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Fractal Approximants on the Circle

María A. Navascués[1], Sangita Jha[2], A. K. Bedabrata Chand[2], and María V. Sebastián[1]

[1] Departmento de Matemática Aplicada, Escuela de Ingeniería y Arquitectura, Universidad de Zaragoza, 500018 Zaragoza, Spain.
(E-mail: manavas@unizar.es, msebasti@unizar.es)
[2] Department of Mathematics, Indian Institute of Technology Madras, 600036 Chennai, India
(E-mail: chand@iitm.ac.in, sangitajha285@gmail.com)

Abstract. A methodology based on fractal interpolation functions is used in this work to define new real maps on the circle generalizing the classical ones. The power of fractal methodology allows us to generalize any other interpolant, both smooth and non-smooth, but the important fact is that this technique provides one of the few methods of non-differentiable interpolation. In this way, it constitutes a functional model for chaotic processes. In this article we study a generalization of some approximation formulae proposed by Dunham Jackson both in classical and fractal cases.
Keywords: Fractal Interpolation, Trigonometric Approximation, Trigonometric Interpolation, Smoothing, Curve Fitting.

1 Introduction

Interpolation and approximation are mostly carried out with smooth functions, but in many practical situations for example we come across sampled signals, which are not smooth. Thus we cannot use classical interpolation in such cases. Fractal interpolatio [2] helps to solve this problem to a large extent as it involves both smooth and non-smooth which are created depending on the choice of the scaling parameters, see for instance ([3], [4], [12], [5]). Barnsley [3] and Navascués [10] observed that by a suitable choice of IFS whose elements are selected in terms of a prescribed continuous function f, an entire family of fractal functions f^α, called the α-fractal functions, can be constructed to interpolate and approximate f. We give here a global deterministic method to model periodic signals by fractal interpolation. This paper generalizes a particular type of approximants defined by Jackson [8] and extend these approximants to its fractals which are smooth or non-smooth in nature depending on the choice of scaling factors. The functions proposed have the advantage of owning an analytical explicit expression in terms of the samples (specific values) of the original function. This fact gives them a particular importance in order to

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obtain mathematical representations of experimental signals, from which only their values at evenly sampled nodes are known. The limits proposed prove the convergence of approximants and intepolants with very weak conditions as the sampling frequency is increased. The fast evolution of the programs of advanced calculus enables the use of fractal functions more complicated that mere polynomial and trigonometric mappings. In this way, from the theoretical point of view, new fractal nodal elements are proposed, that provide a generalization of the trigonometric functions. The density of these mappings in the space of continuous and periodic functions is proved. In some cases, the new fractal elements perform better than the classical Jackson's originals.
The paper is organized as follows. In section 2, we have given some brief description of α-fractal functions and review the required classical results on uniform convergence of polynomial and trigonometric interpolation. A new type of discrete version of Jackson approximants ans its convergence results proposed in section 3. These results have been extended to the fractal version of discrete Jackson approximants in section 4.

2 Preliminaries:

In this section we shall gather some essential materials that can be found in details in the references ([1], [2], [7], [13], [11]).

2.1 Constructions of fractal functions:

In section 2.1, construction of continuous fractal interpolation function based on iterated function system is reviewed. Let $x_1 < x_2 < \ldots < x_N$ be real numbers, and $I = [x_1, x_N]$, be a closed interval that contains them. Let a set of data points $\{(x_i, y_i), i \in \mathbb{N}_N\}$ be given, where $\mathbb{N}_k$ is the first k natural numbers, and $I_i = [x_i, x_{i+1}]$. Let $L_i : I \to I_i, i \in \mathbb{N}_{N-1}$ be contractive homeomorphisms such that

$$L_i(x_1) = x_i,\ L_i(x_N) = x_{i+1}. \tag{1}$$

Let $K = I \times \mathbb{R}$ and $N-1$ continuous mappings, $F_i : K \to \mathbb{R}$ be satisfying

