New aspects regarding to some stochastic concepts needful in the study of the systems

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Abstract. In this paper we shall refer shortly to some aspects regarding to the Markov property and some related problems in a vision of K. Itô. We also emphasize the connection with Bownian motion and the usefulness of such results in the study of the systems. We make reference also to some of our papers published in *Journal of Mathematics and Its Applications*, Transilvania University of Braşov.

Keywords: stochastic calculus, Markov processes, Markov property, Brownian motion, convergence..

1 Introduction

A prominent role in the development of the analytical methods of the probability theory played Abraham DeMoivre, Pierre Simon de Laplace, Karl Friederich Gauss, Siméon Denis Poisson. Beginning with the middle of the 19^{th} century and in the 20^{th} , the development of the probability theory is connected frequently with the name of P.L. Chebyshev, A.A. Markov and A.M. Lyapunov.

On the other hand the initial developments in the design of informationprocessing systems dealt mainly with the problems of hardware design and the integration of the hardware to make it perform a given task. As the hardware problems were solved, however, it became evident that the theoretical techniques necessary for designing the hardware portion of a system were not adequate to study the abstract characteristics of complete digital systems. A whole new area of research had to be undertaken to study these properties. The results appear in a new scientific discipline that can be defined as *the study of the dynamic behavior of information systems*.

Within this area we find not only problems that deal with digital computers but also problems associated with such topics as describing the behavior of nerve networks, the representation of the properties of languages, the analysis of information-transmission systems, and the modeling of how man perceives and reacts to his environment. In this sense we mention that the algorithms of the stochastic approximation type have found applications in new and diverse



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areas. For example, the applications in signal processing can be successfully developed, whether or not they are called stochastic approximations. Such algorithms occur frequently in practical systems for the purpose of noise or interface cancellation, the optimization of "post processing" or "equalization" filters in time varying communication channels, adaptive antenna systems, and many related applications.

On the other hand, when a stochastic differential equation is considered if it is allowed for some randomness in some of its coefficients, it will be often obtained a so-called *stochastic differential equation* which is a more realistic mathematical model of the considered situation. Many practical problems conduct us to this notion. Therefore, it is clear that any solution of a stochastic differential equation must involve some randomness. In other words one can hope to be able to say something about the probability distribution of the solutions.

At the same time, results on almost sure convergence of stochastic approximation processes are often proved by a separation of deterministic (pathwise) and stochastic considerations. The basic idea is to show that a "distance" between estimate and solution itself has the tendency to become smaller. The so-called *first Lyapunov method of investigation* does not use knowledge of a solution. Thus, in deterministic numerical analysis gradient of Newton procedures for minimizing or maximizing F by a recursive sequence (X_n) are investigated by a Taylor expansion of $F(X_{n+1})$ around X_n - a device which has been used in stochastic approximation for the first time by J.R. Blum, H.J. Kushner, Z. Schuss, M.T. Wasan, M.B. Nevel'son & R.S. Hasminskij.

Since the Brownian motion was firstly investigated by L. Bachelier and A. Einstein, and then N. Wiener had the possibility to put it on a firm mathematical foundation, many of the scientific works have been done on their applications in physics, chemistry, communications, population genetics, and other fields.

Many researchers were fascinated by the great beauty of the theory of Brownian motion and many results have been obtained in the last decades. As for example, among other things, in the theory of diffusion processes and related topics by K. Itô and H.P. McKean Jr. ([8]); in the theory of stochastic differential equations and their applications or in stochastic approximation by Z. Schuss ([21]), M.T. Wasan ([24]); and in stochastic calculus and its applications to some problems in finance by J.M. Steele ([22]).

In this line we shall consider together some fundamental problems for the study of the systems, discussed separatelly, in some previous Sessions of *Chaotic Modeling and Simulation* International Conferences.

Among other things the usefulness of such results in the study of the systems is emphasized.

2 On Markov processes. A general vision on some fundamental aspects

It is observed that the usual class of Markov processes which we consider has many times some restrictions which do not cover many interesting processes. This is the reason for which we try often to obtain some extensions of this notion. Research in this direction is due, among others, to Kiyosi Itô. In this context we shall refer here to Markov processes and their impact for some practical problems (see for example [7], [8], [9]).

