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Analysis, simulation and composition using nonlinear techniques

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Abstract. We are interested in the use of prediction algorithms of nonlinear time series analysis in the context of music. In particular, our target is to use prediction algorithms to simulate a musical style, to mimic a particular musical composition or to produce new music. Moreover, we introduce and discuss the use of some algorithms that may improve the quality of predictions both for simulating styles and composing.
Keywords: Nonlinear analysis; Musical Analysis;Time series; Predictions;.

1 Introduction

A large variety of phenomena has been studied using methods of Nonlinear time series analysis (NTSA) (see [4]) and it seems natural to consider a musical composition as a data set and to try to reconstruct and study the underlying dynamical systems. Moreover, it makes sense to study the degree of predictability (or chaoticity, see [2]) of a musical compositions and this can be done with the aid of some indicators such as *Correlation Dimension* and *Lyapunov Exponents*. It is also natural to ask if these tools are useful for cataloguing musical styles (see [2]). The most important goal of NTSA is to make predictions on the future behavior of the system using the available data set. In section 2 below we introduce a prediction algorithm in the context of music. The use of such algorithm was suggested in [3] with the purpose to produce original music for both artistic and pedagogical purposes. We underline that this topic would benefit of further research from the point of view of musical composition and theory. The main original contribution of section 2 is the introduction of multivariate prediction in music, that is, prediction for multidimensional time series (pitches and durations), and a detailed analysis and study of prediction

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algorithms. The main objects of NTSA, the m-histories (m-hs), reveal to be an important tool in developing a "nonlinear musical analysis", since they are capable of describing not only the geometry of patterns but also the connection between them. Section 3 is devoted to this topic, we suggest different tools for developing a NTSA in music and compare their efficiency. We remark that the idea is not to substitute classical musical analysis such as Thematic, Motivic or Shaenkerian analysis, but to support it by nonlinear methods. The interest in making predictions is obvious in the context of natural sciences, while, in the context of music, we propose the idea to use prediction in order to simulate a musical style. To do that it seems not sufficient to reproduce the "qualitative behavior of the system": (that is the intrinsic geometric of the patterns and their connections), we sometimes need some corrections and improvements for the predictions. Section 4 is dedicated to the introduction of several examples of the modified algorithm that improve the quality of the predictions, while in the last section we consider some final remarks.

2 Prediction of musical series

An easy way to predict the future behavior of a data set is to consider the so called Lorenz method of analogues (see [4]). Let x_i with $i = 1, \dots N$ be the elements of the time series and suppose that there exists a continuous function F such that $x_{n+1} = F(x_n)$, then it is possible to predict, within certain error, x_{N+1} from the data set. The idea is to consider the point x_{n_0} in the time series that is the closest to x_N with respect to some predefined norm. Then from the continuity of F we have:

$$x_{n_0} \simeq x_N \qquad \Longrightarrow \qquad x_{n_0+1} \simeq x_{N+1}, \tag{1}$$

in the sense that under the continuity condition, the two points should evolve in a similar manner. If N is large enough, then usually there is a set of points satisfying the criteria of being close to x_N for a given radius of tolerance and it is possible to consider some sort of mean value of their evolution. We will use the following algorithm (see [1]):

$$y^m_{t+1} = \sum_{k=1}^{N} \left[\hat{y}^m_{k+1} - \hat{y}^m_k + y^m_t \right] \omega_k(y^m_t, \hat{y}^m_k), \tag{2}$$

where y^m_{t+1} is the $m - hs$ that we want to predict, y^m_t is the last $m - hs$ of our data set. The points $\{\hat{y}^m_k\}^N_{k=1}$, are the elements of the time series that lie in the neighborhood $B_\varepsilon(y^m_t)$ of the vector y^m_t with a given radius ε. We denote by $\hat{y}^m_{k+1}$ the points in the time series that are next to the neighbors $\hat{y}^m_k$. The parameter ε needs to be chosen in such a way that $B_\varepsilon(y^m_t)$ contains enough elements. The function $\omega_k(\cdot)$ are weight functions and satisfy