$$F_i(x_1, y_1) = y_i, F_i(x_N, y_N) = y_{i+1},\ |F_i(x, y) - F_i(x, y')| \leq |\alpha_i||y - y'|, \tag{2}$$

where $(x, y), (x, y') \in K,\ \alpha_i \in (-1, 1),\ i \in \mathbb{N}_{N-1}$. Now define functions $w_i : \mathbb{R}^2 \to \mathbb{R}^2$ as $w_i(x, y) = (L_i(x), F_i(x, y)), \forall i \in \mathbb{N}_{N-1}$.

Theorem 1. ***(Barnsley[2])*** *The IFS $\mathcal{I} = \{K; w_i, i = 1, 2, \ldots, N-1\}$ admits a unique attractor G. G is the graph of a continuous function $f : I \to R$ which obeys $f(x_i) = y_i$ for $i = 1, 2, \ldots, N$.*
The previous function is called fractal interpolation function(FIF) corresponding to the IFS $\mathcal{I}$. For the implicit representation of FIF, we proceed as follows:

Let $\mathcal{G} = \{g : [x_0, x_N] \to [c, d] \mid g \text{ is continuous and } g(x_0) = y_0,\ g(x_N) = y_N\}$. Then $\mathcal{G}$ is a complete metric space with respect to uniform norm $\|.\|_\infty$.
Define a mapping $T : \mathcal{G} \mapsto \mathcal{G}$ by $(Tg)(x) = F_i(L_i^{-1}(x), g \circ L_i^{-1}(x))\ \forall x \in$

$[x_i, x_{i+1}], \ \ i \in \mathbb{N}_{N-1}$.
Now, T is a contraction mapping on the metric space $(\mathcal{G}, \|.\|_\infty)$, that is,

$$\|Tg - Th\|_\infty \le |\alpha|_\infty \|g - h\|, \ g, h \in \mathcal{G},$$

where $\alpha = (\alpha_1, \dots, \alpha_{N-1})$ and $|\alpha|_\infty = \max\{|\alpha_i| : i = 1, 2, \dots, N-1\} < 1$. Here T possesses a unique fixed point $f \in \mathcal{G}$, which satisfies

$$f(x) = F_i(L_i^{-1}(x), f \circ L_i^{-1}(x)) \ \ \forall x \in [x_i, x_{i+1}].$$

α-fractal functions: Navascués [10] observed that the theory of FIF can be used to generate a family of continuous functions having fractal characteristics from a prescribed continuous function. Consider a partition $\Delta := \{x_1, x_2, \dots, x_N\}$ of I satisfying $x_1 < x_2 < \dots < x_N$, a base function b satisfying $b \in C(I)$, $b \neq f, b(x_1) = f(x_1)$, $b(x_N) = f(x_N)$ and $N-1$ real numbers α_i satisfying $|\alpha_i| < 1$. Define an IFS through the maps

$$L_i(x) = a_i x + d_i, \quad F_i(x, y) = \alpha_i y + f \circ L_i(x) - \alpha_i b(x), \, i \in \mathbb{N}_{N-1}, \tag{3}$$

where L_i and F_i satisfies (1) and (2) respectively and b is defined through the linear map $L : C(I) \to C(I)$ such that L is bounded with respect to sup norm and satisfy $Lf(x_1) = f(x_1)$ and $Lf(x_N) = f(x_N)$.
The corresponding FIF denoted by $f^\alpha_{\Delta,b} = f^\alpha$ is referred as α-fractal function for f with respect to a scale vector α base functions b and partition Δ.

From (3) the following uniform error bound can be found (see for instance [11]),

$$\|f^\alpha - f\|_\infty \le \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|f^\alpha - b\|_\infty. \tag{4}$$

2.2 Some classical results:

Dunham Jackson deduced several inequalities to compute the uniform distance between a continuous (or differentiable) function and the space of trigonometric or algebraic polynomials. For instance in the periodic case we have the following results ([6],[7],[9]).