Thus, we develop some aspects regarding to the Markov processes in the vision of Kiyosi Itô. Firstly, we introduce the transition probabilities and we define the Markov process. Then, we shall refer to the Markov property as a fundamental concept in the study of the systems and, in this context, we consider some aspects regarding the extended Markov property and the strong Markov property.

2.1 Transition probabilities

Let S be a state space and consider a particle which moves in S. Also, suppose that the particle starting at x at the present moment will move into the set $A \subset S$ with probability $p_t(x, A)$ after t units of time, "irrespectively of its past motion", that is to say, this motion is considered to have a Markovian character.

The transition probabilities of this motion are $\{p_t(x, A)\}_{t,x,A}$ and we considered that the time parameter $t \in T = [0, +\infty)$.

The state space S is assumed to be a compact Hausdorff space with a countable open base, so that it is homeomorphic with a compact separable metric space by the Urysohn's metrization theorem. The σ -field generated by the open sets (the topological σ -field on S) is denoted by K(S). Therefore, a Borel set A is a set in K(S).

The mean value

$$m = M(\mu) = \int_R x \, \mu(dx)$$

is used for the center and the scattering degree of a one-dimensional probability measure μ having the second order moment finite, and the *variance* of μ is defined by

$$\sigma^2 = \sigma^2(\mu) = \int_R (x - m)^2 \mu(dx).$$

On the other hand, from the Tchebychev's inequality, for any t > 0, we have

$$\mu(m - t\sigma, m + t\sigma) \le \frac{1}{t^2},$$

so that several properties of 1-dimensional probability measures can be derived.

Note that in the case when the considered probability measure has no finite second order moment, σ becomes useless. In such a case one can introduce the central value and the dispersion that will play similar roles as m and σ for general 1-dimensional probability measures.

Remark 1. We recall that J. L. Doob defined the *central value* $\gamma = \gamma(\mu)$ as being the real number γ which verifies the following relation

$$\int_{R} \operatorname{arctg}(x - \gamma) \mu(dx)) = 0.$$

Here, the existence and the uniqueness of γ follows from the fact that $\operatorname{arctg}(x-\gamma)$ is continuous and decreases strictly from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$, for x fixed, as γ moves from $-\infty$ to $+\infty$.

The dispersion δ is defined as follows

$$\delta = \delta(\mu) = -\log \int \int_{\mathbb{R}^2} e^{-|x-y|} \mu(dx) \mu(dy).$$

We will assume that the transition probabilities $\{p_t(x, A)\}_{t \in T, x \in S, A \in K(S)}$ satisfy the following conditions:

- (1) for t and A fixed,
 - a) the transition probabilities are Borel measurable in x;
 - b) $p_t(x, A)$ is a probability measure in A;
- (2) $p_0(x, A) = \delta_x(A)$ (i.e. the δ -measure concentrated at x);
- (3) $p_t(x, \cdot) \xrightarrow{weak} p_t(x_0, \cdot)$ as $x \to x_0$ for any t fixed, that is

$$\lim_{x \to x_0} \int f(y) p_t(x, dy) = \int f(y) p_t(x_0, dy)$$

for all continuous functions f on S;

- (4) $p_t(x, U(x)) \longrightarrow 1$ as $t \searrow 0$, for any neighborhood U(x) of x;
- (5) the Chapman-Kolmogorov equation holds:

$$p_{s+t}(x,A) = \int_{S} p_t(x,dy) p_s(y,A)$$

The transition operators can be defined in a similar manner. Consider C = C(S) to be the space of all continuous functions (it is a separable Banach space with the supremum norm).

The operators p_t , defined by

$$(p_t f)(x) = \int_S p_t(x, dy) f(y), \quad f \in C$$

are called *transition operators*.

The conditions for the transition probabilities can be adapted to the transition operators, but we do not give the details here (for more details see [7], [8], [9], [1]).

Remark 2. Let us observe that the conditions (1) - (5) above are satisfied for Brownian transition probabilities. One can define

$$p_t(x, dy) = \frac{1}{t\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2t^2}} dy \quad \text{in } R$$
$$p_t(\infty, A) = \delta_\infty A.$$

Definition 21 A Markov process is a system of stochastic processes

$${X_t(\omega), t \in T, \omega \in (\Omega, K, P_a)}_{a \in S}$$

that is for each $a \in S$, $\{X_t\}_{t \in S}$ is a stochastic process defined on the probability space (Ω, K, P_a) .