$$\omega_k(a,b) = \frac{K_h(\|a-b\|)}{\sum_{h=1}^{N} K_h(\|a-b\|)}, \quad K_h(x) = \frac{1}{h} K\left(\frac{x}{h}\right), \quad K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \tag{3}$$

where h is a positive parameter representing the bandwidth of the kernel. Notice that the prediction algorithm described produces as an outcome, vectors of m entries, then a reconstruction process is needed (see [1]).
Now let's consider a first example of prediction. For simplicity we chose the Prelude of the I suite for Cello by J.S. Bach, since the rhythm of the tones is almost the same (a sixteenth note). In order to construct the time series we associate to each pitch a natural number in a chromatic way. So we predict only the tones and put the values of sixteenth to each tone. In figure 1 we present bars 17-20 of the original Bach composition while in figure 2 and 3 we show the prediction of the bars 19-20 for the selection of $\varepsilon = 30$ and $h = 0.5$ and $h = 1$ respectively. After an initial chromatic adjustment, we observe

Fig. 1. Original Bach Music bars 17-20.

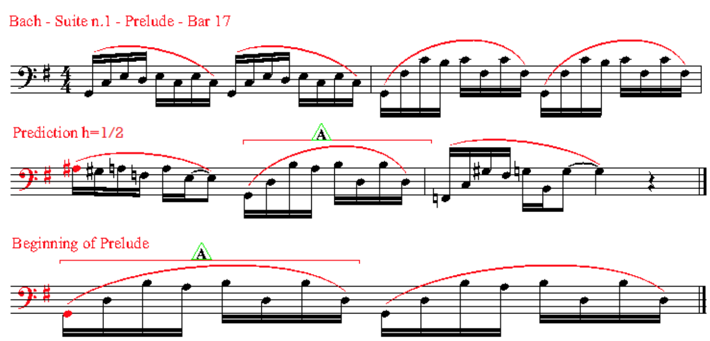

Fig. 2. In the first pentagram we represent bars 17-18 of the original composition while in the second the predicted bars 19 and 20 (using $\varepsilon = 30$ and $h = \frac{1}{2}$). Finally, in third pentagram we report the first bar of the original composition in order to compare Pattern A.

the presence of the first pattern of the suite (for $h = \frac{1}{2}$) and of the first pattern (with a slight modification) of bar 18 (for $h = 1$). For these reasons in both cases we decided to put a rest point after the chromatic pattern to let the following patterns start in the right time subdivision. Actually, the original composition contains a similar rest point in bar 22. We observe that except for

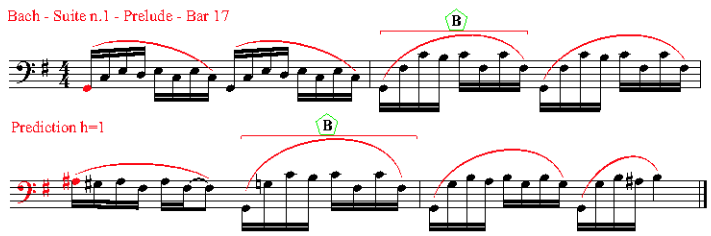

Fig. 3. Bars 17-18 of the original composition and the predicted bars 19 and 20, using $\varepsilon = 30$ and $h = 1$.

the initial chromatic pattern the prediction algorithm is capable of producing the right tonality of G major.

In the case of more complex musical compositions we need to modify the strategy and consider "multivariate" prediction. The time series will be two-dimensional, the first component representing the pitch while the second component represents the duration. In particular, we represent the rhythm by considering the value of a sixteenth note as the unit value (see figure 4):

$$\frac{1}{16} \to 1, \qquad \frac{1}{8} \to 2, \qquad \frac{1}{4} \to 4 \ldots. \tag{4}$$

We consider a two-dimensional time series: $x_t = (x_t^1, x_t^2)$, $t = 1, \ldots N$, where

Fig. 4. The way to associate a natural number to the rhythmic figures.

$\{x_t^1, \quad t = 1, \ldots N\}$ and $\{x_t^2, \quad t = 1, \ldots N\}$ are the one-dimensional time series of the pitches and of the durations respectively. For each time series we compute the embedding dimension being m_1 and m_2 respectively, we predict $(m_1 + m_2)-$dimensional vectors and again a reconstruction methods is needed (see [1] for details).