Theorem 2. *Let $f \in \mathcal{C}[-\pi, \pi]$ and be periodic. If $d_n^*(f) = d(f, \pi_n)$, where $d(f, \pi_n)$ is the minimum distance between f and the space*

$$\pi_n = \Big\{ \sum_{k=0}^{n} \Big(a_k \cos(kx) + b_k \sin(kx) \Big); a_k, b_k \in R \Big\},$$

then $d_n^*(f) \le \omega\Big(\frac{\pi}{n+1}\Big)$, where $\omega(\delta)$ is the modulus of continuity of f.

Theorem 3. *If $f \in \mathcal{C}^2[-\pi, \pi]$ and is periodic, then*

$$|a_k| = \mathcal{O}\Big(\frac{1}{k^2}\Big), \qquad |b_k| = \mathcal{O}\Big(\frac{1}{k^2}\Big)$$

and

$$\|f - S_n\|_\infty = \mathcal{O}(n^{-1}),$$

where S_n is the n-th Fourier sum, and a_k, b_k are its coefficients.

Theorem 4. *If $f \in \mathcal{C}[-\pi,\pi]$, is periodic and satisfies a Dini-Lipschitz condition $\lim\limits_{\delta\to 0}\log(\delta)\omega(\delta)=0$, then the Fourier series converges uniformly to f*

This result is achieved by the Jackson approximants with the single hypothesis of continuity (Section 3), and let us remember that uniform convergence implies convergence in the mean of order p ($\mathcal{L}^p$-norm) on compact intervals for $1 \leq p \leq \infty$.
The advantage of the Jackson approximants is the fact of being explicit in terms of the sampled values of the original function.

3 Generalization of Discrete Jackson approximants

We consider a type of Jackson approximants. They are defined in a discrete way, using only the samples of the function to be approached. We consider arbitrary but positive exponents $\gamma > 0$. In the Jackson case the value of the exponent is $\gamma = 4$ ([**?**], p. 456). Consider $P_{m,i,\gamma}(x) = \left|\frac{\sin\left(\frac{m(x_i-x)}{2}\right)}{m\sin\left(\frac{x_i-x}{2}\right)}\right|^{\gamma}$,

$$D_{m,\gamma}(f)(x) = H_{m,\gamma}\sum_{i=1}^{2m} f(x_i)P_{m,i,\gamma}(x), \tag{5}$$

where

$$H_{m,\gamma}^{-1} = \sum_{i=1}^{2m} P_{m,i,\gamma}(x) \text{ and } x_{i+1}-x_i = \frac{\pi}{m}, \text{ for } i=1,2,\ldots,2m-1. \tag{6}$$

For a positive exponent $\gamma > 0$, $H_{m,\gamma} = H_{m,\gamma}(x)$ depends on the variable x although we will preserve the original notation omitting it. $H_{m,4}$ is a constant.

An important identity involving the basic functions in (5) is ([7], p. 340): For $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\left(\frac{\sin(\frac{nt}{2})}{\sin(\frac{t}{2})}\right)^2 = n + 2\big((n-1)\cos(t) + (n-2)\cos(2t) + \ldots + \cos((n-1)t)\big). \tag{7}$$

Lemma 1. *For all $m = 1,2,\ldots$; $\gamma > 0$, and $v \in R$:*

$$\left|\frac{\sin(mv)}{m\sin(v)}\right|^{\gamma} \leq 1.$$

Proof. Using the identity (7) for $n = m$ and $t = 2v$:

$$\left(\frac{\sin(mv)}{\sin(v)}\right)^2 = m + 2\big((m-1)\cos(2v) + (m-2)\cos(4v) + \ldots + \cos(2(m-1)v)\big).$$

