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Stability in Development of Complex Systems with Time Shifts and their Optimal Control

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Abstract: The stability and critical levels in development of the complex systems of different nature with time shifts are studied. Mathematical modeling and analysis is presented for revealing and investigation of the critical levels in systems for various natures associated with diverse complicated factors, in particular with shifted arguments. Intensive research in this direction may optimize management of the complex systems in financial-economic, natural and other fields. Construction of adequate mathematical models for development of complex systems, critical modes and their effective control are shown in paper on examples.
Keywords: Model, Development equation, Time shifts, Critical levels, Stability, Complex system, Optimal control.

1 Introduction

Development of the complex systems of various natures associated with diverse complex factors, in particular with the shifted arguments in the systems, have revealed critical processes and regimes in the contemporary area of research in this field during the last few decades [1-17]. The techniques to optimize the control of those systems, for example financial-economic, natural and others have been developed by Zhirmunsky & Kuzmin, Allen, Kazachkov, etc. [1, 2, 6, 9, 12, 16, 17]. The systems' development and their harmonization were in focus of many philosophers and scientists from different fields of science since ancient times, e.g. the Pythagoras adherents believed that planet motion is governed by the same numbers as the harmony of spheres (Fulier) [8], rhythmic unity of processes at diverse organizational levels were considered by ancient China and India (Sima Xjan [18], Mahabharata [19]).

Regularities of the critical levels, the Napierian number *e* as a module of the geometric progression, issues related to general problems of systems' development in different branches of science and practical activities were considered in a number of books and papers, e.g. the ones by Zhirmunsky & Kuzmin (1990) [1], where a historical aspect was laid out together with the results of their own researches. The effective research method for a number of different developing systems has been elaborated and many examples of diverse origins, nature and scales were analyzed in detail and amazing features revealed

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from just simple analysis of the features of equations, without their solution. Development and investigation of the physical and mathematical models for complex systems with shifted arguments and critical modes, their effective control are important for a wide range of modern problems, which are of paramount interest. Critical levels in a development of economic, banking, industry, technical, political and other systems, which may cause instability of the system and destroy a stable development of the system due to growing oscillations in a system, must be revealed and accounted in the strategic and tactical planning.

The problem stated is vital for study, mathematical modeling and simulation. The theory of nonlinear dynamical systems with shifted (deviated) arguments provides a powerful mathematical tool for the study of complex systems and determination of critical levels in their development. Properties of many real objects essentially depend on the after-effects due to which their behavior in the next moment of time depends on the previous history of development, and not only on the current state of the object. The simplest cases of such systems have been studied in the theory of functional differential equations with shifted arguments (delay and forecast terms by time): [2, 3, 6]. Then during the recent times: [4, 5, 9-15]. It should be noted that real objects are more complicated and the mathematical models describing them, even with simplification described by systems of differential equations, contain the arguments depending on many deviating arguments, which, moreover, can themselves be time-dependent and to be linked: [1, 9, 16, 17].

2 Statement by Mathematical Modeling of Complex Systems

Over several decades the fundamental results in the theory of dynamical systems with delayed and forecasting arguments formed the theory of differential equations with shifted arguments applied during the last 30 years to modeling of complex systems from a wide range of science, technology, wildlife, economics, and the like. Development of numerical algorithms and their application to problem's solution devoted a lot of effort [20-26]. But almost no attention was given to the equations with shifted arguments. Only Baker, et al. [3] and Yan [26] provided classification task with forecasting arguments. In relation to nonlinear dynamical systems with delayed and forecasting arguments, they were considered for modeling of potentially hazardous objects of nuclear energy [4, 9-12, 17]. Also in some papers the dynamics of populations' crashing in biological systems and high voltage power lines were considered. Interestingly, in the theory of motion control with delay in time the application of necessary optimality conditions in the form of Pontryagin's maximum principle leads to the conjugate system of the equations with forecasting arguments [22].