The transition probabilities of a Markov process will be denoted by $\{p(t, a, B)\}$, and we will denote by $\{H_t\}$ the transition semigroup and by R_{α} be the resolvent operator of $\{H_t\}$.

The next results shows that p(t, a, B), H_t and R_{α} can be expressed in terms of the process as follows (we do not insist on this aspect):

Theorem 21 Let f be a function in C(S). Then

 $\begin{array}{l} i) \ p(t,a,B) = P_a(X_t \in B).\\ ii) \ For \ E_a(\cdot) = \int_{\Omega} \cdot P_a(d\omega) \ one \ has \ H_t f(a) = E_a(f(X_t)).\\ iii) \ R_\alpha f(a) = E_a\left(\int_0^\infty e^{-\alpha t} f(X_t) dt\right). \end{array}$

2.2 On Markov property - A concept useful in the study of the systems

The Markov property is given in the theorem below:

Theorem 22 Let $\Gamma \in K$ be given. The following is true

$$P_a(\theta_t \omega \in \Gamma | K_t) = P_{X_t(\omega)}(\Gamma) \quad a.s.(P_a),$$

that is to say

$$P_a(\theta_t^{-1}\Gamma|K_t) = P_{X_t(\omega)}(\Gamma).$$

Remark 3. The following notation can be used

$$P_{X_t(\omega)}(\Gamma) = P_b(\Gamma)|_{b=X_t(\omega)}.$$

Proof. It will be suffice to show that

$$P_a(\theta_t^{-1}\Gamma \cap D) = E_a(P_{X_t}(\Gamma), D) \tag{1}$$

for $\Gamma \in K$ and $D \in K_t$.

We distinguish the following three cases.

I. Let us consider Γ and D as follows:

$$\Gamma = \{X_{s_1} \in B_1\} \bigcap \{X_{s_2} \in B_2\} \bigcap \dots \bigcap \{X_{s_n} \in B_n\}$$

and

$$D = \{X_{t_1} \in A_1\} \bigcap \{X_{t_2} \in A_2\} \bigcap \dots \bigcap \{X_{t_m \in A_m}\}$$

with

$$0 \le s_1 < s_2 < \dots < s_n$$

$$0 \le t_1 < t_2 < \dots < t_m \le t$$

and $B_i, A_j \in K(S)$.

Now it will be observed that the both sides in (1) are expressed as integrals on S^{m+n} in terms of transition probabilities. Thus, one can see that they are equal.

II. Let now be Γ as in the case **I** and let us denote by D a general member of K_t . For Γ fixed the family \mathcal{D} of all D's satisfying (1) is a Dynkin class. If \mathcal{M} is the family of all M's in the case **I** then, this family is multiplicative and $\mathcal{M} \subset \mathcal{D}$. In this way it follows

$$\mathcal{D}(\mathcal{M}) \subset \mathcal{D} = K(\mathcal{M}) = K_t$$

and one can conclude that, for Γ in the case I and for D general in K_t , the equality (1) holds.

III. (General case) This case can be proved in the same manner as case II by fixing an arbitrary $D \in K_t$.

It will follows that $P_a(\Gamma)$ is Borel measurable in a for any $\Gamma \in K$.

Corollaire 21

$$E_a(G \circ \theta_t, D) = E_a(E_{X_t}(G), D) \quad \text{for } G \in \mathbf{B}(K), D \in K_t, \\ E_a(F \cdot (G \circ \theta_t)) = E_a(F \cdot E_{X_t}(G)) \text{ for } G \in \mathbf{B}(K), F \in \mathbf{B}(K_t), \\ E_a(G \circ \theta_t | K_t) = E_{X_t}(G) \text{ a.s. } (P_a) \text{ for } G \in \mathbf{B}(K).$$

2.3 The extended Markov property

It is interesting to see that the Markov property can be extended. We will refer to the following theorem as the *extended Markov property*.

Theorem 23 Let $\Gamma \in K$ be given. The following is true

$$P_a(\theta_t \omega \in \Gamma | K_{t+}) = P_{X_t}(\Gamma)$$
 a.s. (P_a) .