Then

$$\Big|\frac{\sin(mv)}{m\sin(v)}\Big|^{\gamma} = \frac{1}{m^{\gamma}}|m+2\big((m-1)\cos(2v)+(m-2)\cos(4v)+\ldots+\cos(2(m-1)v)\big)|^{\gamma/2}$$
$$\leq \frac{1}{m^{\gamma}}(m+2(\frac{1+(m-1)}{2})(m-1))^{\gamma/2}$$
$$\leq \frac{1}{m^{\gamma}}(m^2)^{\gamma/2}.$$

$\diamond$

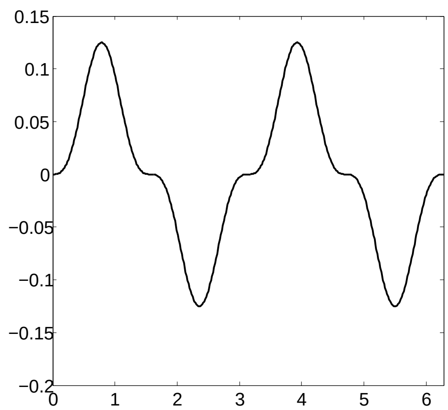

Fig. 1. Graph of the function $f(x) = (\sin x \cos x)^3$ in the interval $[0, 2\pi]$.

Theorem 5. *Let $f \in \mathcal{C}[-\pi,\pi]$ be Hölder continuous such that for $x, x' \in [-\pi,\pi]$*

$$|f(x) - f(x')| \leq K|x - x'|^q, 0 < q \leq 1. \tag{8}$$

Then for $\gamma > q+1$,
$\|D_{m,\gamma}(f) - f\|_\infty \leq \frac{K}{2}(\frac{\pi}{2})^{\gamma}(\frac{\pi}{2m})^q\Big(1 + 2^{\gamma}\big(\zeta(\gamma-q)+\zeta(\gamma)\big)\Big),$
where $\zeta(s)$ is the Riemann zeta function:

$$\zeta(s) = \sum_{i=1}^{+\infty}\frac{1}{i^s}. \tag{9}$$

Proof. According to the definitions of $H_{m,\gamma}$ and $D_{m,\gamma}$, and the change

$$x_i = x + 2u_i,$$

we have:

$$|E_{m,\gamma}(f)(x)| = |D_{m,\gamma}(f)(x) - f(x)| \leq H_{m,\gamma}\sum_{i=1}^{2m}|f(x+2u_i) - f(x)|P_{m,i,\gamma}(u). \tag{10}$$

Using the Lipschitz constant and exponent (5)

$$|E_{m,\gamma}(f)(x)| \leq H_{m,\gamma} K 2^q \sum_{i=1}^{2m} |u_i|^q |P_{m,i,\gamma}(x_i - 2u_i),$$

where due to periodicity, we can assume $u_i \in [-\pi/2, \pi/2]$. Considering increasing order in $|u_i|$ and denoting them by $v_0, v_1, \ldots, v_{2m-1}$:

$$|E_{m,\gamma}(f)(x)| \leq H_{m,\gamma} K 2^q \sum_{i=0}^{2m-1} v_i^q P_{m,i,\gamma}(x_i - 2v_i), \tag{11}$$

where,

$$\frac{\pi i}{4m} \leq v_i \leq \frac{\pi(i+1)}{4m} \leq \frac{\pi}{2}, \tag{12}$$

for $i = 0, 1, \ldots, 2m-1$ ([?], p. 458). Here $P_{m,i,\gamma}(x_i - 2v_i) = |\frac{\sin(mv_i)}{m\sin(v_i)}|^\gamma$. Now from Lemma 1

$$\Big|\frac{\sin(mv_0)}{m\sin(v_0)}\Big|^\gamma \leq 1. \tag{13}$$

For $i \geq 1$, using the inequality

$$\sin(v) \geq \frac{2v}{\pi}, \tag{14}$$

for $v \in [0, \pi/2]$,

$$m\sin(v_i) \geq m\sin\Big(\frac{\pi i}{4m}\Big) \geq m2\frac{i}{4m} \geq \frac{i}{2}.$$

