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Numerical Simulation for Dynamic and Chaotic behaviour of Coupled Biological Oscillators through Extended Poincaré-Cartan Invariants

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Abstract: This article deals with the numerical analysis of linearly coupled oscillators, which are connected in the form of a ring. This system is being used as a model of biological system of intestine or may be the liver tissues exhibiting a tube of oscillators having cells both around the periphery and along the axis of the tube. The biological oscillators may be represented as Van der Pol oscillators, which are coupled with non-trivial couplings, where damping parameters are also included in system. The dynamics is determined analytically through extended Poincaré -Cartan theorem of umbra-Lagrangian theory, which examines the invariance of Poincare integral in *m*-dimensional phase space. The synchronization dynamics is achieved with full and cluster mode in terms of coupling parameters. The effects of injected signal on the stability boundaries of the synchronized states are analyzed using numerical simulations, which exhibit interesting beating phenomena. This analysis is corroborated with simulation results to visualize the chaotic behaviour of the coupled oscillator.

Keywords: Coupled oscillators, Umbra-Lagrangian theory, Biological system, Poincaré – Cartan Invariant, Chaotic behaviour.

Notation:

q_i Generalized displacement
$\dot{q}_i =$ Generalized velocity
t = Real-time
x_i = Concentration of the oscillator, where $i = (1,2,3,4)$
$\dot{x}_i$ = Rate of concentration of the oscillator, where $i = (1,2,3,4)$
$A_{K_{ij}}$ = Kinematic area projection in $i-j$ th plane
$A_{P_{ij}}$ = Phase area projection in $i-j$ th plane
$A_{T_{ij}}$ = Trajectorial area projection $i-j$ th plane
K = Solute concentration gradients of all oscillators

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K_{cp}^{A} = Solute concentration gradients of the inhibitory coupling

K_{cp}^{B} = Solute concentration gradients of the excitatory coupling

L = Lagrangian of the system

$L*$ = Umbra-Lagrangian of the system

m= Mass of the all oscillators

R_{cp} = Resistance to the flow of solute

μ = Nonlinear damping coefficient

η = Umbra-time

1. Introduction

The dynamics of non-linearly coupled oscillators has received much attention around the world and dynamical systems are currently the most popular and complex research areas. This is specifically true that non-linear dynamics of these systems have been widely investigated by [1-3] and now a day, attention is focused on dissipative dynamical systems [4-5]. Coupled non-linear oscillators have been source of growing interest in different research areas, ranging from physics, chemistry, engineering to biology and even in social sciences [6-7]. Non-linearity in these systems is essential to attain self-sustained characteristics. These systems reach as stable oscillatory dynamics as consequences of an internal balance between amplification and dissipation. The one of the most interesting features displayed by coupled oscillators is synchronization, due to an adjustment of rhythms due to weak interactions. This phenomenon of synchronization is observed in nature and offers the possible applications in electronics, communications engineering, information technology, biological and chemical structures. Wang et. al.[8] have studied the synchronous bursts on scale-free neuronal networks considering an attractive and repulsive coupling. In another research, Want et al [9] also investigated synchronization transitions on scale-free neuronal networks. In this study, they considered the transitions due to finite information transmission delays. In similar study by Sun et. al.[10], the burst synchronization has been attempted in neuronal networks. In another study by Volman et. al.[11], the gap junctions and epileptic seizures were analyzed in view of synchronization of neuronal ensembles. All such studies have attempted the synchronization phenomena in neuronal networks of brain.

The synchronization phenomena have been successfully applied in chemical cells as revealed in the study of Perc et. al. [12], where periodic calcium waves in coupled cells due to internal noise has been investigated. In another study by Gosak et. al. [13], spatial coherence resonance in excitable biochemical media induced by internal noise has been examined.

Recently, research has shifted towards the study of dynamical behaviour of a system of coupled limit cycle oscillators forming a ring with main focus of synchronization criteria [14-15]. The behaviour of a pair of coupled limit cycle oscillators display a significant wider range of phenomenon than a single limit cycle oscillators and may be used to model systems such as cardiac cells, gas flux control in plant, vibrations of heat exchangers, human body, biomedical engineering etc. Many researchers have contributed in these areas. Bi [16] studied the dynamical behaviour of two coupled parameter excited Vander Pol's oscillators. Leung [17] investigated the dynamics process of synchronizing chaotic Vander Pol systems driven periodic force. Such situations are very prominent in biological systems such as animal and human bodies. Collins and Richmond [18] have shown that quadrupedal mammals, the four oscillators controlling the limb of animals may be coupled to form a ring. Similarly McClellan [19] has also provided the organization of spinal locomotors networks. A lot of works in this field are contributed by Woafo and his team [20]. Woafo and Enjieu Kadji [21] had investigated the synchronized states in a ring of mutually coupled self –sustained electrical oscillators and later Enjieu Kadji et al. [22] have applied the same theory in human intestine. But their model has several constraints, as they have not considered the damping coefficient of the coupling, which otherwise, in biological systems find wider applications from duodenum, liver tissues, kidney, and heartbeat, and also affects the synchronization dynamics of the oscillators. Present paper deals with the synchronization dynamics through a novel approach of extended Poincare-Cartan invariant, which was not attempted in any archival literature.

Integral invariants of dynamical systems play an important role in mechanics. Primarily, the study of integral invariants is limited only to conservative systems. Some researchers have tried to extend the applications of integral invariants to non-conservative systems and non-holonomic systems [23]. However, they introduced integral invariants of Poincaré-Cartan's type for non-holonomic constrained systems by extending the Hamilton's stationary principle to nonholonomic-constrained systems. Benavent and Gomis [24] had extended the Poincaré-Cartan integral to constrained systems and found sufficient conditions for systems whose equations of motion contain arbitrary functions to be Hamiltonian. Liu et al. [25] reported that no basic integral invariants exist for non-conservative systems from traditional viewpoint and put forward the integral variants of the non-conservative system.

Some significant works in this direction were reported by Gou et al. [26, 27, 28] in their several papers. They have proposed the Poincaré-Cartan integral invariants for non-conservative dynamical systems on an extended phase space, where the set of dynamical equations and their adjoint equations are canonical. In another work, Gou et al. [27] further constructed Poincaré and Poincaré-Cartan integral invariants of a generalized Hamiltonian system and also for non-holonomic constrained system [28], where non-holonomy of the constraints and non conservative forces acting on the system is derived from D'Alembert-Lagrange principle.