As an example, we apply the algorithm to the Prelude of the II Suite which contains many rhythmic patterns. We consider the first 47 bars which corresponds to an homogeneous part ending at bar 48 with a rest point. Using the method of the false neighbors (see [2]) we get the values of the embedding $m_1 = 7$ and $m_2 = 3$ of the series of pitches and duration respectively. We observe that the prediction (see fig 5) respects the tonality of the composition and the chromatic tones of the prediction, are also present in the original composition.

In order to check wether the algorithm is capable of simulating the original rhythmic patterns we start by representing in figure 6 the four rhythmic patterns of the Prelude (until bar 47) and we denoted them by the letters A, B, C

Fig. 5. Prediction of the Prelude of the II Suite for Cello by J.S. Bach. Parameters: $\varepsilon = 30$ and $h = 0.125$.

Fig. 6. The four rhythmic patterns which appear in the Prelude until bar 47.

and D respectively. We observe that patterns A and B appear 6 times, pattern C appears 115 times and pattern D appears 8 times. The prediction algorithm is capable of producing 3 of the 4 different rhythmic patterns. We note that, the missing pattern B can be written as the union of two patterns: a quarter tone pattern, denoted by E, and pattern C. This suggest that the prelude is completely unbalanced towards the pattern C and for this reason it is difficult to predict the pattern B (that is, the half pattern E).

3 Analysis and Simulation of Musical Styles

The idea considered in this section consists in using the prediction algorithms to simulate a musical style or to mimic a particular musical composition. Notice that it is not our objective to ask how close is the prediction to the prosecution of the original composition. Our objective here is to see how close are the two compositional styles. We consider the first 18 bars of the Prelude of the I Suite and apply the prediction multivariate algorithm discussed in the previous section. We recall that the embedding dimension for the series of pitch is 7 while for the series of the duration is 1.

3.1 Analysis of the predictions

We start with a simple pattern analysis of the predictions obtained by setting the parameters as in 7. All the predictions (see figure 7) share an initial 6-notes chromatic pattern which is of the two types: C1 (for predictions 1-6) and C2 (for predictions 7-10). Most of the patterns belong to two families: A and B (see figure 7). We indicate with apexes, stars and circles the modifications of the basic patterns A and B. Moreover the predictions contain also patterns

Fig. 7. Comparison of the multivariate predictions of Prelude of Suite I for $\varepsilon = 30$, $h_k = 0.125k$ and $k = 1, \ldots, 10$.

D, E, F and G. All these patterns (with exception of C1, C2 and of G which is a chromatic variation of A) appear in first 18 bars of the Prelude (see 2). In particular: pattern A' appears two times in bar 3 and two times in bar 18; patterns B and B' appear in bar 4; patterns D and E appear in bar 5; and finally pattern F appears two times in bar 6. In the following table we consider the percentage of patterns of each prediction which belong to the original composition (without considering the first chromatic pattern):

Prediction	1	2	3	4	5	6	7	8	9	10
Percentage	4/5	0/5	0/5	4/5	4/5	4/5	4/5	4/5	0/5	3/5

Moreover, in table 1 we give a detailed description of each prediction in terms of patterns (see also figure 7). Prediction 1,4,9,10 are very similar: they contains only variations of patterns A and B. They are homogeneous prediction and contains only tones from the original tonality of G Major (except prediction 10 which contains a chromaticism and the alternation of pattern A' and A"). Predictions 2 and 3 present the repetitions of pattern A raised up a semitone (in the tonality of $A\flat$). Prediction 5 and 6 are more heterogeneous with the first one containing an extra sound $C\sharp 2$ (two times). Prediction 7 and 8 contain two chromatic patterns (C2 and G). Prediction 7 continues with the repetition of pattern A' (1 note of difference with A) while pattern 8 continues with pattern A' and the repetition of pattern B. From a musical point of view the best predictions correspond to the values of h between $h = 0.5$ and $h = 0.75$. The prediction obtained with $h = 0.5$ is more homogeneous, while the ones corresponding to $h = 0.625$ and $h = 0.75$ present more variations. We observe that predictions corresponding to $h = 0.25$ and $h = 0.375$ are homogeneous, they present the pattern A raised one semitone (then the prediction appear in the "wrong tonality" of $A\flat$). The tonality of the prelude is G Major and the first 18 bars contain the following chromatic tones: $A\flat$, $C\sharp$, $E\flat$ and $F\natural$. The only missing sound is $B\flat$ and it appears in the predictions. In table 2 we describe which and how many chromatic tones appear in each prediction.