The effective numerical methods and methods of averaging the differential, integral, as well as the integro-differential operators allowing performing mathematical simulation in a wide range of complex processes and systems are applied for solution of differential equations with delayed and forecasting arguments. In particular, modeling the dynamics of behavior of potentially hazardous industries, based on statistical information about the objects. Since such complex objects in most cases do not allow constructing the precise deterministic mathematical models due to big or even huge number of influential parameters and often unknown links between them, then the aggregate model built on statistics about the object can be useful for studying the nature and behaviors of the objects.

Development of the models for studying the evolution of complex systems (economic, political, social, banking, physical, combined by different nature and properties, etc.) in a rather general setting requires consideration of the delayed and forecasting temporal arguments. This is because development of the system is really accompanied by some delays compared to the planned indicators due to various reasons, as well as orientation for leading indicators what is known as the term "foreseen adaptation". Such phenomena have actually been observed in a number of different processes and systems in wildlife and technical origin [1].

3 Development of the mathematical models for complex systems

One of the first equation for system development was considered in 18th century by T.R. Malthus [25, 27] who was a pivotal figure in development of the empirical study of human populations, well known for his *Essay on the Principle of Population*, the central theme of which was tendency for human numbers to `outstrip the means of subsistence', based on solution of the equation

$$\frac{dy}{dt} = ky, \quad (1)$$

where y,t are the function describing evolution of some magnitude and time, respectively, k is the growth factor of the system, which generally can be function of time.

The solution of equation (1) for the simplest case with a constant growth rate k has shown that a law of population growth was seen as threatening to society and spawned the philosophical course of Malthusianism, which justified the war as a necessary mechanism for regulation of population growth. It was a big mistake based on a simplified equation of the system's growth. In fact, later on it was shown that the system can start development following the law (1), but further the coefficient k depends on time, e.g., decreases according the hyperbolic law (so called allometric process). Thus, the function's increase rate is going down with time. Since that time, there were many attempts to use this as a simple equation of development, and many others, more accurately given the characteristics of the systems under consideration.

The observation of a number of systems has shown the development processes better described in several other equations: over time, in some cases, coefficient k is being gradually or abruptly falling down, and this leads to solutions, in which there is no exponential growth function. This behavior corresponds better to the realities of different systems of diverse nature: first there is an intensive development process, which is adjusted according to the results of the development and analysis of the development's needs. In the real systems some time delays and forecasting terms are met.

The linearity of the systems' development may be broken in many cases (systems with more complex behavior), which in the example of equation (1) can be demonstrated as follows:

$$\frac{dy}{dt} = ky(A - y), \quad (2)$$

where *A* is the maximum possible value of *y* under consideration. Limit values are defined by natural or artificial means in the specific system. For example, it may be a limit of a number of population for existing conditions with respect to given power supplies, the number of workers for the industry, if the industry is considered as a complex system and the limiting possible value is known, a limited amount of financial provision in development of the bank, and the like. As seen from (2), the growth stops at the level $y=A$ upon reaching the limit.

Another kind nonlinearity of the system (1) is available due to dependence of the coefficient *k* against time and the function *y* against deviating arguments:

$$\frac{dy}{dt} = ky(t - \tau), \quad (3)$$

where τ is a value of time delay of the system. For solving the equation (3) the initial conditions on the time interval τ that precedes the starting point are needed, to specify in time or to examine the mechanism of delay (lag) only after a period of time τ. This is a significant feature of the equations with time delay, which greatly complicates solution (the numerical solution of differential equations with automatic selection of time step must approximate point values with a shift in time, which may not be available by automatic partitioning of the time interval). Equation (3) has much broader applications to modeling the development of complex systems because it accounts a possibility of time delays in the system's development relative to a current state of the system.