The key to the generalization of integral invariants to other systems lies in the search for new Lagrangian or Hamiltonian for those systems. Apart from all previous extensions of classical Lagrange's equation, Mukherjee [30] suggested an alternative method for finding invariants of motion or possible invariant trajectories for dynamical systems. Mukherjee *et al.* [31] proposed a modified Lagrange's equation introducing an additional time like variable called 'umbra time'. This notion was extended to all type of energies as well as the Lagrangian itself, which was termed as the "umbra-Lagrangian". The umbra-Lagrange's equation for general class of systems with *n* generalized co-ordinates [29, 33] and [34] may be written as

$$\frac{d}{dt}\left[\lim_{\eta \to t} \frac{\partial L^*}{\partial \dot{q}_i(\eta)}\right] - \lim_{\eta \to t} \frac{\partial L^*}{\partial q_i(\eta)} = 0 \text{ ,for } i = 1..n\,. \qquad (1)$$

where L^* is umbra-Lagrangian, *t* and η are real and umbra time respectively and q_i and $\dot{q}_i$ are generalized displacements and velocities. Umbra-Lagrangian may be given as

$$L^*\left(t,\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t),\boldsymbol{q}(\eta),\dot{\boldsymbol{q}}(\eta)\right) = T_c^*\left(t,\dot{\boldsymbol{q}}(\eta)\right) - V^*\left(t,\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t),\boldsymbol{q}(\eta)\right), \qquad (2)$$

where a bold face letter represents a vector quantity, with $T^*(...)$ as the umbra – co-kinetic energy and $V^*(...)$as the umbra-potential energy. The details expression of umbra-co kinetic energy and umbra-potential energy has been outlined in ref. [33, 34] and briefly presented in *Appendix 'A'*.

In present work, the model of Collins and Richmond [18] and Enjieu Kadji et al. [22] have been refined by incorporating the damping parameter in the system for modeling the duodenum or intestine. This ring model has been investigated through extended Poincare-Cartan theorem [29] of umbra-Lagrangian theory formulated by Mukherjee [30] and Mukherjee et. al. [33, 34]. A candid commentary on this umbra-Lagrangian theory has been given by Brown [31]. The intestine model comprises a tube of oscillators having cells both around the periphery and along the axis of the tube. In this physiological model, the enzyme does not migrated in the product, so that it may be separated easily. In this model, the ring connections of these four oscillators have been considered with four protomers of the oligomer of an allosteric enzyme. The coupling parameters have been considered non-identical between the oscillators enrolled in the process. Moreover, earlier model of promoters [22] is also refined by incorporating damping coefficient of the coupling, which also contributes some interesting results as a promoter may also bind a single molecule of an effector ligand present in the surrounding solution and molecule may also exert some resistance to the flow of solute.

In this paper, different aspects of the dynamical behaviour of coupled oscillators are discussed. In section two, Poincaré-Cartan theorem of umbra-Lagrangian theory is presented and mathematical analysis of coupled biological oscillators is obtained. Section three presents the dynamical behaviour of the coupled biological oscillators. This behaviour is quite obvious through simulation results with changing the value of coupling parameters. One may also obtain the chaos conditions and beating phenomena, when local injected signals in forms of irregularity are incorporated in the model and analyzed using numerical simulations.

2. Poincaré -Cartan Theorem for umbra-Lagrangian theory

To enlarge the scope of Lagrangian-Hamiltonian mechanics, a new proposal of additional time like variable 'umbra-time' was made by Mukherjee [30] and this new concept of umbra-time leads to a peculiar form of equation, which is termed as umbra-Lagrange's equation [31]. The details of this theory are reported in reference [33-34]; however some specific proof may be reproduced here. This theorem is an extension of Poincare-Cartan theorem, which has an action integral, where initial and terminal terms are varied as functions of a parameter α [35]. The modulatory and the trajectories displacements are also varied through the variation of the indices, which select the members of the homotopic family of these functions. The homotopy may now be extended up to the varied initial and terminal instants of time.
The integral may be written as

$$W = \int_{t_0(\alpha)}^{t_1(\alpha)} L\left(\bar{q}(\xi, a+\alpha), \dot{\bar{q}}(\xi, a+\alpha), \bar{q}(\xi, b+\alpha), \dot{\bar{q}}(\xi, b+\alpha), \xi\right) d\xi. \tag{3}$$

The index *a* corresponds to modulatory and *b* corresponds to trajectorial member. The parameter $\alpha=0$ corresponds to unvaried function with initial and terminal time. Assuming δt as a small variation in the parameter α from the zero value, one may write

$$\Delta W = \int_{t_0(\alpha)}^{t_1(\alpha)} L(\ldots) d\xi - \int_{t_0(0)}^{t_1(0)} L(\ldots) d\xi\,, \tag{4}$$

After following the procedure as in reference [35], one obtains integral invariant as

$$\frac{dI(t)}{dt} = -\oint_{C_t} \sum_{i=1}^{n} \left[\left(\lim_{\eta \to t} \frac{\partial L^*}{\partial \dot{q}_i(t)} \right) \delta \dot{q}_i + \left(\lim_{\eta \to t} \frac{\partial L^*}{\partial q_i(t)} \right) \delta q_i \right]. \tag{5}$$

Applying Green’s theorem, Eq. (5) may be rewritten by taking $\dot{q}_i$ and q_i in cyclic order as

$$\begin{aligned}\frac{dI(t)}{dt} = &-\sum_{i=1}^{n}\sum_{j=1}^{n}\iint_{A_{K_{ij}}}\left\{\frac{\partial}{\partial \dot{q}_i(t)}\left(\underset{\eta\to t}{Lim}\frac{\partial L^*}{\partial \dot{q}_j(t)}\right)-\frac{\partial}{\partial \dot{q}_j(t)}\left(\underset{\eta\to t}{Lim}\frac{\partial L^*}{\partial \dot{q}_i(t)}\right)\right\}\delta\dot{q}_i\delta\dot{q}_j \\ &-\sum_{i=1}^{n}\sum_{j=1}^{n}\iint_{A_{Pij}}\left\{\frac{\partial}{\partial \dot{q}_i(t)}\left(\underset{\eta\to t}{Lim}\frac{\partial L^*}{\partial q_j(t)}\right)-\frac{\partial}{\partial q_j(t)}\left(\underset{\eta\to t}{Lim}\frac{\partial L^*}{\partial \dot{q}_i(t)}\right)\right\}\delta\dot{q}_i\delta q_j \\ &-\sum_{i=1}^{n}\sum_{j=1}^{n}\iint_{A_{T_{i,j}}}\left\{\frac{\partial}{\partial q_i(t)}\left(\underset{\eta\to t}{Lim}\frac{\partial L^*}{\partial q_j(t)}\right)-\frac{\partial}{\partial q_j(t)}\left(\underset{\eta\to t}{Lim}\frac{\partial L^*}{\partial q_i(t)}\right)\right\}\delta q_i\delta q_j .\end{aligned} \tag{6}$$

gives the rate of change of a Poincaré integral and contains three area projections in *i*- j^{th} plane, which are, the velocity-velocity area projection $A_{K_{ij}}$ termed as the *kinematic projection*, velocity-displacement area projection $A_{P_{i,j}}$ termed as the *phase projection,* and displacement-displacement area projection $A_{T_{i,j}}$ termed as the *trajectorial* or *orbital projection*. For purely Hamiltonian systems with no dissipative, gyroscopic or regenerative forces and fields, the right hand side terms vanish, and lead to the classical result as