Prediction	Patterns					
1	C1	A	B	B	B	B
2	C1	A^*	A^o	A^o	A^o	A^o
3	C1	A^*	A^o	A^o	A^o	A^o
4	C1	A	B'	B'	B'	B'
5	C1	A	B	D	E	F
6	C1	A	B'	B	D	E
7	C2	G	A'	A'	A'	A'
8	C2	G	A'	B	B	D
9	C2	B"	B"	B"	B"	B"
10	C2	A'	A"	A'	A"	A'

Table 1. Patterns analysis for the 10 predictions of figure 7. In the last column we represent a portion of the pattern.

Prediction	**A$\flat$**	**C$\sharp$**	**E$\flat$**	**F$\natural$**	**B$\flat$**	**Total**
1	1	0	0	0	3	4
2	5	13	0	0	3	21
3	5	12	0	0	4	21
4	1	0	0	0	3	4
5	1	2	0	0	3	7
6	1	0	0	0	3	4
7	0	0	0	2	2	4
8	0	0	0	2	2	4
9	0	0	0	1	1	2
10	0	0	0	5	1	6

Table 2. Chromatic sounds in the Predictions.

3.2 Simulation of a musical style

In order to see if predictions preserve the styles of the original composition we introduce two mathematical tools. Instead of considering 8 dimensional vectors, as usual in patterns analysis, we consider $8 - hs$. The first method we consider is qualitative and tries to analyze if the style of the composition is preserved by the prediction. We transform the $8 - hs$ of the original series (until bar 18) into fundamental patterns (indicated by FP8). Given an $m - hs$, $\mathbf{x} = (x_1, \ldots, x_8)$, let $\alpha(\mathbf{x})$ denote the minimum of the values of x, then we define its fundamental pattern as

$$\mathbf{x}_{FP8} = (x_1 - \alpha(\mathbf{x}) + 1, \ldots, x_8 - \alpha(\mathbf{x}) + 1). \tag{5}$$

In this way at least one entry of every fundamental pattern contains the number 1. For example:

$$(4, 5, 9, 13, 12, 7, 11, 19)_8 \quad \rightarrow \quad (1, 2, 6, 10, 9, 4, 8, 16)_{FP8}. \tag{6}$$

This method let us perform a pattern analysis, without considering transposition or change of tonality, by only comparing their geometry. In the following table we report how many fundamental patterns of the prediction do not belong to the set of fundamental patterns of the original composition. We note that predictions 2-8 completely preserve the geometry of the original composition.

h	0,125	0,25	0,375	0,5	0,625	0,75	0,875	1	1.125	1,25
FP	5	0	0	0	0	0	0	1	8	8

A more "accurate" method consists in considering $8-hs$ $h=(h_1,\ldots,h_8)$ instead of fundamental patterns. Then we compute the following distance

$$d(h,V):=\min_{v\in V} d(h,v), \tag{7}$$

where V is the set of all $8-hs$ of Prelude of Suite I (until bar 18) and the distance $d(h,v)$ between two $m-hs$, h and v is defined as:

$$d(h,v)=|h_1-v_1|+|h_2-v_2|+\ldots+|h_8-v_8|. \tag{8}$$

Denote by D, the sum all the distances for any predicted $8-hs$ h:

$$D=\sum_{h\in H} d(h,V), \tag{9}$$

where H is the set of $8-hs$ of the prediction. We will compare the values of the parameter D in terms of different values of h. We already know how many patterns in the predictions belong to the original composition but this calculation can also evaluate how patterns are connected between them (since we are considering $8-hs$ instead of just $8-$note patterns). In the following table we present the computation of the distance D between the prediction and the original series till bar 18.