By positive factor *k* in the equation (3), the solution is increasing function. If *k* is negative – attenuation system (decrease of the function *y*, that is, the drop development, extinction of populations, reducing the Bank's funding, etc. - depending on the nature of the complex system that is modeled). The equations with shifted arguments have more complex modes and features, in particular, critical stages of development and the possible instabilities that rapidly lead to a destruction of the system. Such regimes are particularly important and must be thoroughly investigated. In complex real systems the type (3) can be many different and each of the plurality of interrelated system parameters may have its own delay, which significantly complicates the mathematical model of the system. The solution of (3) with the constant τ and *k* can be sought in the form:

$$y = y_0 e^{zt} = y_0 e^{ut}(cosvt + isin\, vt), \quad (4)$$

where y_0 is the initial value *y* at $t = 0$, $z = u + iv$ are the Eigen values of the differential operator, *u, v* are, respectively, the real and imaginary part of the Eigen values, $i = \sqrt{-1}$ is the imaginary unit. By substitution of (4) into (3) the equation for calculation of the Eigen values is got (after deletion by e^{zt}):

$$z = ke^{-z\tau}. \quad (5)$$

Applying the Euler's formula to (5), for the real and imaginary parts of the equation (5) yields the following equation array:

$$u = ke^{-u\tau}\cos v\tau, \;\; v = -ke^{-u\tau}\sin v\tau, \quad (6)$$

where from follows: at *v=0* two processes are possible: monotonously exponentially increasing (*u>0*) and decreasing (*u<0*) in time. Obviously the region by $v \neq 0$ is $v\tau \in \left(0 + 2\pi n, \frac{\pi}{2} + 2\pi n\right) \cup \left(\frac{3\pi}{2} + 2\pi n, 2\pi + 2\pi n\right)$, *n*=0,1,2, … for *u*>0. Because *v* is considered positive, from the second equation (6) yields the attainable region by parameter *v*: $v\tau \in (\pi + 2\pi n, 2\pi + 2\pi n)$, therefore the common region by v is $v\tau \in \left(\frac{3\pi}{2} + 2\pi n, 2\pi + 2\pi n\right)$. Thus, the oscillating solution growing in time is realized by $v\tau = \frac{3\pi}{2} + 2\pi n$, which means that it starts from $v\tau = \frac{3\pi}{2}$. For unstable regime the equation (4) results in

$$y = y_0 e^{ut} cos \frac{3\pi}{2\tau} t, \quad (7)$$

where $\omega = \frac{3\pi}{2\tau}$ is a frequency of oscillations depending on the time delay τ. Thus, the bigger is a time delay, the lower is frequency of oscillations in a system. From (7) for the time t_*, which corresponds to argument by cosine going from 0 to $\frac{\pi}{2}$, when $y = 0$ (system fall abruptly down), is $\frac{3\pi}{2\tau} t_* = \frac{\pi}{2}$, $t_* = \frac{\tau}{3}$.

The corresponding critical values of the system achieved during the stable development are estimated by (6) as follows:

$$u^2 + v^2 = k^2 e^{-2u\tau}, \quad (8)$$

where from follows that before starting *v≠0* the achieved by $v = 0$ status of the system is $u^2 = k^2 e^{-2u\tau}$. The last is solved with $k\tau = v\tau = 1.5\pi$, $u\tau = 1.5\pi e^{-u\tau}$, where from $u = 1.293/\tau$ is estimated as critical level of the parameters. Consequently, (8) results in a circle $u^2 + v^2 = k^2$, when $\tau = 0$ and circle $u^2 + v^2 \approx 1.684/\tau^2$ for $\tau \neq 0$. The higher is time delay, the lower is growing rate of the system and the narrower is region by parameters *u*,*v*.

Thus, when *v≠0*, the oscillating modes of system's development with exponentially decreasing (*u<0*) and exponentially increasing (*u>0*) amplitudes become available. In the first case, the oscillating system decays with time and is sustainable (can lead to degeneration of the system, the cessation of its operation), whereas in the second case, growing over time, the oscillation process will quickly destroy the system (growing in time instability cause catastrophe). For example, in a case of financial systems this means that the rocking of growth and decrease leads to a total collapse. So one needs to find the conditions, under which prevention of mode oscillations growing with time is possible (control of the system). Thus, it has to evolve, growing smoothly, without oscillations. From such general speculations one can come to conclusion that the financing of the project should be closely monitored for features of the development, which are modeled by the appropriate equations and scheduled for possible delays in time.