As a consequence, for $i \geq 1$,

$$\Big|\frac{\sin(mv_i)}{m\sin(v_i)}\Big|^\gamma \leq \Big(\frac{2}{i}\Big)^\gamma. \tag{15}$$

Using (11) we obtain

$$|E_{m,\gamma}(f)(x)| \leq H_{m,\gamma} K 2^q \sum_{i=0}^{2m-1} v_i^q \Big|\frac{\sin(mv_i)}{m\sin(v_i)}\Big|^\gamma \leq H_{m,\gamma} K 2^q \Big[v_0^q + \sum_{i=1}^{2m-1} v_i^q \Big(\frac{2}{i}\Big)^\gamma\Big]. \tag{16}$$

By definition of v_i (7):

$$v_0^q \leq \Big(\frac{\pi}{4m}\Big)^q, \tag{17}$$

$$\sum_{i=1}^{2m-1} v_i^q \Big(\frac{2}{i}\Big)^\gamma \leq 2^\gamma \Big(\frac{\pi}{4m}\Big)^q \sum_{i=1}^{2m-1} (i+1)^q \frac{1}{i^\gamma}.$$

For $0 \leq q \leq 1$, $(i+1)^q \leq (i^q + 1)$,

$$\sum_{i=1}^{2m-1} v_i^q \Big(\frac{2}{i}\Big)^\gamma \leq 2^\gamma \Big(\frac{\pi}{4m}\Big)^q \sum_{i=1}^{2m-1} (i^q+1)\frac{1}{i^\gamma} = 2^\gamma \Big(\frac{\pi}{4m}\Big)^q \sum_{i=1}^{2m-1} \Big(\frac{1}{i^{\gamma-q}} + \frac{1}{i^\gamma}\Big)$$

and thus

$$\sum_{i=1}^{2m-1} v_i^q\Big(\frac{2}{i}\Big)^{\gamma} \le 2^{\gamma}\Big(\frac{\pi}{4m}\Big)^q\Big(\zeta(\gamma-q)+\zeta(\gamma)\Big), \tag{18}$$

where $\zeta(s)$ is the Riemann zeta function (9), convergent for $s>1$.
If $\gamma > q+1$, collecting all the inequalities ((16), (17), (18)):

$$|E_{m,\gamma}(f)(x)| \le H_{m,\gamma}K2^q\Big[\Big(\frac{\pi}{4m}\Big)^q + 2^{\gamma}\Big(\frac{\pi}{4m}\Big)^q\Big(\zeta(\gamma-q)+\zeta(\gamma)\Big)\Big]. \tag{19}$$

Let us bound now $H_{m,\gamma}$ (6).

$$H_{m,\gamma}^{-1} = \sum_{i=0}^{2m-1}\Big|\frac{\sin(mv_i)}{m\sin(v_i)}\Big|^{\gamma} > \Big|\frac{\sin(mv_0)}{m\sin(v_0)}\Big|^{\gamma} + \Big|\frac{\sin(mv_1)}{m\sin(v_1)}\Big|^{\gamma}. \tag{20}$$

Since $mv_i \le \frac{\pi}{2}$ for $i=0,1$, from (7) and (14),

$$\frac{\sin(mv_i)}{m\sin(v_i)} \ge \frac{2mv_i}{m\pi\sin(v_i)} \ge \frac{2}{\pi},$$

and thus (20),

$$H_{m,\gamma}^{-1} > 2\Big(\frac{2}{\pi}\Big)^{\gamma}. \tag{21}$$

Finally (19),

$$|E_{m,\gamma}(f)(x)| \le \frac{K}{2}(\frac{\pi}{2})^{\gamma}(\frac{\pi}{2m})^q\big(1+2^{\gamma}(\zeta(\gamma-q)+\zeta(\gamma)),$$

if $\gamma > q+1$. ⋄

The former result is refined in the next Theorem.