h	0,125	0,25	0,375	0,5	0,625	0,75	0,875	1	1.125	1,25
D	759	780	780	761	710	710	778	766	752	785

This method allows us to evaluate the prediction globally and as a consequence the best results are obtained by prediction 5 and 6 that, after the chromatic pattern, present 5 different patterns. The heterogeneity makes the prediction closer to a standard musical composition which, in general, does not consists in the repetition of one or two patterns. For this reason higher values of D are attained by the more homogeneous prediction (compare the above table with 1). This is in full accordance to what is suggested by musical analysis and we consider that this is the proper way to evaluate the "quality" of a prediction. In fact, if we want to analyze in a "non-reductionistic way" the character of a classical musical composition, we need not only a pattern (or sub pattern) analysis but we also need to know how patterns are connected.

4 Improvement of predictions algorithms

In this section we propose some algorithms and basic ideas in order to improve the quality of predictions for simulating musical styles. Suppose we have a multivariate prediction of two variables (pitch and duration) of the following form $(p_1, d_1), (p_2, d_2), \ldots, (p_N, d_N)$, where $\{p_i : \quad i = 1, \ldots, N\}$ is the 1-dimensional series of the pitches while $\{d_i : \quad i = 1, \ldots, N\}$ denotes the series of the durations, with embedding dimensions m_1 and m_2 respectively. Having in mind the Preludes of Suite I and II as main examples, we present two different algorithms.

Simple Case: Prelude of Suite I

We start by considering only the time series of the pitches. We compare the predicted $m_1 - hs$ with the $m_1 - hs$ contained in the original composition (until bar 18). The algorithm looks for similarities between the $m_1 - hs$ belonging to the two sets and, when a match has been found then, it looks at the series of the durations. The algorithm modifies the predicted durations $\{d_i : \quad i = 1, \ldots, N\}$ if they differ "too much" from the original ones. In details : consider a $m_1 - h$ of the prediction $p = (p_{k+1}, \ldots, p_{k+m_1})$, and look for the nearest $m_1 - h$ of the original composition with respect to the following distance:

$$d(u, v) = |u_1 - v_1| + \ldots + |u_{m_1} - v_{m_1}|. \tag{10}$$

Suppose that the nearest $m_1 - h$ is $h = (h_{l+1}, \ldots, h_{l+m_1})$. Then, for a fixed radius of tolerance $\mathcal{R}$ previously chosen, we have:

CASE 1: $d(p, h) > \mathcal{R}$. The two $m_1 - hs$ are too far and we cannot conclude that they correspond to the same pattern. The algorithm passes to analyze the next $m_1 - h$.

CASE 2: $d(p, h) \leq \mathcal{R}$. The two $m_1 - hs$ correspond to the same pattern. Then, the algorithm checks if the two $m_1 - hs$ start at the same subdivision. If not, we add a proper value to d_k in order to let p start at the same subdivision of h. Then the algorithm starts to analyze the next $m_1 - h$. In the examples that we consider, that is, the Preludes of Suite I and II, the bars contains 4 and 3 movements respectively. Each movement is made up of 4 subdivisions of $\frac{1}{16}$ notes. Each bar of the preludes has 16 and 12 subdivisions respectively, in figure 8 below we represent the case of the Prelude of Suite I.

Fig. 8. The 16 Subdivisions of the 4 movements of the bars of the Prelude of Suite I.

In details: let $\{t_i : \; i = 1, \ldots, \tilde{N}\}$ be the original series of durations, then

$$Sub(h) := \left(\sum_{i=1}^{l} t_i \right) + 1 \qquad (Mod\ 4), \tag{11}$$

represents the Starting Subdivision of the $m_1 - h$, $h = (h_{l+1}, \ldots, h_{l+m_1})$. We make the same computation for the predicted $m_1 - h$, $p = (p_{k+1}, \ldots, p_{k+m_1})$ with the corresponding predicted series of durations $\{d_i : \quad i = 1, \ldots, N\}$:

$$Sub(p) := \left(\sum_{i=1}^{k} d_i \right) + 1 \qquad (Mod\ 4), \tag{12}$$

and compare the two results. If the starting subdivision of p is not the same as that of h then we change d_k in order to obtain the same subdivision. That is, we modify the k^{th} element of $\{d_i : \quad i = 1, \ldots, N\}$ in the following way:

$$\tilde{d}_k = d_k + |Sub(h) - Sub(p)|. \tag{13}$$

In the following examples, we decided to consider $8 - hs$ of the tones due to the shape of the patterns which are mainly made up of 8 sixteenth tones. In figures 9 and 10 we represent some examples, corresponding to $h = 0.625$ and $h = 1$ respectively, with values of $\mathcal{R}$ as indicated. We observe that for $h = 0.625$

