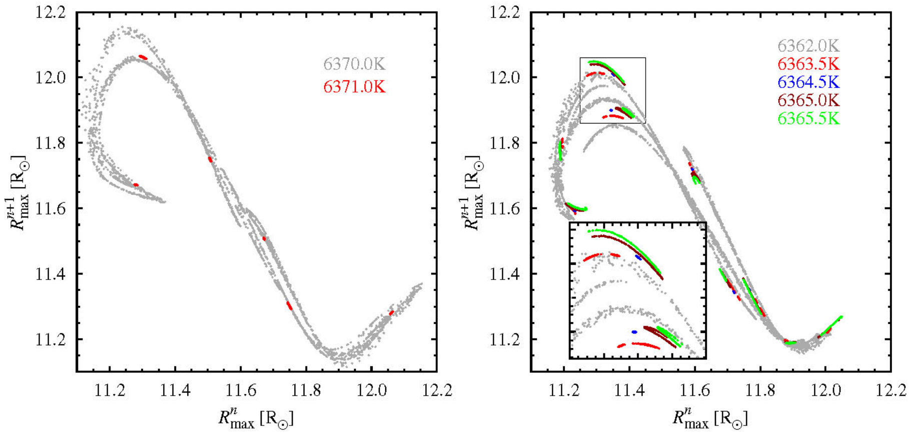

Fig. 3. First return map for a period-6 model (left) and four models from period-9 window. In both cases first return maps for directly neighbouring, slightly cooler, chaotic models are plotted with grey dots, for a reference.

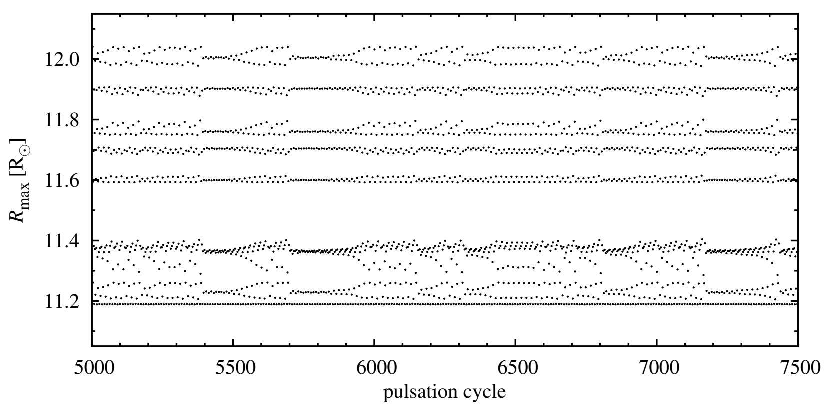

Fig. 4. Type III intermittency in a model with $T_{\rm eff} = 6365.0\,$K.

I intermittency is observed with the stages of almost periodic behaviour rapidly shrinking with the growing distance from the bifurcation point.

- **Interior crisis and crisis induced intermittency.** These phenomena are present on the hot side of the period-3 window. There, at $T_{\rm eff} \approx 6435\,$K, a period doubling cascade forms three separated chaotic bands. As effective temperature is increased, these three bands hit the unstable period-3 cycle created in the tangent bifurcation at the cool edge of the period-3 window, expand, and merge into one chaotic band ($T_{\rm eff} \approx 6438\,$K). A crisis induced intermittency is well visible in slightly hotter models and is illustrated in Fig. 6.
- **Remerging Feigenbaum tree.** The period-3 and period-6 windows are tightly connected, as is well visible in the bifurcation diagram (Fig. 1), and form a period-3 bubble or remerging Feigenbaum tree (Bier & Bountis[21]). The scenarios at the cool and at the hot side of the chaotic band separating