Proof. Returning to the equality (1) above, we will prove it now for $D \in K_{t+}$. To this end, we will show the following equality:

$$E_a(f_1(X_{s_1}(\theta_t\omega))\cdot\ldots\cdot f_n(X_{s_n}(\theta_t\omega)), D) = E_a(E_{X_t}(f_1(X_{s_1})\cdot\ldots\cdot f_n(X_{s_n})), D)$$
(2)

for $f_i \in C(S)$, $D \in K_{t+}$ and $0 \le s_1 < s_2 < \ldots < s_n$.

But $D \in K_{t+h}$ for h > 0, so that by Corollary 21 we obtain

$$E_a(f_1(X_{s_1}(\theta_{t+h}\omega))\cdots f_n(X_{s_n}(\theta_{t+h}\omega)), D) = E_a(E_{X_{t+h}}(f_1(X_{s_1})\cdots f_n(X_{s_n})), D)$$
(3)

It can be seen that

$$E_a(f_1(X_{s_1})\cdot\ldots\cdot f_n(X_{s_n}))$$

is a continuous function of a, since

$$E_a(f_1(X_{s_1}) \cdot \ldots \cdot f_n(X_{s_n})) = H_{s_1}(f_1 \cdot \ldots \cdot (H_{s_{n-1}-s_{n-2}}(f_{n-1} \cdot H_{s_n-s_{n-1}}f_n)))$$

and $H_s: C \longrightarrow C$.

Since $X_t(\omega)$ is right continuous in t, we obtain

$$f_i(X_{s_i}(\theta_{t+h}\omega)) = f_i(X_{s_i+t+h}(\omega)) \longrightarrow f_i(X_{s_i+t}(\omega)) = f_i(X_{s_i}(\theta_t\omega))$$

as $h \searrow 0$.

The equality (2) follows now by taking the limit in (3) with $h \searrow 0$. In this way, for G_i open in S, from (2) we obtain the following equality

$$E_a(X_{s_i}(\theta_t\omega) \in G_1, \cdots, X_{s_n}(\theta_t\omega) \in G_n, D) =$$

= $E_a(P_{X_t}(X_{s_1} \in G_1, \cdots, X_{s_n} \in G_n), D),$ (4)

and therefore we can use Dynkin's theorem, concluding the proof.

2.4 The strong Markov property

As it is known, the intuitive meaning of a Markov process (for example X(t)) is the fact that such a processes "forgets" the past, provided that t_{n-1} is regarded as the present.

Now, the intuitive meaning of the Markov property is that under the condition that the path is known up to time t, the future motion would be as if it started at the point $X_t(\omega) \in S$.

The following question arises: what will happen if the time t is replaced by a random time $\sigma(\omega)$?

Definition 22 A random time $\sigma : \Omega \to [0, \infty]$ is called a "stopping time" with respect to $\{\overline{K_t}\}$ if

$$\{\sigma \le t\} \in \overline{K}_t \tag{5}$$

for every t.

Note that the above condition is equivalent to the condition

$$\{\sigma < t\} \in K_t \tag{6}$$

for every t.

Remark 4. Note that if the condition (5) holds, then

$$\{\sigma < t\} = \bigcup_{n} \left\{ \sigma \le t - \frac{1}{n} \right\} \in \overline{K}_{t},$$

and if the condition (6) holds, then

$$\{\sigma \le t\} = \bigcap_{n \ge m} \left\{ \sigma < t + \frac{1}{n} \right\} \in \overline{K}_{t + \frac{1}{m}}$$

for every m.

Therefore

$$\{\sigma < t\} \in \bigcap_m \overline{K}_{t+\frac{1}{m}} = \overline{K}_t$$

by right continuity of \overline{K}_t . A trivial example is the "deterministic time" $\sigma \equiv t$. **Theorem (Dynkin's formula).** Let us suppose that σ is a stopping time with

 $E_a(\sigma) < \infty$. Then, for $u \in \mathcal{D}(A)$ it follows:

$$E_a\left(\int_0^\infty Au(X_t)dt\right) = E_a(u(X_\sigma)) - u(a).$$

We will consider now the strong property (for more details and proofs, see [7], [8], [9], [14], [1]).

The strong Markov property is contained in the theorem below

Theorem 24 The following equalities hold a.s.