Fig. 9. Modification of prediction with $h = 0.625$ and $\mathcal{R}$ as indicated.

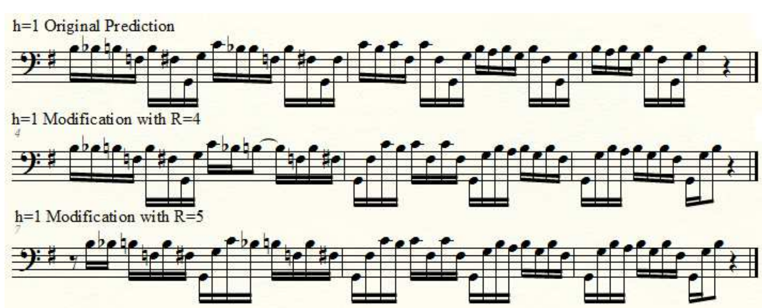

Fig. 10. Modification of prediction with $h = 0.625$ and $\mathcal{R}$ as indicated.

the choice $\mathcal{R} = 3, 4$ let all patterns start in the right subdivision (see figure 1).

This is obtained by the modification of pattern $C1$. The difference between $\mathcal{R} = 3$ and $\mathcal{R} = 4$ is that in the latter case the pattern $C1$ is modified in a way that his second half is made similar to the pattern of the A family, while for $\mathcal{R} = 3$ the algorithm does not recognize this similarity. In the case $h = 1$ and $\mathcal{R} = 4$ the algorithm modifies the pattern G (see figure 7) . For $\mathcal{R} = 5$ the pattern $C2$ is modified by putting a rest before it, as a consequence all the patterns start in the right subdivision and this results to be the best choice. In both cases $h = 0.625$ and $h = 1$ the highest value of $\mathcal{R}$ corresponds to the best modification of the prediction.

Prelude of Suite II and more general cases

For the Prelude of Suite II (and for more general situations) we will use a slightly different procedure since we need to consider also the durations in the computation. We consider again subdivision made up of sixteenth tones, each one of the three movements is made up of 4 subdivisions. Suppose that the two dimensional time series has e embedding dimensions m_1 (pitches) and m_2 (durations) respectively and consider a multivariate prediction $\{(p_k, d_k) : k = 1, \dots N\}$, where $\{p_k\}_k$ and $\{d_k\}_k$ are the series of predicted pitches and durations respectively. To each $m_1 - h$ of $\{p_k\}_k$ we associate a $m_1 - h$ from $\{d_k\}_k$ obtaining a set of $2m_1 - hs$:

$$q_{k+1} = (p_{k+1}, \dots, p_{k+m_1}, d_{k+1}, \dots, d_{k+m_1}) := (P_{k+1}, D_{k+1}). \tag{14}$$

We consider an analogous arrangement for the two dimensional series of the original composition:

$$w_{s+1} = (n_{s+1}, \dots, n_{s+m_1}, o_{s+1}, \dots, o_{s+m_1}) := (N_{s+1}, O_{s+1}), \tag{15}$$

where $\{n_s\}_s$ and $\{o_s\}_s$ are the original series of pitches and durations respectively. We note that we are not using the embedding dimension m_2. Then we have constructed two sets of $2m_1$-dimensional vectors:

$$Q := \{(P_k, D_k) : \quad k = 1, \dots N\}, \qquad W := \{(N_k, O_k) : \quad k = 1, \dots \tilde{N}\}. \tag{16}$$

We start the procedure with the first element $q_1 = (P_1, D_1) \in Q$ and we look for the nearest $w = (N, O) \in W$ according to the following procedure.