$$\frac{dI(t)}{dt} = 0 . \tag{7}$$

2.1 Physical Model and Problem Statement

Any biological system can be easily model with Ferro-electricity behaviour in brain waves [36-37] and may be represented by classical Vander Pol equation as

$$m\ddot{x} - \mu\left(x^2 - 1\right)\dot{x} + Kx = 0 . \tag{8}$$

In this paper, a general approach for analyzing ring structure of coupled biological oscillators is presented. The study may be applied to mammalian intestinal activity and how the enzymes flow from one cell to other cells. However, four coupled oscillators are presented in context of animal locomotion [43]. Quadruped gaits closely resemble the natural patterns of four oscillators systems. As example of rabbit bounds its move its front leg together then its back legs. There is a phase difference of zero between the two front legs and one half between the front and back legs. Four oscillators may be coupled in seven ways such as (a) Synchrony each one fourth of a cycle out of phase with the others (b) Three synchronous and one with unrelated phase (c) Peculiar rhythm of two oscillators, anti-synchronous with each other.

As already stated that this system is already interacting with their surroundings and neighboring enzymes, so there will be cooperating interactions between them. In this problem, coupled enzymes with ring connection consisting of four mutually coupled Vander Pol oscillators can be shown in Fig.1, which may be expressed by following equations:

$$\begin{aligned}
m\ddot{x}_1 - \mu\left(x_1^2 - 1\right)\dot{x}_1 + Kx_1 &= K_{cp}^A(x_4 - x_1) + K_{cp}^B(x_2 - x_1) + R_{cp}(\dot{x}_4 - \dot{x}_1) + R_{cp}(\dot{x}_2 - \dot{x}_1) \\
m\ddot{x}_2 - \mu\left(x_2^2 - 1\right)\dot{x}_2 + Kx_2 &= K_{cp}^A(x_3 - x_2) + K_{cp}^B(x_1 - x_2) + R_{cp}(\dot{x}_3 - \dot{x}_2) + R_{cp}(\dot{x}_1 - \dot{x}_2) \\
m\ddot{x}_3 - \mu\left(x_3^2 - 1\right)\dot{x}_3 + Kx_3 &= K_{cp}^A(x_2 - x_3) + K_{cp}^B(x_4 - x_3) + R_{cp}(\dot{x}_2 - \dot{x}_3) + R_{cp}(\dot{x}_4 - \dot{x}_3) \\
m\ddot{x}_4 - \mu\left(x_4^2 - 1\right)\dot{x}_4 + Kx_4 &= K_{cp}^A(x_1 - x_4) + K_{cp}^B(x_3 - x_4) + R_{cp}(\dot{x}_1 - \dot{x}_4) + R_{cp}(\dot{x}_3 - \dot{x}_4)
\end{aligned} \tag{9}$$

In this set of equation, the over dot denotes the time derivative, μ, the parameter of non-linearity and $x_i(1,2,3,4)$ the concentration of i^{th} biological system. It is a fact when two or more oscillators are coupled, the equation governing the behaviour tends to become intractable and possible behaviour becomes much more complex. In this formation of a ring, each oscillator may be coupled only to a few immediate neighbours. This ring may be the same as the neuro-muscular oscillators in the small intestine. The force of coupling between the oscillators is significant parameter to decide the synchronization behaviour. In this system, there is one positive and one negative direction of the solute flow, which represents the inhibitory and excitatory couplings. In this way, the directions of the solute flow may be represented as K_{cp}^A and K_{cp}^B, which may be positive or negative. R_{cp} in the model represents the resistance to solute flow due to any irregularity and may be expressed as damping coefficient of couplings. Now, from these equations, one may easily obtain Umbra-Lagrangian Equation [33-34], which may be written as

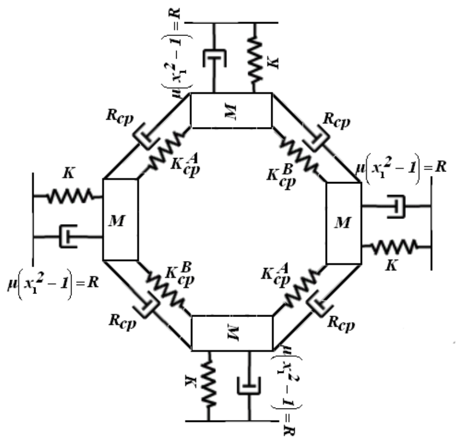

Fig.1: Four Biological oscillators connected in a ring with non-trivial coupling

$$
\begin{aligned}
L^{*} = {} & \frac{1}{2}m\dot{x}_1^2(\eta) + \frac{1}{2}m\dot{x}_2^2(\eta) + \frac{1}{2}m\dot{x}_3^2(\eta) + \frac{1}{2}m\dot{x}_4^2(\eta) - \frac{1}{2}Kx_1^2(\eta) - \frac{1}{2}K_{cp}^{A}\{x_4(\eta) - x_1(\eta)\}^2 \\
& - \frac{1}{2}K_{cp}^{B}\{x_2(\eta) - x_1(\eta)\}^2 - \frac{1}{2}Kx_2^2(\eta) - \frac{1}{2}K_{cp}^{A}\{x_3(\eta) - x_2(\eta)\}^2 - \frac{1}{2}K_{cp}^{B}\{x_1(\eta) - x_2(\eta)\}^2 \\
& - \frac{1}{2}Kx_3^2(\eta) - \frac{1}{2}K_{cp}^{A}\{x_2(\eta) - x_3(\eta)\}^2 - \frac{1}{2}K_{cp}^{B}\{x_4(\eta) - x_3(\eta)\}^2 \\
& - \frac{1}{2}Kx_4^2(\eta) - \frac{1}{2}K_{cp}^{A}\{x_1(\eta) - x_4(\eta)\}^2 \\
& - \frac{1}{2}K_{cp}^{B}\{x_3(\eta) - x_4(\eta)\}^2 - \mu(x_1^2 - 1)\dot{x}_1(t)x_1(\eta) - \mu(x_2^2 - 1)\dot{x}_2(t)x_2(\eta) - \mu(x_3^2 - 1)\dot{x}_3(t)x_3(\eta) \\
& - \mu(x_4^2 - 1)\dot{x}_4(t)x_4(\eta) - R_{cp}\{\dot{x}_2(t) - \dot{x}_1(t)\}\{x_2(t) - x_1(t)\} - R_{cp}\{\dot{x}_3(t) - \dot{x}_2(t)\}\{x_3(t) - x_2(t)\} \\
& - R_{cp}\{\dot{x}_4(t) - \dot{x}_3(t)\}\{x_4(t) - x_3(t)\} - R_{cp}\{\dot{x}_1(t) - \dot{x}_2(t)\}\{x_1(t) - x_2(t)\}
\end{aligned}
\tag{10}
$$