If the time delay exceeds an upper limit for the system, there may be fluctuations in the parameters growing over time. The growing amplitude of oscillations may destroy the system. Therefore with the time delays in the development up to an extreme level nothing dangerous happens. But when the limits are exceeded, the system quickly collapses due to growing oscillations.

4 Critical levels of the system and its control in the pre-crisis regime

The study of the equations (6) by $v\neq 0$ concerning an occurrence of values $u>0$ leads to the conditions: $v\tau = (1.5 + 2n)\pi$, $k\tau = v\tau = (1.5 + 2n)\pi$, where n=0,1,2,... Analysis shows that the first critical value causes fluctuations of the system with growing amplitude is $k\tau=3\pi/2$. From (6) is got: $u\tau = 1.5\pi e^{-u\tau}$, which leads to the numerical solution $u\tau = 1.293$. Thus, a stable development of the system is possible only to a limit in the time delay $\tau = 1.293/u$.

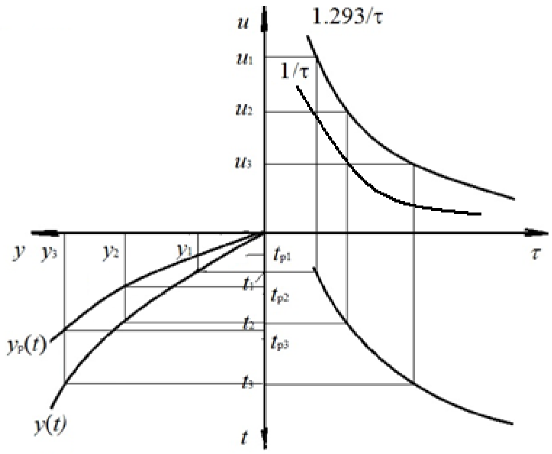

Fig. 1. Critical levels of u versus τ and corresponding values y of the real system's development τ and forecasting system without delays

This means for a given u the constant increase in time delay is permitted only up to a critical value $\tau = 1.293/u$ followed by rapid destruction of the system.

Based on the above, the strategy for sustainable development with time delay requires managing the system starting from any value of rate $u = u_1$. Once it attains a critical value of delay $\tau = \tau_1 = 1.293/u_1$, the system's control must reduce the time delay or reduce the rate of the system's development. These features can be clearly traced in Fig. 1 done according to [1], where the critical levels of a growth rate of the system and associated delay in the first quadrant show the critical dependence of a rate of time delay for the system's development. And in the fourth quadrant the dependence of the lag (time delay) is shown, while in the third – y versus time. Regimes of stable development of the system according to Fig. 1 are below the curve $u = 1.293/\tau$.

The development can be optimally predicted in the following way. If the growth of the system starts with a rate u_1 at the beginning of such system's grow rate, it may be maintained only until the time $t = t_1$, when the increasing delay of the system at the time $\tau_1(t_1)$ becomes critical (line $u = u_1$ crosses a critical curve $= \frac{1.293}{\tau}$) at the point $\tau_1(t_1)$. Then further growth of the system with the specified rate is impossible, provided the further growth with the same time delay. It is necessary to reduce the rate of growth of the system, e.g. to some $u = u_2$, then there is an additional system resource regarding the increase in the delay to the intersection of the critical curve at a lower level of growth. If possible, for sustainable development it is necessary to reduce the growth of time delay, which in many cases is poorly controlled or not controlled at all. Under the case of uncontrolled time delay one can improve the situation and control the system reducing the growth rate every moment when the critical values of time delay are got. In reality, the shift points of the control system according to the above described scenario can be realized in the following way. For example, the Bank finances the project, which for the equation of development has been identified in a specified form, whose solution was found.

Assume that *y(t)* for sustainable development is known. If one needs to take into account some possible time delays in the system and manage it for optimal development and lack of critical modes, which destroy the system, it is necessary to determine a delay by comparing the characteristics of the system in real-time calculated, as shown in Fig. 1. The forecasted system's parameters $y_p(t)$ obtained without taking into account the possible time delay in comparison with *y(t)* obtained for a given delay are compared to identical values, resulting in delays expressed by the following expressions: $\tau_1 = t_1 - t_{p1}$, $\tau_2 = t_2 - t_{p2}$, $\tau_3 = t_3 - t_{p3}$, and so on. Thus, it is possible to determine the actual delay, if the results of the behavior of a real system are known.