1. We compute the distance between D_1 and all O with $(N, O) \in W$ and consider the elements that minimizes the following distance:

$$dd_2(D_1, O) = |d_1 - o_1| + \dots + |d_{m_1} - o_{m_1}|. \tag{17}$$

2. Among the minimizers $w \in W$, we select the one that minimizes:

$$dd_1(P_1, N) = |p_1 - n_1| + \dots + |p_{m_1} - n_{m_1}|. \tag{18}$$

   If there is more than one minimizer, then we choose the first in the order.

Then we have associated to $q_1 = (P_1, D_1) \in Q$ the nearest element $w = (N, O) \in W$. We fix the value of two positive parameters $\mathcal{R}_1$ and $\mathcal{R}_2$. Then, if

$$dd_1(P_1, N) < \mathcal{R}_1, \qquad \textit{and} \qquad dd_2(D_1, O) < \mathcal{R}_2, \tag{19}$$

we conclude that q_1 and w represent the same patterns. Now we pass to compare their starting subdivision and possibly change that of q_1 as we have explained in the previous subsection. If the above conditions are not verified, the algorithm passes to analyze the next element q_2.
We tested this algorithm for the prediction with $h = 0.25$ (see figure 11). This prediction presents a prevalence of pattern C and only one pattern D.

Fig. 11. The prediction of the Prelude of Suite II for $h = 0.25$.

Moreover, in bar 6 there is the beginning of a sequence of pattern A starting at the wrong subdivision. Then, to summarize, the quality of the prediction is affected by two main problems: lack of Pattern B, misplacement of pattern A (in several bars). We tested the utility of the algorithm for $\mathcal{R}_1 = 0, 1, 2,$ and $\mathcal{R}_2 = 0, 1, 2, 3, 4, 5$. We first observe that for the following values of the parameters, the algorithm does not change anything:

$$\begin{array}{lll} (\mathcal{R}_1, \mathcal{R}_2) = (a, b), & a \in \{0, 1\}, & b \in \{0, 1, 2, 3, 4, 5\}, \\ (\mathcal{R}_1, \mathcal{R}_2) = (a, b), & a \in \{3, 4\}, & b \in \{0, 1, 2, 3, 4\}, \\ (\mathcal{R}_1, \mathcal{R}_2) = (5, b), & b \in \{0, 1, 2, 3, 4\}. & \end{array}$$

For the remaining values we obtain only three different modifications:
Modification 1 $\mathcal{R}_1 = 2$ and $\mathcal{R}_2 \in \{0, 1, 2, 3, 4, 5\}$. The algorithm modifies the fifth tone of the fifth bar by adding a value of $\frac{7}{16}$. This gives rise to a rhythmic configuration, that is a 2/4 tone, which is not present in the original composition. However this change makes the patterns of bar 6-8 starting at the right subdivision. We remark that this change create a “connection pattern” that we regard as CA, made up of half pattern C and half pattern A, which starts the sequence of A patterns at bar 6. Moreover, a value of 1/16 is added to the last tone of the last bar giving rise to a rhythmic configuration, that we called E, that is present in the original composition since it is the first half of pattern B. We remark that neither pattern B nor pattern E are present in the original prediction. Finally, the algorithm improved the prediction, all the patterns A,C,D (and also E) start at the right subdivision.
Modification 2 $\mathcal{R}_1 = 3, 4$, $\mathcal{R}_2 = 5$. The algorithm modifies the first bar by adding a value of 1/4 to the fifth tone. This gives rise to the pattern B which is in the original composition and it is not in the original prediction. Also in this case, the fifth tone of the fifth bar is modified by adding the value of 3/16, the produced pattern (E) belongs to the original composition. This change makes

all patterns of bars 6-8 start at the right subdivision. Pattern CA is created as a connection for the sequence of patterns A. Finally, the value of 1/16 is added to the last tone of the last bar producing pattern E.To sum up, in this case the algorithm not only made the patterns start at the right subdivision but was able to produce the missing pattern B.

Modification 3 $\mathcal{R}_1 = \mathcal{R}_2 = 5$. The algorithm produces a lot of changes and introduce patterns that are not present in the original composition.