Now through Eq. (6), one may finally obtains

$$
\frac{dI}{dt} = \begin{bmatrix}
-\mu \iint\limits_{A(P\dot{x}_1x_1)} (x_1^2(t) - R_{cp} - 1)\,\delta\dot{x}_1(t)\delta x_1(t) - \mu \iint\limits_{A(P\dot{x}_2x_2)} (x_2^2(t) - R_{cp} - 1)\,\delta\dot{x}_2(t)\delta x_2(t) \\
-\mu \iint\limits_{A(P\dot{x}_3x_3)} (x_3^2(t) - R_{cp} - 1)\,\delta\dot{x}_3(t)\delta x_3(t) - \mu \iint\limits_{A(P\dot{x}_4x_4)} (x_4^2(t) - R_{cp} - 1)\,\delta\dot{x}_4(t)\delta x_4(t) \\
- \iint\limits_{A(P\dot{x}_1x_2)} R_{cp}\,\delta\dot{x}_1(t)\delta x_2(t) - \iint\limits_{A(P\dot{x}_2x_1)} R_{cp}\,\delta\dot{x}_2(t)\delta x_1(t) - \iint\limits_{A(P\dot{x}_2x_1)} R_{cp}\,\delta\dot{x}_2(t)\delta x_3(t) \\
- \iint\limits_{A(P\dot{x}_3x_2)} R_{cp}\,\delta\dot{x}_3(t)\delta x_2(t) - \iint\limits_{A(P\dot{x}_3x_4)} R_{cp}\,\delta\dot{x}_3(t)\delta x_4(t) - \iint\limits_{A(P\dot{x}_4x_3)} R_{cp}\,\delta\dot{x}_4(t)\delta x_3(t) \\
- \iint\limits_{A(P\dot{x}_4x_1)} R_{cp}\,\delta\dot{x}_4(t)\delta x_1(t) - \iint\limits_{A(P\dot{x}_1x_4)} R_{cp}\,\delta\dot{x}_1(t)\delta x_4(t)
\end{bmatrix}
\tag{11}
$$

Equation (11) gives the oscillations of coupled Vander Pol's oscillators in a 12 dimensional phase space. When the coupling terms R_{cp} and K_{cp} are equal to zero, the system consist of four independent oscillators each evolving on a limit cycle that is orbitally stable. In this way, Equation (11) just reduces to

$$
\frac{dI}{dt} = \begin{bmatrix}
-\mu \iint\limits_{A(P\dot{x}_1x_1)} (x_1^2(t) - 1)\,\delta\dot{x}_1(t)\delta x_1(t) - \mu \iint\limits_{A(P\dot{x}_2x_2)} (x_2^2(t) - 1)\,\delta\dot{x}_2(t)\delta x_2(t) \\
-\mu \iint\limits_{A(P\dot{x}_3x_3)} (x_3^2(t) - 1)\,\delta\dot{x}_3(t)\delta x_3(t) - \mu \iint\limits_{A(P\dot{x}_4x_4)} (x_4^2(t) - 1)\,\delta\dot{x}_4(t)\delta x_4(t)
\end{bmatrix}.
\tag{12}
$$

It is assumed that the coupled oscillators are identical and let any stage of time t, the system be initiated on a circle of radius r as $x(t), y(t) = r\cos t$ and $\dot{x}(t), \dot{y}(t) = r\sin t$, after simplification, it yields

$$
\frac{dI}{dt} = 2\left[\mu r^2 \pi \left(1 - \frac{r^2}{4}\right)\right].
\tag{13}
$$

It is clear that $\frac{dI}{dt}=0$ if *r=2*, $\frac{dI}{dt}<0$ if *r>2* and $\frac{dI}{dt}>0$ if *r<2*. Therefore, *r=2* leaves the Poincare-Cartan integral I invariant and hence is the radius of the limit cycle as shown in *Fig. 2 a-2d*.

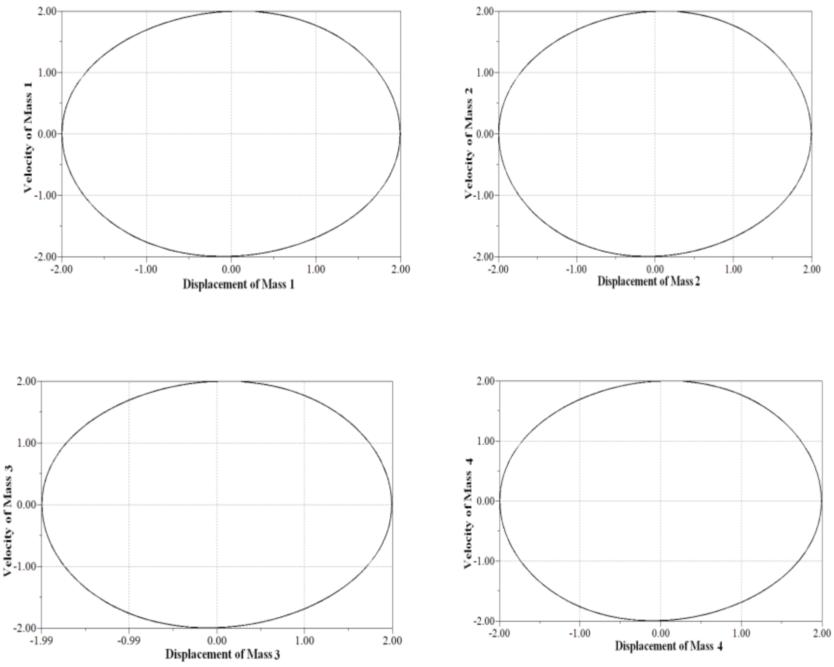

Fig.2 (a-d): Limit cycle of first, second, third and fourth biological oscillator at $\mu = 0.05$

When non-trivial couplings exists due to some internal parameters of the system, it is possible that the nearest neighbouring oscillator are coupled to a considered oscillator through the same or different energy activation due to a conformational spread. In this way, coupled oscillators interact altogether resulting in trajectories embedded in 12 dimension phase space, which may be obtained as Eq. 11.