Any control of complex systems has not only to consider the time lag, but it is also based on the forecast, and therefore the mechanism of adjustment of the managed system, its sustainability must be proactive. Therefore, in many cases the development management of the systems requires simultaneous and consistent taking into account both the time delay and the switching mechanisms of adjustment (adaptation) of the system under the future features. To study characteristic mechanisms of timing and their impact on sustainable development of systems it can be considered, for example, the following mathematical model:

$$\frac{dy}{dt} = ky(t+\tau), \tag{9}$$

where τ is the time forecasting term of the system.

Equation (9) means that the rate of development of the system focuses not on current performance as in equation (1) and not on previous performance as in equation (3), but on a future performance. To solve (9), the initial conditions on the time interval from zero to τ must be specified. For example, funded project and its execution at each moment of time has a rate that does not match the current level of funding, and one that meets the future, a higher level, which is the head. This is a case of development in advance. Modeling the development of complex systems with forecasting terms has interesting features that are worth exploring and are used for optimal control of systems' development.

Considering solution of (9) in the form (4) and substituting it in (9), the equation for the Eigen values is: $z\tau = k\tau e^{z\tau}$. Researching possible solutions of this equation yields the highest value *kτ*, by which solution exists:

$$u\tau = 1,\ k\tau = 1/e, \tag{10}$$

where it is clear that as similarly to equation with time delay, when $v=0$ there are two cases: exponentially increasing ($u>0$) and exponentially decreasing ($u<0$) in time. When $v\neq 0$, the oscillating modes of development in a system with exponentially decreasing ($u<0$) and exponentially increasing ($u>0$) amplitudes take place. In the first case, the process of oscillation is damped over time and is sustainable (can lead to the degeneration of the system, the cessation of its operation). The rates of development of the system, taking into account forecasting in time that are below $u = 1/\tau$, will lead to increasing time-oscillations of the system, leading to its destruction (see in Fig. 1).

According to the above and (10), similar to the previous case, here parameters of the growing system satisfy the same circle $u^2+v^2 = k^2$, when $\tau = 0$ and circle $u^2+v^2 = 1/\tau^2$ for $\tau \neq 0$. The higher is time delay, the narrower is region by parameters *u,v*. Thus, the strategy of development is in the range between the circles $u^2+v^2 = 1/\tau^2$ and $u^2+v^2 \approx 1.684/\tau^2$. By the fixed time delay and time forecast, parameters of oscillations are interconnceted. The above considered forces to conclude that for sustainable development of the system both processes of the time delay and time forecasting must be correlated in a system development. A stable development is possible only between these two strategies and the actual behavior of the system can only be inside the region between them. Example of building the control strategy for system with prevention of critical regimes in steady development is considered below.

Given the peculiarities of behavior of system's development with shifted arguments, one can construct a control strategy in the following way. Let at the initial moment of time ($t = 0$) the intensity of development equal to u_0, delay is absent ($\tau=0$), so the system according to (4) develops following the law

$$y = y_0 e^{u_0 t} \tag{11}$$

at time $t = t_1$, where the delay is equal to $\tau_1(t_1) = 1.293/u_0$, then it must switch the rate of development to the border region, which is leading to the arguments $u_1 = 1/\tau_1 = u_0/1.293$. Next, the system continues stable growth law (9) with the new rate u_1 point-in-time $\tau_2(t_2) = 1.293/u_1$when the delay reaches a critical level of development for a strategy with delay (Fig. 1). Then again, one can go to the lower critical curve, which corresponds to the development strategy with forecasting $u_2 = 1/\tau_2 = u_1/1.293$. And so on to provide the desired level of system development. For the control of a time delay it is useful to have function $\tau(t)$ obtained for example as a polynomial satisfying the conditions: $t = 0, \tau = 0$; $t = t_1, \tau = \tau_1$; $t = t_2, \tau = \tau_2$:

$$\tau = \frac{1.293t}{t_1 t_2 (t_2 - t_1)} \left[\frac{t_1}{u_1}(t - t_1) + \frac{t_2}{u_0}(t_2 - t) \right]. \quad (12)$$