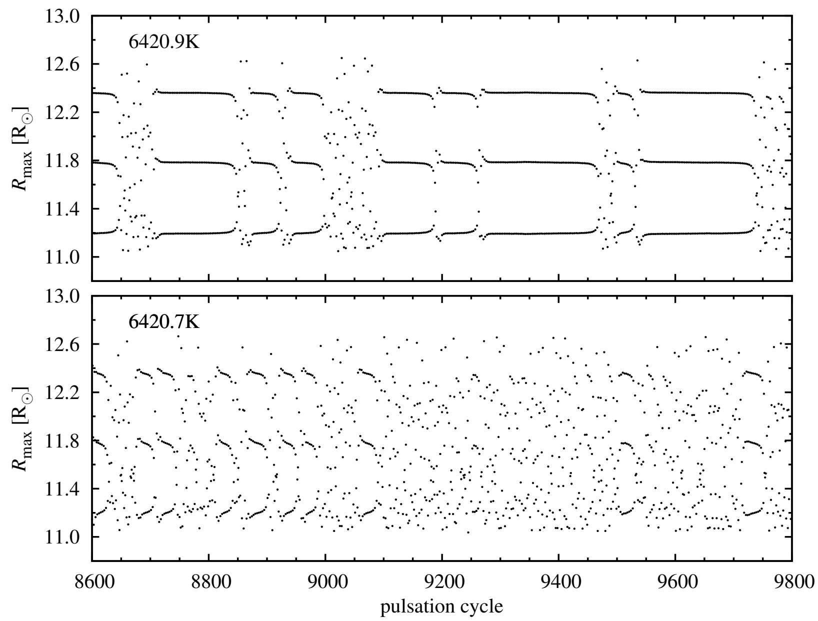

Fig. 5. Type I intermittency in a model with $T_{\rm eff} = 6420.9\,$K (top) and in a model 0.2 K cooler (bottom).

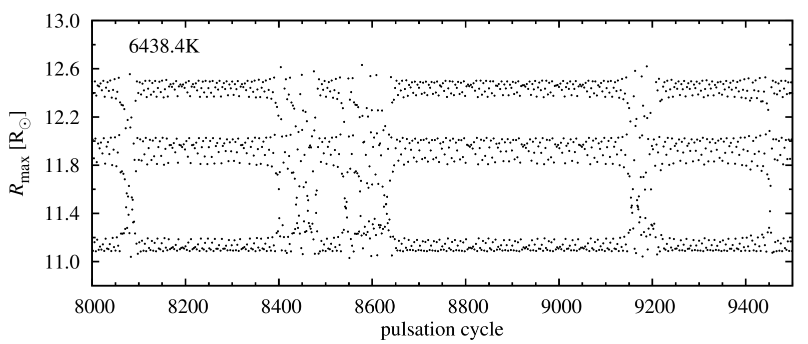

Fig. 6. Crisis induced intermittency in a model with $T_{\rm eff} = 6438.4\,$K.

these two windows are mutual mirror images. In addition the three chaotic bands that are formed in the two windows (as temperature is increased within period-3 window, and as temperature is decreased in period-6 window) do not disappear as they merge into one band in the crisis bifurcation. They sustain their identity and smoothly merge within the chaotic domain (between 6438 K and 6459 K) as dark grey bands in Fig. 1 indicate.

5 Discussion

Most of the chaotic phenomena detected in the models were not yet detected in pulsating stars. In BL Her stars only period doubling effect was found in one star. Nevertheless chaotic dynamics is present in more luminous type II Cepheids of RV Tau type and in semi-regular and Mira-type variables. Based on our models we expect, that the wealth of dynamical behaviours well known in classical dynamic systems, like Lorenz or Rössler systems, may also be present in pulsating stars. Detection of these effects is difficult however, as long, regular and precise monitoring of stellar variability is necessary. With the growing amount of high quality data from massive sky surveys, like Optical Gravitational Lensing Experiment (Udalski et al.[22]), we hope that discovery of the reported effects, like intermittency or period-k pulsation (with k other than 2), is only a matter of time.