3. Dynamics of Coupled Oscillators

First of all, it is considered that the coupled oscillators are identical i.e. $x_1(t) = x_2\,(t) = x_3(t) = x_4(t)$, and there is no coupling between the oscillators, then one may easily obtain in phase mode to be exist in the system by simple inspection of Eq. (12)

$$\frac{dI}{dt}=\begin{bmatrix} -\mu \iint\limits_{A(P\dot{x}_1 x_1)} (x_1^2(t)-1)\,\delta\dot{x}_1(t)\delta x_1(t)-\mu \iint\limits_{A(P\dot{x}_1 x_1)} (x_1^2(t)-1)\,\delta\dot{x}_1(t)\delta x_1(t) \\ -\mu \iint\limits_{A(P\dot{x}_1 x_1)} (x_1^2(t)-1)\,\delta\dot{x}_1(t)\delta x_1(t)-\mu \iint\limits_{A(P\dot{x}_1 x_1)} (x_1^2(t)-1)\,\delta\dot{x}_1(t)\delta x_1(t) \end{bmatrix}, \qquad (14)$$

which reduces into four uncoupled Vander Pol equations. In this way, the in phase mode gives an exact solution to the coupled equations.The dynamic behaviour of coupled oscillators include in phase/full synchronization and cluster synchronization. This synchronization phenomenon is of great interest in neural systems and related with several important issues in neurology and science. Moreover, these results are well proven by experiments and analysis in several neurological and heart disease [38-39]. The other significant characteristics of these oscillators lie in the fact that they are highly sensitive to their initial conditions. If these four oscillators are excited at different initial conditions, their trajectory finally circulates on the same limit cycle but with different phase, which is termed as phase synchronization. Simulation results also reveal that there exits a very small region in the coupling parameter space, where the synchronization dynamics are found to be chaotic. The next subsection provides the detailed analysis of in phase or full synchronization.

3.1. In-Phase synchronization / Full Synchronization

In-phase mode is a periodic motion and satisfies the condition $x_1(t)=x_2(t)=x_3(t)=x_4(t)$. Substituting $x_1(t)=x_2(t)=x_3(t)=x_4(t)$ into Eq. (11), one may find the integral invariant as

$$\frac{dI}{dt}=-4\left[\mu \iint\limits_{A(P\dot{x}_1 x_1)} (x_1^2(t)-R_{cp}-1)\,\delta\dot{x}_1(t)\delta x_1(t)+2 \iint\limits_{A(P\dot{x}_1 x_1)} R_{cp}\,\delta\dot{x}_1(t)\delta x_1(t)\right]. \qquad (15)$$

In phase mode, if it exists, resides in the 2-dimensional space $x_1-\dot{x}_1$, which is itself as invariant manifold in 12-dimensional space $x_1-\dot{x}_1-x_2-\dot{x}_2-x_3-\dot{x}_3-x_4-\dot{x}_4$. This mode gives four similar limit cycles oscillating in two dimensional phase space. Thus in-phase mode can further be visualized for any value of R_{cp} and any of the coupling parameter K_{cp}^{A} or $K_{cp}^{B}>0$. Numerical simulation may be advantageous to visualize the phase synchronization as shown in Fig.3, where time history of biological oscillators is shown at value

($\mu = 2.03, R_{cp} = 0.06, K_{cp}^{A} = -0.294, K_{cp}^{B} = 0.294$). The initial conditions for all such oscillator are assumed to the same or $x_{01} = x_{02} = x_{03} = x_{04} = 0.0$.

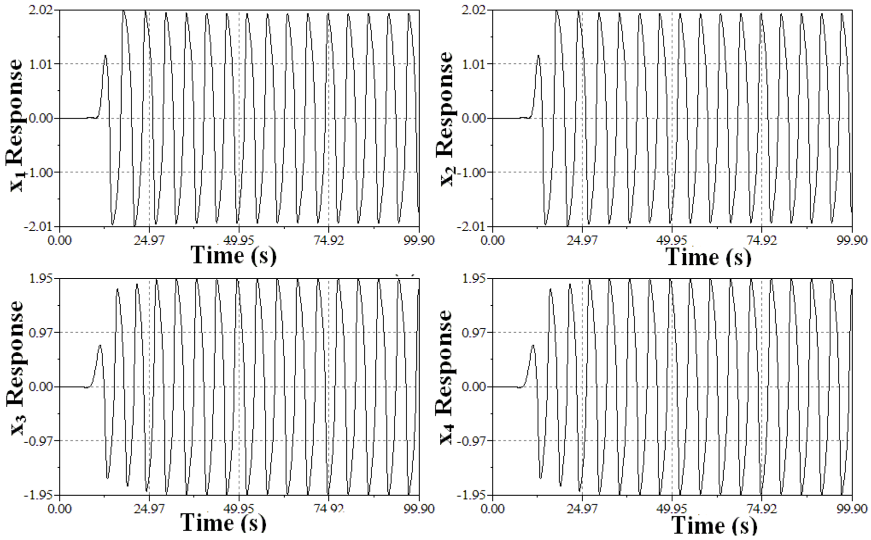

Fig. 3: Time history of oscillators at
$\mu = 2.03, R_{cp} = 0.06, K_{cp}^{A} = -0.294, K_{cp}^{B} = 0.294$

The Fig. 4 and Fig. 5 reveal that all oscillators cross zero (with a positive slope or negative slope) at the same time, which indicates that all oscillators are in–phase synchronization. The full synchronization is also apparent either in positive or negative direction, which means that the full synchronization is having full phase identical whenever the solute flow in positive direction or vice versa. The four biological oscillators display same dynamics at this instant. In the process of phase synchronization, the oscillators oscillate with a repeating sequence of relative phase angles. It may be concluded that the all four oscillators are synchronized having the identical phase.

3.2 Cluster Synchronization

The out of phase is a periodic motion, which satisfy $x_1(t) = x_2(t)\ and\ x_3(t) = x_4(t).$ Putting this condition into Eq. (11), one obtains

$$\frac{dI}{dt}=-2\left[\begin{array}{l}\mu\iint\limits_{A(P\dot{x}_1x_1)}(x_1^2(t)-R_{cp}-1)\,\delta\dot{x}_1(t)\delta x_1(t)+\mu\iint\limits_{A(P\dot{x}_3x_3)}(x_3^2(t)-R_{cp}-1)\,\delta\dot{x}_3(t)\delta x_3(t)\\+\iint\limits_{A(P\dot{x}_1x_1)}R_{cp}\,\delta\dot{x}_1(t)\delta x_1(t)+\iint\limits_{A(P\dot{x}_2x_3)}R_{cp}\,\delta\dot{x}_3(t)\delta x_3(t)\end{array}\right] \tag{16}$$

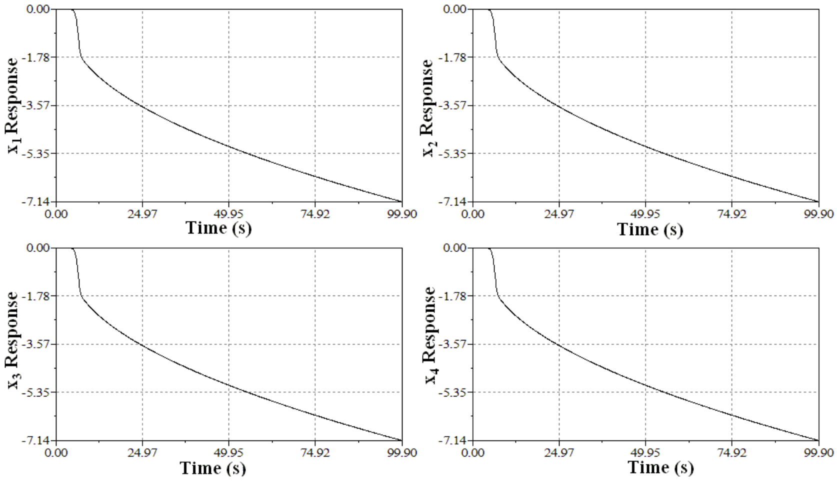

Fig.4: Full synchronization in negative direction at $\mu=0.84, R_{cp}=0.954, K_{cp}^{A}=-1.064, K_{cp}^{B}=-0.316$