5 Peculiarities of the non-linear systems

Equation (1) is linear; equation (2) is nonlinear. The nonlinearity in the system by the mathematical model (2) is relatively simple. But the equation (3) with delayed arguments seems linear, but it contains the worst type of nonlinearity (2). If in equation (3) replace the function of delayed argument *y(t-τ)* according to the Elsholtz's theorem [6], which is satisfied for monotone functions, then the function *y(t-τ)* is represented in the Taylor series relative to the point *t* by *τ* with accuracy to the linear terms, since the linear approximation is the most precise in this case: $y(t - \tau) \approx y(t) - \tau dy/dt$, then equation (3) takes the form:

$$(1 + k\tau)\frac{dy}{dt} = ky(t). \quad (13)$$

Equation (13) differs relatively little from (1) and delay affects only the time deformation. But when the function is non-monotonic, such solutions can be oscillatory, the theorem of Elsholtz is broken, and the left side (13) is complete Taylor series to the derivative of *y* and the powers of *τ*. The last case is exactly critical, examples of which have just been considered. Significantly stronger effect of the time delay in case of the nonlinear equation (2) is:

$$\frac{dy}{dt} = ky(t - \tau)(A - y(t - \tau)), \quad (14)$$

which even for the case of monotonous growing, when Elsholtz theorem satisfies, leads to the strong nonlinear equation of the form

$$(1 + Ak\tau - 2k\tau y)\frac{dy}{dt} + \left(\tau \frac{dy}{dt}\right)^2 = ky(A - y), \quad (15)$$

containing the nonlinear terms of different type.

The nonlinearity of the systems cause unpredictable properties and characteristics of their behavior including the existence of various special and critical parameters and system modes. Possible points of bifurcation in a system correspond to the situations, where the system abruptly jumps from one mode to another (usually completely different from the previous one). There are available also strange attractors (sets of trajectories in the phase space of the system in which all other trajectories approach under any initial conditions), and the other features. Solution of equation (2) by initial condition: $t = 0, y = y_0$ is:

$$y = \frac{Ay_0 e^{Akt}}{(A - y_0) + y_0 e^{Akt}}. \quad (16)$$

Because $A > y_0$ the denominator in (16) is non-zero, the non-linear system, in contrast to the linear one, is growing in time smoothly. The limit value $y = A$ is achieved at $t = \infty$. Now the equation (14) is considered in the following simplified form with initial condition: $t = 0$, $y = y_0$:

$$\frac{dy}{dt} = k(A - y)y(t - \tau), \qquad (17)$$

where time delay is accounted only in the main term to the right but not in the multiplier *A-y* as far as this term controls the current state of the system and its closeness to the limit. Solution of the (17) is found in the form similar to the above obtained (16) for the equation without time shifts:

$$y = \frac{Ay_0 e^{zt}}{(A-y_0)+y_0 e^{zt}}, \quad z=u+iv. \qquad (18)$$

Substituting (18) into (17) yields characteristic equation similar to (5), (6), with only difference multiplier *A*:

$$z = Ake^{-z\tau}, \quad u = Ak\,e^{-u\tau}\cos v\tau, \quad v = -Ake^{-u\tau}\sin v\tau. \qquad (19)$$

Surprisingly the result for non-linear system with the time delay is absolutely similar to the above obtained for the linear development equation. The solution (18) can be presented in the following complete form

$$y = Ay_0 e^{ut}\,\frac{(A-y_0+y_0e^{ut}cos vt)cos vt + sin^2 vt}{(A-y_0+y_0e^{ut}cos vt)^2+y_0^2e^{2ut}sin^2 vt}. \qquad (20)$$

Imaginary part is omitted in (20). Here $Ak\tau = v\tau = (1.5 + 2n)\pi$, *n*= 0,1,2,..., where from $u = 1.293/\tau$ is estimated as critical level of the parameter *u* for transformation of the monotonous solution to the oscillating one. The frequency of oscillation in non-linear system $\omega = \frac{3\pi}{2\tau}$ is the same as in the linear one. Despite the non-oscillating solution following to (19) is similar to the above considered linear development equation with time delay, its solution (20) is stable for both *v=0*, as well as for *v≠0*.