- i) $P_{\mu}(\theta_{\sigma}^{-1}\Gamma|\overline{K}_{\sigma}) = P_{X_{\sigma}}(\Gamma)$ a.s. (P_{μ}) on $\{\sigma < \infty\}$, where $\Gamma \in \overline{K}$. ii) $E_{\mu}(F(\theta_{\sigma}\omega|\overline{K}_{\sigma}) = E_{X_{\sigma}}(F)$ a.s. (P_{μ}) on $\{\sigma < \infty\}$, where F is bounded and \overline{K} -measurable.

Remark 5. The following conclusions are true.

i) $\theta_{\sigma}\omega = \theta_{\sigma(\omega)}\omega$ for any ω with $\sigma(\omega) < \infty$, and

$$\theta_{\sigma}^{-1}\Gamma = \{\omega : \sigma(\omega) < \infty \text{ and } \theta_{\sigma}\omega \in \Gamma\}.$$

ii) We notice that μ is arbitrary, and for this reason both $F(\theta_{\sigma}\omega)$ and $E_{X_{\sigma}}(F)$ are \overline{K} -measurable.

A version of the strong Markov property is given below. It is referred to as the time-dependent strong Markov property.

Theorem 25 If $F(t, \omega)$ is bounded and $K[0, \infty] \times \overline{K}$ – measurable, then the following equality holds

$$E_{\mu}(F(\sigma,\theta_{\sigma}\omega)|\overline{K}_{\sigma}) = E_{a}(F(t,\omega))|_{t=\sigma,a=X_{\sigma}} \quad a.s. \ (P_{\mu}). \tag{7}$$

Proof. Since σ is \overline{K}_{σ} -measurable, it follows that

$$F(t,\omega) = f(t)G(\omega).$$

Considering the additivity of both sides of (7) with respect to F, the general case follows.

Remark 6. One can observe that such a study implies stochastic calculus and approximation. Various applications, results and comment has been developed, among others by B. Øksendal ([14]), B. Øksendal & A. Sulem ([15]), J.M. Steele ([22]), Z. Schuss ([21]), K. Itô ([9]). At the same time, more details and related topics can be found also in G.V. Orman ([16], [17]). And a general presentation of Markov processes t the perspective of Kiyosi Itô is developed by D.W. Stroock ([23]).

3 From chaotic motion to Brownian motion - new aspects of the study

In a previous CHAOS Conference we have discussed about this subject which is very useful in the random systems analysis. Now, we come back to this subject and complete it with new aspects.

Thus, as in the previous our study, let us imagine a chaotic motion of a particle of colloidal size immersed in a fluid. Such a chaotic motion of a

particle is called, usually, *Brownian motion* and the particle which performs such a motion is referred to as a *Brownian particle*. Such a chaotic perpetual motion of a Brownian particle is the result of the collisions of particle with the molecules of the fluid in which there is.

But this particle is much bigger and also heavier than the molecules of the fluid which it collide, and then each collision has a negligible effect, while the superposition of many small interactions will produce an observable effect.

On the other hand, for a Brownian particle such molecular collisions appear in a very rapid succession, their number being enormous. For a so high frequency, evidently, the small changes in the particle's path, caused by each single impact, are too fine to be observable. For this reason the exact path of the particle can be described only by statistical methods.

Thus, the influence of the fluid on the motion of a Brownian particle can be described by the combination of two forces in the following way.

1. The considered particle is much larger than the particle of the fluid so that the cumulated effect of the interaction between the Browni-an particle and the fluid may be taken as having a hydrodynamical character. Thus, the first of the forces acting on the Brownian particle may be considered to be the forces of *dynamical friction*. It is known that the frictional force exerted by the fluid on a small sphere immersed in it is determined from the Stockes's law: the drag force per unit mass acting on a spherical particle of radius a is given by $-\beta \mathbf{v}$, with $\beta = \frac{6\pi a\eta}{m}$, where m is the mass of the

particle, η is the coefficient of dynamical viscosity of the fluid, and **v** is the velocity of particle.

- 2. The other force acting on the Brownian particle is caused by the individual collisions with the particles of the fluid in which there is. This force produces instantaneous changes in the acceleration of the particle. Furthermore, this force is *random both in direction and in magnitude*, and one can say that it is a *fluctuating force*. It will be denoted by $\mathbf{f}(\mathbf{t})$. For $\mathbf{f}(\mathbf{t})$ the following assumptions are made:
 - *i* The function $\mathbf{f}(\mathbf{t})$ is statistically independent of $\mathbf{v}(t)$.
 - *ii* $\mathbf{f}(\mathbf{t})$ has variations much more frequent than the variations in $\mathbf{v}(t)$.
 - *iii* $\mathbf{f}(\mathbf{t})$ has the average equal to zero.