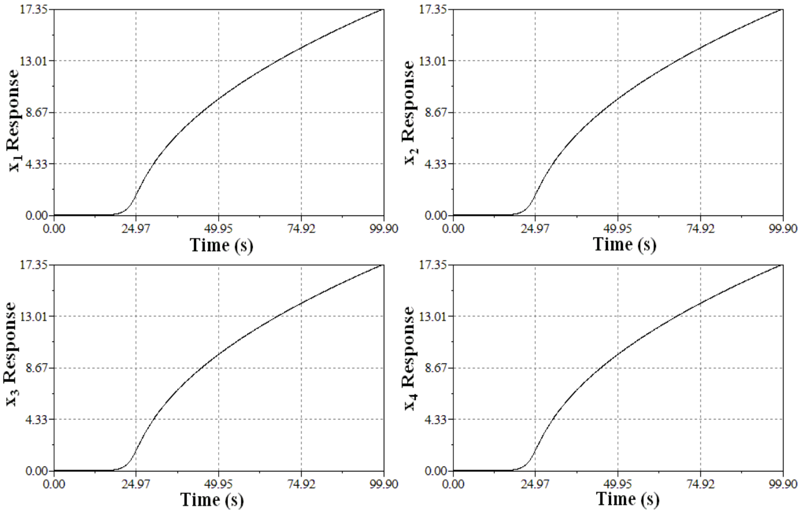

Fig.5: Full synchronization in positive direction at $\mu=2.03, R_{cp}=0.06, K_{cp}^{A}=-1.064, K_{cp}^{B}=0.316$

Here the two oscillators in group are shifted with the zero crossings at a constant time. This phase lag φ may be defined as

$$\varphi = \frac{\Delta t}{T}.2\pi$$

where *Δ t* is the time difference between the zero crossings. The two oscillators are out of phase, when φ =180°. The equation (16) determines that two oscillators $x_1(t),\ x_2(t)$ and $x_3(t),\ x_4(t)$ will synchronize together and this group of two oscillators will oscillate in out of phase synchronization with phase lag of 180°. This phenomenon is termed as cluster synchronization, where oscillators cleave such that all oscillators within one cluster move in perfect synchrony but the motion of two clusters is not synchronized at all. This phenomenon of cluster may also be achieved with $x_1(t) = x_4(t)\ and\ x_2(t) = x_3(t)$.

The reason is quite oblivious for oscillator with neighboring interactions. However, it will not be achieved with the configuration $x_1(t) = x_3(t)\ and\ x_2(t) = x_4(t)$ due to non interacting coupling between them. It may be shown in Figure 6 the time history of out of phase motion corresponding to the value ($\mu = 2.03, R_{cp} = 0.30, K_{cp}^{A} = -0.28, K_{cp}^{B} = -1.4$). The two oscillators are shifted with the zero crossings at a constant time. This phase lag ϕ may be 180 during out of phase synchronization.

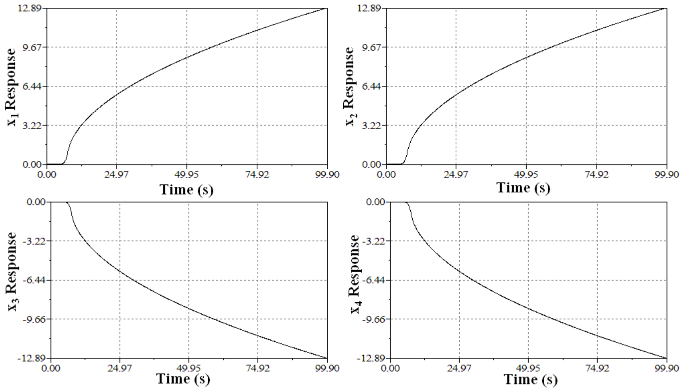

Fig.6: Cluster synchronization of oscillators at $\mu = 2.03, R_{cp} = 0.30, K_{cp}^{A} = -0.28, K_{cp}^{B} = -1.4$

The ranges of coupling parameter including damping coefficient leading to full synchronization and lack of synchronization can be shown in the Fig.7, where the regions of in-phase and out of phase are clearly marked to obtain the optimal output of synchronization phenomena.

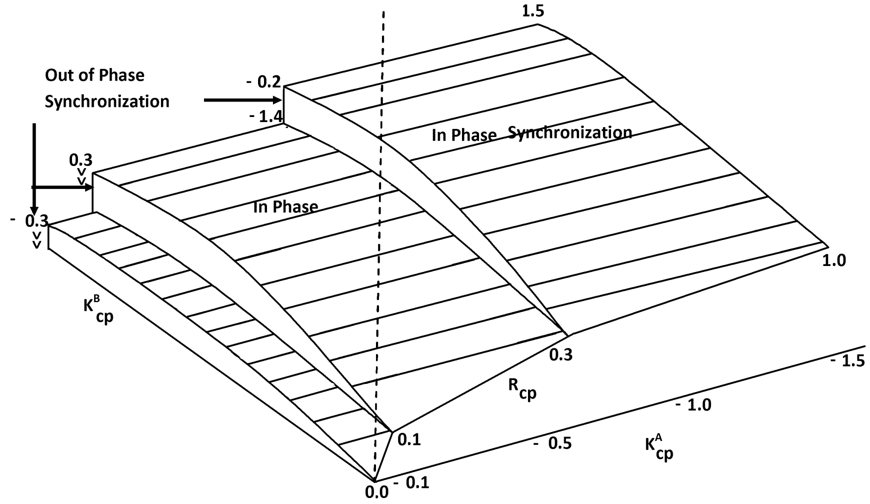

Fig.7: Reigns of in phase and out of phase synchronization of oscillators

4. Influence of Locally Injected Signal on Synchronization Phenomena

As mentioned in previous subsections that different dynamical states are observed in several identical coupled biological oscillators like clustering synchronization (synchronization in sub groups) or complete synchronization (in phase synchronization). But there are some situations in nature in which the complete system may be coupled with an external independent oscillator or excitation. This is a matter of fact that all real system contains impurity, irregularity in the form of natural fluctuations, which is associated with dissipation as well as random external environment.