Parameters *u,v* according to (19) are interconnected as follows

$$u^2+v^2 = A^2k^2e^{-2u\tau}. \qquad (21)$$

The equation (21) differs from (8) only with additional multiplier *A* in this non-linear case. At *v=0* two processes are possible: monotonous with exponentially increasing in time (*u>0*) term but not exponentially growing, just monotonously rapidly achieving the limit *A*, and oscillating process with increasing amplitude but just to limit *A* (at *u<0*). Obviously from (19), similar to the above, the oscillating solution growing in time up to limit *A* is realized by $v\tau = \frac{3\pi}{2} + 2\pi n$, which means that it starts from $v\tau = \frac{3\pi}{2}$, when (20) gets oscillations growing only up to limit *A* due to non-linear effects. For the monotonous regime (*v=0*)

$$y = \frac{Ay_0 e^{ut}}{A-y_0+y_0e^{ut}}. \qquad (22)$$

Thus, from (22) is seen that *y* is going with time to limit value *A*. The oscillating regime according to (20) by $t \gg 1$ is going to limit $y = Acos^2\frac{3\pi}{2\tau}t$.

6 Complex systems with many parameters and shifted arguments

Complex systems with a large number of governing parameters and delays or forecasting terms are difficult to investigate in the above-mentioned way. But the general patterns are similar on a qualitative level and must be accounted. The aggregated mathematical model of a potentially hazardous object in nuclear energy developed in [17], studied in detail in [6, 9-12] has the following form:

$$\frac{dz_i}{dt} = [b_{i0} + \sum_{j=1}^{6} b_{ij} z_j (t - \tau_{ij})] z_i (t - \tau_{i0}), \quad i = 1 \div 6, \qquad (23)$$

where τ_{ij} is the time delay for the corresponding parameters. Here z_i- system's parameters, b_{ij}- coefficients of the mathematical model, which are determined for each model based on the results of its functioning.

The equation array (23) is more complex than the equations discussed above and allows analyzing the critical levels by described methods. It was investigated numerically [6, 9-12] for many different situations. Surprisingly despite 6 interconnected nonlinear equations and a lot of shifted arguments the solutions did not reveal instability in contrast to the linear equations with shifted arguments. One of typical solution in presented in Fig. 2:

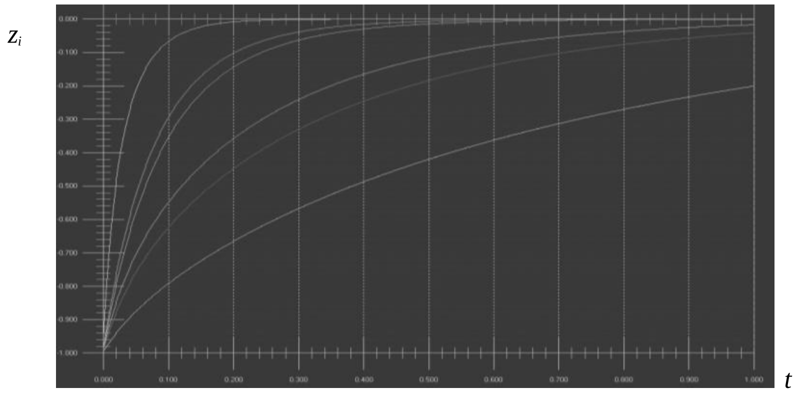

Fig. 2 Parameters of the potentially hazardous object modeled on computer

7 Conclusions

The equations for the development of systems with delayed and forecasting arguments have been considered. The critical levels revealed allow building an optimal strategy for the development. Nonlinear equations with limit levels of the development revealed absolutely stable regimes in contrast to the linear systems having unstable regimes due to shifted arguments.

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