In these conditions, the Newton's equations of motion are given by the following stochastic differential equation

$$\frac{dbfv(t)}{dt} = -\beta \mathbf{v}(t) + \mathbf{f}(t) \tag{8}$$

which is called the Langevin's equation.

From the Langevin's equation, the statistical properties of the function $\mathbf{f}(\mathbf{t})$ can be obtained if its solution will be in correspondence with known physical laws. One can observe that the solution of (8) determines the *transition probability density* (in brief *the transition density*) $\rho(\mathbf{v}, t\mathbf{v}_0)$ of the random process $\mathbf{v}(t)$, which verifies the equation

$$P(\mathbf{v}(t) \in A) \,|\, \mathbf{v}(0) = \mathbf{v}_0) = \int_A \rho(\mathbf{v}, t, \mathbf{v}_0) d\mathbf{v}. \tag{9}$$

Now, the initial velocity \mathbf{v}_0 can be supposed to be given. Then, one gets

$$\rho(\mathbf{v}, t, \mathbf{v}_0) \to \delta(\mathbf{v} - \mathbf{v}_0)$$

as $t \to 0$ where δ is the *Dirac's* δ -function. On the other hand, from the statistical physics it is known that the transition density $\rho(\mathbf{v}, t, \mathbf{v}_0)$ must approach the Maxwell's density for the temperature T of the surrounding medium and this, independently of \mathbf{v}_0 as $t \to \infty$. We come to the limit

$$\rho(\mathbf{v}, t, \mathbf{v}_0) \to \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{m|\nu|^2}{2kT}} \tag{10}$$

as $t \to \infty$. This means, in other words, that the fluctuating force $\mathbf{f}(t)$ has certain statistical properties. For the formal solution is as follows (according to (8))

$$\mathbf{v}(t) = \mathbf{v}_0 e^{-\beta t} + \int_0^t e^{-\beta(t-s)} \mathbf{f}(z) dz.$$
(11)

Therefore, the integral and the difference $\mathbf{v}(t) - \mathbf{v}_0 e^{-\beta t}$ must have the same statistical properties. Since

$$\mathbf{v}(t) - \mathbf{v}_0 \, e^{-\beta t} \approx \mathbf{v}(t)$$

for latge values of t, it results that the integral must have in the limit a normal density. But the integral can be written as a finite Riemann sum in the following way

$$\int_{0}^{t} e^{-\beta(t-s)} \mathbf{f}(z) dz \approx$$
$$\equiv e^{-\beta t} \sum_{n} e^{\beta n \,\Delta t} \mathbf{f}(n \,\Delta t) \Delta t \equiv e^{-\beta t} \sum_{n} e^{\beta n \,\Delta t} \,\Delta \mathbf{g}_{n}$$

where was denoted $\Delta \mathbf{g}_n = \mathbf{f}(n \Delta t) \Delta t$. Hence, for large values of t, the following approximation is found

$$\mathbf{v} \approx \sum_{n} e^{\beta(n\,\Delta t - t)} \,\Delta \mathbf{g}_{n}.$$
(12)

Here $\Delta \mathbf{g}_n$ is a random variable which gives the random accelerations transmited to a Brownian particle in an inteval of time $(n \Delta t), (n+1)\Delta t$. Therefore, the random variables $\Delta \mathbf{g}_n$ can be assumed to be statistically independent of each other, the successive collisions being completely chaotic.

One can assume that, in comparison with the average period of a single fluctuation of the function \mathbf{g}_n , the time intervals Δt are enough large. The function \mathbf{g}_n has a period of fluctuation of the order of the time between successive collisions which appear between the Brownian particle and the molecules of the fluid.

Thus, if $\Delta \mathbf{g}_n$ is choosen to be a normal random variable with mean zero, it follows that $\nu(t)$ will be also a normal random variable, as it is desired. By means of 12, and setting $D^2(\Delta \mathbf{g}_n) = 2q\Delta t$ one gets

$$E|\mathbf{v}|^{2} = \sum_{n} 2q\Delta t \, e^{2\beta(n\,\Delta t - t)} \rightarrow$$
$$\rightarrow 2q \int_{0}^{t} e^{2\beta(z-t)} \, dz = \frac{q}{\beta} \left(1 - e^{-2\beta t}\right) \tag{13}$$

as $\Delta t \to 0$.