In self-sustained oscillators are always prone to different causes of irregularity in a form of some external strength like ΔK (sometime termed as detuned parameter). This example may be further ascertained by considering the example of cardio-vascular system of humans, displaying self-sustained vibration. When there is any impact of any irregularity, the cardiac vibrations [40-41] are not perfectly periodic. That is why some locally injected signal in the form of ΔK may be added in the systems in first biological oscillator. Present case of biological system deals with an impulse transmission in neurons and their synchronization dynamics is analyzed further. Including the locally injected parameter ΔK , one may obtain the equations as

$$\begin{aligned}
m\ddot{x}_1-\mu\left(x_1^2-1\right)\dot{x}_1+Kx_1&=K_{cp}^{A}\left(x_4-x_1\right)+K_{cp}^{B}\left(x_2-x_1\right)+R_{cp}\left(\dot{x}_4-\dot{x}_1\right)+R_{cp}\left(\dot{x}_2-\dot{x}_1\right)+\Delta K(x_1-x_c)\\
m\ddot{x}_2-\mu\left(x_2^2-1\right)\dot{x}_2+Kx_2&=K_{cp}^{A}\left(x_3-x_2\right)+K_{cp}^{B}\left(x_1-x_2\right)+R_{cp}\left(\dot{x}_3-\dot{x}_2\right)+R_{cp}\left(\dot{x}_1-\dot{x}_2\right)\\
m\ddot{x}_3-\mu\left(x_3^2-1\right)\dot{x}_3+Kx_3&=K_{cp}^{A}\left(x_2-x_3\right)+K_{cp}^{B}\left(x_4-x_3\right)+R_{cp}\left(\dot{x}_2-\dot{x}_3\right)+R_{cp}\left(\dot{x}_4-\dot{x}_3\right)\\
m\ddot{x}_4-\mu\left(x_4^2-1\right)\dot{x}_4+Kx_4&=K_{cp}^{A}\left(x_1-x_4\right)+K_{cp}^{B}\left(x_3-x_4\right)+R_{cp}\left(\dot{x}_1-\dot{x}_4\right)+R_{cp}\left(\dot{x}_3-\dot{x}_4\right)
\end{aligned} \tag{17}$$

In Eq. (17), x_c represents the dynamic parameter of the external oscillator and also defined as controlling signal.

The umbra-Lagrangian equation for Eq.(17) may be written as

$$\begin{aligned}
L^* =\ &\frac{1}{2}m\dot{x}_1^2(\eta)+\frac{1}{2}m\dot{x}_2^2(\eta)+\frac{1}{2}m\dot{x}_3^2(\eta)+\frac{1}{2}m\dot{x}_4^2(\eta)-\frac{1}{2}Kx_1^2(\eta)-\frac{1}{2}K_{cp}^{A}\{x_4(\eta)-x_1(\eta)\}^2\\
&-\frac{1}{2}K_{cp}^{B}\{x_2(\eta)-x_1(\eta)\}^2-\frac{1}{2}\Delta K\{x_c(\eta)-x_1(\eta)\}^2-\frac{1}{2}Kx_2^2(\eta)-\frac{1}{2}K_{cp}^{A}\{x_3(\eta)-x_2(\eta)\}^2\\
&-\frac{1}{2}K_{cp}^{B}\{x_1(\eta)-x_2(\eta)\}^2-\frac{1}{2}Kx_3^2(\eta)-\frac{1}{2}K_{cp}^{A}\{x_2(\eta)-x_3(\eta)\}^2-\frac{1}{2}K_{cp}^{B}\{x_4(\eta)-x_3(\eta)\}^2\\
&-\frac{1}{2}Kx_4^2(\eta)-\frac{1}{2}K_{cp}^{A}\{x_1(\eta)-x_4(\eta)\}^2-\frac{1}{2}K_{cp}^{B}\{x_3(\eta)-x_4(\eta)\}^2-\mu(x_1^2-1)\dot{x}_1(t)x_1(\eta)\\
&-\mu(x_2^2-1)\dot{x}_2(t)x_2(\eta)-\mu(x_3^2-1)\dot{x}_3(t)x_3(\eta)-\mu(x_4^2-1)\dot{x}_4(t)x_4(\eta)-R_{cp}\{\dot{x}_2(t)-\dot{x}_1(t)\}\\
&\{x_2(t)-x_1(t)\}-R_{cp}\{\dot{x}_3(t)-\dot{x}_2(t)\}\{x_3(t)-x_2(t)\}\\
&-R_{cp}\{\dot{x}_4(t)-\dot{x}_3(t)\}\{x_4(t)-x_3(t)\}-R_{cp}\{\dot{x}_1(t)-\dot{x}_2(t)\}\{x_1(t)-x_2(t)\}
\end{aligned} \tag{18}$$

Now through Eq. (6), one may finally obtains Poincaré-Cartan integral invariant similar to Eq. (11) as coupling parameter does not appear in the mathematical expression of the integral invariant. One may visualize the parametric variation of coupling and injecting signal parameter to obtain the significant dynamics, which may be explained in next subsections.

4.1 Beating Phenomena

When the value of injecting signal ΔK is very small, $\Delta K < 0.1$, the four oscillators are in phase to some extent as observed in Fig. 8.

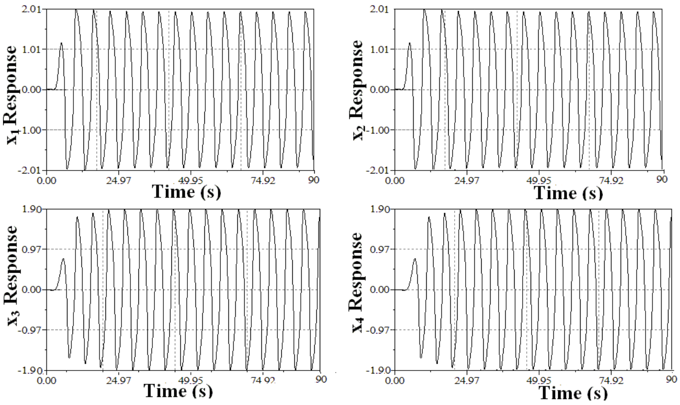

Fig.8: In phase synchronization with local injected signal at

$$\mu = 0.426, R_{cp} = 0, K_{cp}^{A} = -3.16, K_{cp}^{B} = 1.208, \Delta K = 0.08$$

However, at the larger value of ΔK, $\Delta K > 0.1$, in phase mode of vibration vanishes and amplitude of each oscillators beats while the general motion of the response remains similar to in-phase mode. This phenomena is beating phenomena as shown in Fig.9 at $\Delta K = 0.396$. This beating phenomenon may be observed for a bigger time window also.