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References

1. G. Wallerstein. The Cepheids of Population II and Related Stars. *PASP*, **114**, 689–699, 2002.
2. I. Soszynski et al. The Optical Gravitational Lensing Experiment. The OGLE-III Catalog of Variable Stars. XIV. Classical and Type II Cepheids in the Galactic Bulge. *Acta Astronomica*, **61**, 285–301, 2011.
3. J.P. Cox. Theory of stellar pulsation. 1980, Princeton University Press.
4. M.R. Templeton, A.A. Henden. Multicolor Photometry of the Type II Cepheid Prototype W Virginis. *AJ*, **134**, 1999–2005, 2007.
5. R. Smolec et al. Discovery of period doubling in BL Herculis stars of the OGLE survey. Observations and theoretical models. *MNRAS*, **419**, 2407–2423, 2012.
6. J.R. Buchler and P. Moskalik. Pulsational Study of BL Her Models. I. Radial Velocities. *ApJ*, **391**, 736–749, 1992.
7. R.A. Gingold. The evolutionary status of Type II Cepheids. *Mem. Soc. Ast. It.*, **56**, 169–191, 1985.
8. J.R. Buchler, Z. Kolláth, T. Serre. Nonlinear Analysis of the Light Curve of the Variable Star R Scuti. *ApJ*, **462**, 489–501, 1996.
9. Z. Kolláth, J.R. Buchler, T. Serre, J. Mattei. Analysis of the irregular pulsations of AC Herculis. *A&A*, **329**, 147–154, 1998.
10. J.R. Buchler, Z. Kolláth, R.R. Cadmus. Evidence for Low-dimensional Chaos in Semiregular Variable Stars. *ApJ*, **613**, 532–547, 2004.
11. L.L. Kiss and K. Szatmáry. Period-doubling events in the light curve of R Cygni: Evidence for chaotic behaviour. *A&A*, **390**, 585–596, 2003.
12. J.R. Buchler and G. Kovács. Period doubling bifurcations and chaos in W Virginis models. *ApJL*, **320**, L57–L62, 1987.
13. G. Kovács and J.R. Buchler. Regular and irregular nonlinear pulsations in population II Cepheid models. *ApJ*, **334**, 971–994, 1988.
14. R. Smolec and P. Moskalik. Convective Hydrocodes for Radial Stellar Pulsation. Physical and Numerical Formulation. *Acta Astronomica*, **58**, 193–232, 2008
15. R. Kuhfuß. A model for time-dependent turbulent convection. *A&A*, **160**, 116–120, 1986.
16. R. Smolec and P. Moskalik. Period doubling and Blazhko modulation in BL Herculis hydrodynamic models. *MNRAS*, **426**, 108–119, 2012
17. R. Smolec and P. Moskalik. Chaos in hydrodynamic BL Herculis models. *MNRAS*, **441**, 101–115, 2014

18. M.T. Rosenstein, J.J. Collins, C.J. De Luca 1993, A practical method for calculating largest Lyapunov exponents from small data sets, *Physica D*, **65**, 117–134, 1993.
19. Y. Pomeau and P. Mannevile. Intermittent Transition to Turbulence in Dissipative Dynamical Systems. *Comm. Math. Phys.*, **74**, 189–197, 1980.
20. C. Grebogi, E. Ott and J.A. Yorke. Chaotic Attractors in Crisis, *Phys. Rev. Lett.*, **48**, 1507–1510, 1982.
21. M. Bier and T.C. Bountis. Remerging Feigenbaum Trees in Dynamical Systems. *Phys. Lett. A*, **104**, 239–244, 1984.
22. A. Udalski, M.K. Szymanski, I. Soszynski and R. Poleski. The Optical Gravitational Lensing Experiment. Final Reductions of the OGLE-III Data. *Acta Astronomica*, **58**, 69–87, 2008.