Theorem 6. *Let $f \in \mathcal{C}[-\pi,\pi]$ be Hölder continuous such that for $x, x' \in [-\pi,\pi]$,*

$$|f(x)-f(x')| \le K|x-x'|^q,\ 0<q\le 1.$$

Then for $\gamma > q+1$,
$\|D_{m,\gamma}(f)-f\|_{\infty} \le \frac{K}{2}(\frac{\pi}{2})^{\gamma}(\frac{\pi}{2m})^q\Big(1+2^q+2^{\gamma}\big(\frac{1}{\gamma-(q+1)}+\frac{1}{\gamma-1}\big)\Big).$

Proof. We proceed as in the previous Theorem until (11),

$$|E_{m,\gamma}(f)(x)| \le H_{m,\gamma}K2^q\sum_{i=0}^{2m-1} v_i^q\Big|\frac{\sin(mv_i)}{m\sin(v_i)}\Big|^{\gamma}. \tag{22}$$

Bearing in mind the inequality (15) for $i \ge 2$, one obtains

$$|E_{m,\gamma}(f)(x)| \le H_{m,\gamma}K2^q\sum_{i=0}^{2m-1} v_i^q\Big|\frac{\sin(mv_i)}{m\sin(v_i)}\Big|^{\gamma} \le H_{m,\gamma}K2^q\Big(v_0^q+v_1^q+\sum_{i=2}^{2m-1} v_i^q\Big(\frac{2}{i}\Big)^{\gamma}\Big).$$

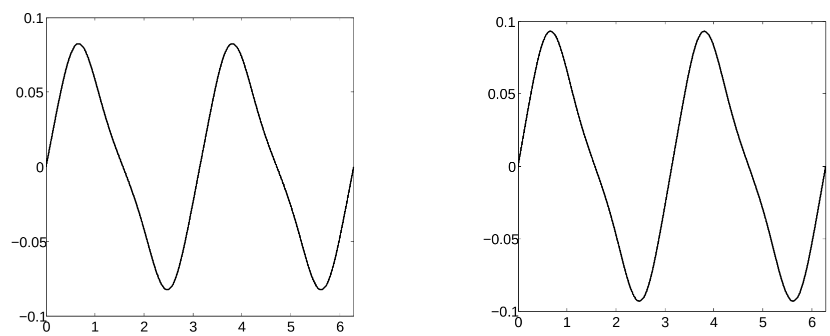

Fig. 2. Approximant $D_{m,\gamma}(f)$ of a sampled signal for $m = 5$, $\gamma = 4$(Jackson case) and $\gamma = 5$ in the interval $[-\pi, \pi]$.

By definition of v_i (7),

$$v_0^q \leq \left(\frac{\pi}{4m}\right)^q, \tag{23}$$

$$v_1^q \leq \left(\frac{\pi}{2m}\right)^q, \tag{24}$$

and

$$\sum_{i=2}^{2m-1} v_i^q \left(\frac{2}{i}\right)^\gamma \leq 2^\gamma \left(\frac{\pi}{4m}\right)^q \sum_{i=2}^{2m-1} (i+1)^q \frac{1}{i^\gamma}.$$

As before

$$\sum_{i=2}^{2m-1} v_i^q \left(\frac{2}{i}\right)^\gamma \leq 2^\gamma \left(\frac{\pi}{4m}\right)^q \sum_{i=2}^{2m-1} (i^q+1) \frac{1}{i^\gamma} = 2^\gamma \left(\frac{\pi}{4m}\right)^q \sum_{i=2}^{2m-1} \left(\frac{1}{i^{\gamma-q}} + \frac{1}{i^\gamma}\right). \tag{25}$$