But, at the same time, one has

$$E|\mathbf{v}|^2 \to \frac{kT}{m}$$

as $t \to \infty$, so that q is given by the equality below

$$q = \frac{\beta kT}{m}.\tag{14}$$

If $\mathbf{x}(t)$ is the notation for the displacement of the Brownian particle then, we have

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{v}(z) dz.$$
(15)

Now substituting (11) in (15) one gets

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \left(\mathbf{v}_0 e^{-\beta z} + e^{-\beta z} \int_0^z e^{\beta y} f(y) dy \right) dz.$$

If the order of integration is changed the following estimation follows

$$\mathbf{x}(t) - \mathbf{x}_0 - \frac{\mathbf{v}_0(1 - e^{-\beta t})}{\beta} = \\ = -e^{-\beta t} \int_0^t \frac{e^{\beta z} \mathbf{f}(z) dz}{\beta} + \int_0^t \frac{\mathbf{f}(z) dz}{\beta} \equiv \int_0^t g(z) \mathbf{f}(z) dz, \tag{16}$$

where $g(z) = \frac{1 - e^{\beta(z-t)}}{\beta}$. If a finite sum approximation to the integral is used again then, we come to the conclusion that

$$\mathbf{x}(t) - \mathbf{x}_0 - \frac{\mathbf{v}_0(1 - e^{-\beta t})}{\beta}$$

is a normal random variable with the mean equal to zero and the variance given by the equality

$$\sigma^2 = 2q \int_0^t g^2(z)dz = \frac{q}{\beta^3} \left(2\beta t + 4e^{-\beta t} - e^{-2\beta t} - 3\right).$$
(17)

Regarding to the probability density of the displacement $\mathbf{x}(t)$, it is given by the following equality

$$p(\mathbf{x}, t, \mathbf{x}_0, \mathbf{v}_0) = \left[\frac{m\beta^2}{2kT\left(2\beta t + 4e^{-\beta t} - e^{-2\beta t} - 3\right)}\right]^{\frac{3}{2}} \times \\ \times e^{-\frac{m\beta^2 \left|\mathbf{x} - \mathbf{x}_0 \frac{1 - e^{-\beta t}}{\beta}\right|^2}{2kT\left(2\beta t + 4e^{-\beta t} - e^{-2\beta t} - 3\right)}}.$$
(18)

Finally, for sufficiently large values of t it results

$$p(\mathbf{x}, t, \mathbf{x}_0, \mathbf{v}_0) \approx \frac{1}{(4\pi D t)^{\frac{3}{2}}} e^{-\frac{|\mathbf{x} - \mathbf{x}_0|^2}{4Dt}}$$
 (19)

where D is

$$D = \frac{kT}{m\beta} = \frac{kT}{6\pi a\eta}.$$
 (20)

and is referred to as the *diffusion coefficient*.

Therefore, it results that $p(\mathbf{x}, t, \mathbf{x}_0, \mathbf{v}_0)$ satisfies the diffusion equation given below

$$\frac{\partial p(\mathbf{x}, t, \mathbf{x}_0, \mathbf{v}_0)}{\partial t} = D\Delta p(\mathbf{x}, t, \mathbf{x}_0, \mathbf{v}_0).$$
(21)

The expression of D in (20) was obtained by A. Einstein.

Observation 31 From physics it is known the following result due to Maxwell: Let us suppose that the energy is proportional to the number of particles in a gas and let us denoted $E = \gamma n$, where γ is a constant independent of n. Then,

$$\begin{split} P\{a < v_i^1 < b\} &= \frac{\int\limits_a^b \left(1 - \frac{x^2 m}{2\gamma n}\right)^{\frac{3n-3}{2}} dx}{+\left(\frac{2\gamma n}{m}\right)^{\frac{1}{2}} \left(\frac{1 - x^2 m}{2\gamma n}\right)^{\frac{3n-3}{2}} dx} \rightarrow \\ & -\left(\frac{2\gamma n}{m}\right)^{\frac{1}{2}} \left(\frac{1 - x^2 m}{2\gamma n}\right)^{\frac{3n-3}{2}} dx \\ & \rightarrow \left(\frac{3m}{4\pi\gamma}\right)^{\frac{1}{2}} \int\limits_a^b e^{-\frac{3mx^2}{4\gamma}} dx. \end{split}$$

Now, for $\gamma = \frac{3kT}{2}$ the following Maxwell's result is found

$$\lim_{n \to \infty} P\{a < v_i^1 < b\} = \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \int_{a}^{b} e^{-\frac{mx^2}{2kT}} dx.$$

T is called the "absolute temperature", while k is the "Boltzmann's constant".