In figure 12 we present the three modified predictions. As we have observed

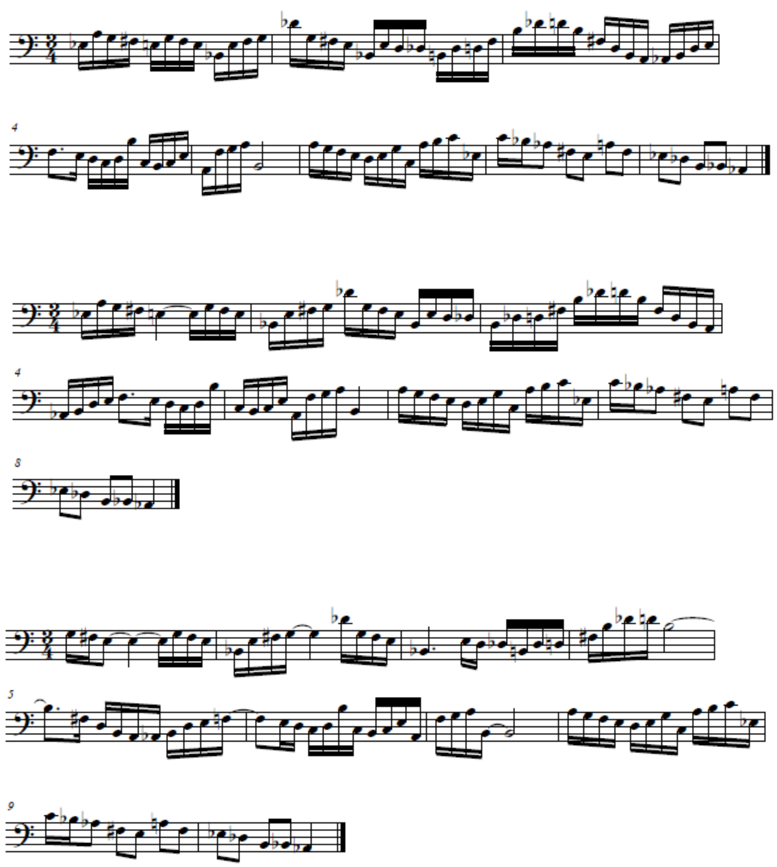

Fig. 12. The three modified predictions of the Prelude of Suite II for $h = 0.25$ and$(\mathcal{R}_1, \mathcal{R}_2) = (2, 0), (3, 5), (5, 5)$ respectively.

above, the second modification is the best choice. This conclusion is supported by the following table (second column), where we compare the distance of the original duration series with that of original and modified predictions. If we consider the normalized value, (by subtracting the values added by the algorithm from the distance) we obtain a better understanding of how the algorithm works (third column).

	$\mathbf{dd_2}$	**Normalized** $\mathbf{dd_2}$
Original prediction	31	
Modification 1	38	38-7-1=30
Modification 2	38	38-4-3-1=30
Modification 3	68	68-43=25

We observe that the first way of computing is useful if we want to obtain an improvement with as few changes as possible: if the values of dd_2 have a small increase we obtain a better prediction. If we look at the third column of the previous table, we obtain that we can minimize the normalized distance but paying a lot in terms of number of changes. This could lead to the introduction of many patterns not present in the original composition (as for the 3rd improvement in 12).

5 Conclusion

All the results presented here must be considered as a first step of a more complete work. We are aware that it is not easy to develop a theory of nonlinear studies in music by using NTSA tools. The results presented here in [2] and [3] suggest a possible strategy. Further work is needed for cataloguing purposes and also in musical analysis, here we have just used basic techniques that can be implemented or supported by more sophisticated ones. In particular $m - hs$ have revealed to be more suitable objects for analysis purposes than simple patterns. The use of prediction algorithms to simulate musical styles has been presented by introducing some corrections algorithm, of course another approach would be possible, for example by modifying directly the prediction algorithms by introducing some other constraints or rules to be satisfied. We remark that prediction algorithms could have a very profitable use in composing new music (see [3]). Further investigations of this topic could be of interest for musical theoreticians, musicians and artists.

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