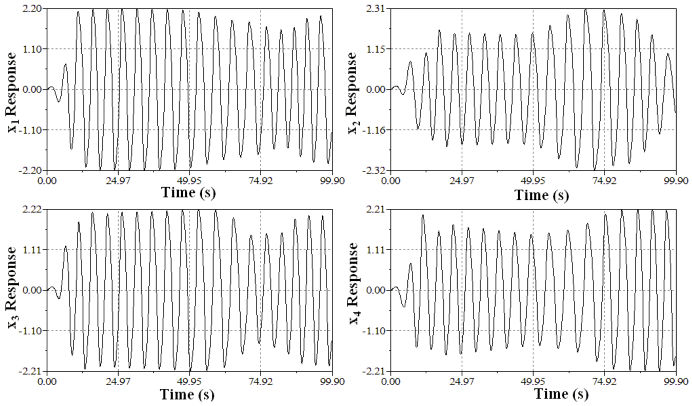

Fig.9: Beating Phenomenon with local injected signal at

$$\mu = 0.426, R_{cp} = 0, K_{cp}^{A} = -3.16, K_{cp}^{B} = 1.208, \Delta K = 0.396$$

4.2 Chaotic behaviour

Chaotic behaviour is also obtained in the system of coupled oscillators, when the strength of local injection signal is enhanced. In this case, damping effect is not taken into consideration because it has a very limited role in determining the chaotic behaviour. As shown in Fig. 10, the phase plot of first oscillator is showing the chaotic behaviour for the value of $\mu = 1.6, R = 0, K_{cp}^{A} = -.172, K_{cp}^{B} = 0.1728, \Delta K = .761.$

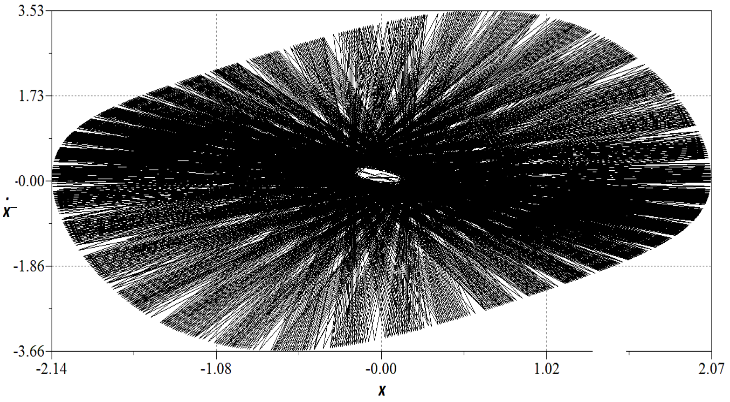

Fig.10: Phase plot of first biological oscillators with $\Delta K = 0.761$

The chaotic behaviour may also be determined by showing the Poincaré maps of the all oscillators as shown in Fig.11, which shows that the chaos is dependent on initial conditions of the oscillators and this sensitive dependency on initial conditions has been verified by computational methods.

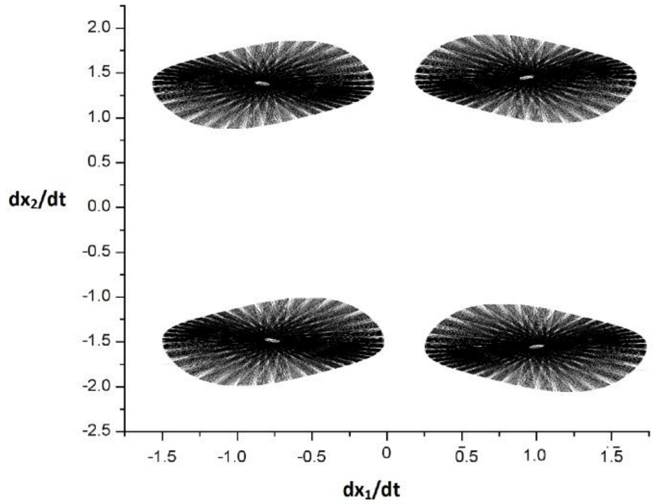

Fig. 11: Projections of second Poincaré map for the section $x_1=0$, and $x_2=0$ for the first oscillators with $\Delta K = 0.761$

The chaotic behaviour may also be shown, when the time step used is reduced at the order of 10^{-1} to 10^{-2}. The Poincaré session contains a series of points by which the periodicity is entrained. But as soon as the pattern vanishes and leads to a structure, which is known as torus. In this case, the overall periodicity of the oscillator is totally disappeared and a torus is achieved as shown in Fig.12.

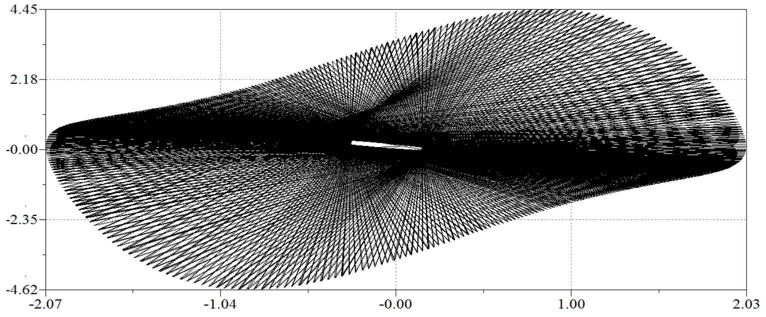

Fig. 12: Torus structure for one oscillators, when simulation time t>10^5 with $\Delta K = 0.761$

The chaotic behaviour may also be visualized by Lorenz maps, which are drawn for all the oscillators as shown in Fig. 13. The trajectory leaves one spiral only after exceeding some critical distance from the centre point. In this case, the intensity of injecting signal is kept at 0.761.

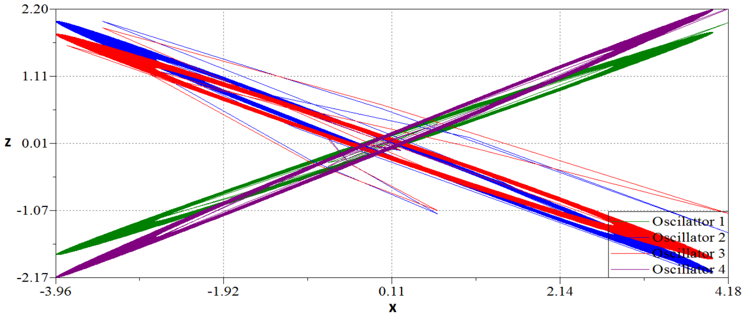

Fig. 13: Phase plot for all the oscillators, when simulation time t>10^5 with $\Delta K = 0.761$

5. Conclusions

In this paper, the analysis of mutually coupled oscillators has been obtained theoretically and numerically. The extended Poincaré-Cartan theorem has been applied in theoretical analysis. It has been shown that in absence of any kind of non-trivial coupling, oscillators oscillate in stable and independent limit cycle. However, presence of this coupling, the oscillators interact resulting in trajectories in 12-dimensional phase spaces, which has been supported by the mathematical formulation of integral invariant. Beside this theoretical analysis, full and cluster synchronization of biological oscillator are also discussed, which has been supported by simulation study of the system. These results relate to phenomena occurring in physiological experiments, such as the periodic stimulation of neural and cardiac cells and in the non-regular function of organs and organism. The effect of local injected signal has been presented, which shows the beating phenomena and chaotic behaviour of the oscillators. The phenomena of stability and bifurcation may also be deal in near future as a separate case to show the chaos control of oscillators.

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