Conclusion 31 We think that when, in various problems, we say "chaos" or "chaotic and complex systems" or we use another similar expression to define the comportment of some natural phenomena, in fact we imagine phenomena similarly to a Brownian motion which is a more realistic model of such phenomena.

Remark 7. More details, proofs and other aspects can be found in K. Itô ([7], [9]), K. Itô and H. P. McKean Jr. ([8]), A. T. Bharucha-Reid ([1]), B. Øksendal ([14]). And a general presentation of Markov processes from the perspective of Kiyosi Itô is developed by D.W. Stroock ([23]).

References

- 1. A. T. Bharucha-Reid, *Elements Of The Theory Of Markov Processes And Their Applications*, Dover Publications, Inc., Mineola, New York, 1997.
- Feller, W., An Introduction to Probability Theory and its Applications, vol. I. John Wiley & Sons, Inc., New York, London, 1960.
- Feller, W., An Introduction to Probability Theory and its Applications, vol. II. John Wiley & Sons, Inc., New York, London, 1966.
- Gihman, I.I. and Skorohod, A.V., Stochastic Differential Equations. Springer-Verlag, Berlin, 1972.
- 5. Gnedenko, B.V., The Theory of Probability. Mir Publisher, Moscow, 1976.
- Gray, K.B., The application of stochastic approximation to the optimization of random circuits. American Mathematical Society, Proceedings of Simposium in Applied Mathematics, 16, 1962, pp. 178-192.9++
- 7. Itô, K. Selected Papers. Springer-Verlag, Berlin Heidelberg, 1987.
- Itô, K. and McKean, H.P. Jr., Diffusion Processes and their Sample Paths. Springer-Verlag, Berlin Heidelberg, 1996.
- Itô, K., Stochastic Processes. Eds. Ole E. Barndorff-Nielsen and Ken-iti Sato. Springer, 2004.
- Kushner, H.J. On the weak convergence of interpolated Markov chains to a diffusion. Ann. Probab., 2, 1974, 40-50.
- Kushner, H.J. and Yin, G.G. Stochastic Approximation Algorithms and Applications. Springer-Verlag New York, Inc., 1997.
- Loève, M., Probability theory I. Springer-Verlag, New York, Heidelberg Berlin, 1977.
- Loève, M., Probability theory II. Springer-Verlag, New York, Heidelberg Berlin, 1978.
- Øksendal, B., Stochastic Differential Equations: An Introduction with Applications. Sixth Edition. Springer-Verlag, 2003.
- B. Øksendal, A. Sulem, Applied Stochastic Control of Jump Diffusions, Springer, 2007.
- 16. Orman, G.V., Lectures on *Stochastic Approximation Methods and Related Topics*. Preprint. "Gerhard Mercator" University, Duisburg, Germany, 2001.
- 17. Orman, G.V., Handbook of Limit Theorems and Stochastic Approximation. "Transilvania" University Press, Brasov, 2003.
- Orman, G.V., Capitole de matematici aplicate. Ed. Albastră, MicroInformatica, Cluj-Napoca, 1999.
- Orman, G.V., Aspects of convergence and approximation in random systems analysis. Lambert Academic Publishing, Germany, 2012.

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- Orman, G.V., Basic Probability Theory, Covergence, Stochastic Processes And Applications. *Concepts and Stochastic Models useful in the Study of the Systems*. Transilvania University Press, Braşov, 2016.
- 21. Schuss Z., Theory and Application of Stochastic Differential Equations. John Wiley & Sons, New York, 1980.
- 22. Steele, J. M. Stochastic Calculus and Financial Applications. Springer-Verlag New York, Inc., 2001.
- 23. Stroock D. W., Markov Processes from K. Itô Perspective. Princeton Univ. Press, Princeton, 2003.
- 24. Wasan, M.T., Stochastic Approximation. Cambridge University Press, 1969.