The last sum is part of lower Riemann sums of the functions $1/x^{\gamma-q}$ and $1/x^\gamma$, in the inteval $[1, +\infty)$ with unit step, respectively:

$$\sum_{i=2}^{2m-1} \frac{1}{i^{\gamma-q}} \leq \int_1^\infty \frac{dx}{x^{\gamma-q}} = \frac{1}{\gamma-(q+1)},$$

if $\gamma > q+1$. Likewise

$$\sum_{i=2}^{2m-1} \frac{1}{i^\gamma} \leq \int_1^\infty \frac{dx}{x^\gamma} = \frac{1}{\gamma-1},$$

if $\gamma > 1$. Collecting all the inequalities ((23), (24), (25)):

$$|E_{m,\gamma}(f)(x)| \leq H_{m,\gamma} K 2^q \Big[\Big(\frac{\pi}{4m}\Big)^q + \Big(\frac{\pi}{2m}\Big)^q + 2^\gamma \Big(\frac{\pi}{4m}\Big)^q \Big(\frac{1}{\gamma-(q+1)} + \frac{1}{\gamma-1}\Big)\Big].$$

Finally,

$$|E_{m,\gamma}(f)(x)| \leq \frac{K}{2}(\frac{\pi}{2})^\gamma(\frac{\pi}{2m})^q\Big(1 + 2^q + 2^\gamma(\frac{1}{\gamma-(q+1)} + \frac{1}{\gamma-1})\Big),$$

if $\gamma > q+1$. $\diamond$

4 Fractal version of discrete Jackson approximants

Some easy and elegant observations on fractal functions in conjunction with the theorems from section 3, provide their corresponding fractal versions. Such kind of fractal approximants possess many desirable properties as their traditional counterparts and also reassures the ubiquity of fractal functions.
As described in section 2, to get the fractal version we have to perturb the basis function $P_{m,i,\gamma}(x)$ using suitable base functions $b_{i,m,\gamma}$, suitable scaling vector α_m and partition of $[-\pi,\pi]$ and define the fractal discrete Jackson approximants as

$$D^\alpha_{m,\gamma}(f)(x) = H_{m,\gamma} \sum_{i=1}^{2m} f(x_i) P^\alpha_{m,i,\gamma}(x).$$

The following generalizes the Theorem 6.

Theorem 7. *Let $f \in \mathcal{C}[-\pi,\pi]$ be Hölder continuous such that for $x, x' \in [-\pi,\pi]$,*

$$|f(x) - f(x')| \leq K|x-x'|^q, 0 < q \leq 1.$$

Then for $\gamma > q+1$,

$$\|D^\alpha_{m,i,\gamma}(f) - f\|_\infty \leq \tfrac{K}{2}(\tfrac{\pi}{2})^\gamma(\tfrac{\pi}{2m})^q\Big(1 + 2^\gamma\big(\zeta(\gamma-q) + \zeta(\gamma)\big)\Big) + m(\tfrac{\pi}{2})^\gamma \|f\|_\infty \tfrac{|\alpha|_\infty}{1-|\alpha|_\infty} \max_{1\leq i\leq 2m} \|P^\alpha_{m,i,\gamma} - b_{m,i,\gamma}\|_\infty,$$

where $\alpha, b_{m,i,\gamma}$ are the suitable scaling vector and basis function used to construct the fractal perturbation of $P_{m,i,\gamma}$.

Proof. From the triangle inequality we have:

$$\|D^\alpha_{m,\gamma}(f) - f\|_\infty \leq \|D^\alpha_{m,\gamma}(f) - D_{m,\gamma}(f)\|_\infty + \|D_{m,\gamma}(f) - f\|_\infty.$$

Under the stated hypothesis from Theorem 5 we have:

$$\|D_{m,\gamma}(f) - f\|_\infty \leq \frac{K}{2}(\frac{\pi}{2})^\gamma(\frac{\pi}{2m})^q\Big(1 + 2^\gamma\big(\zeta(\gamma-q) + \zeta(\gamma)\big)\Big). \tag{26}$$

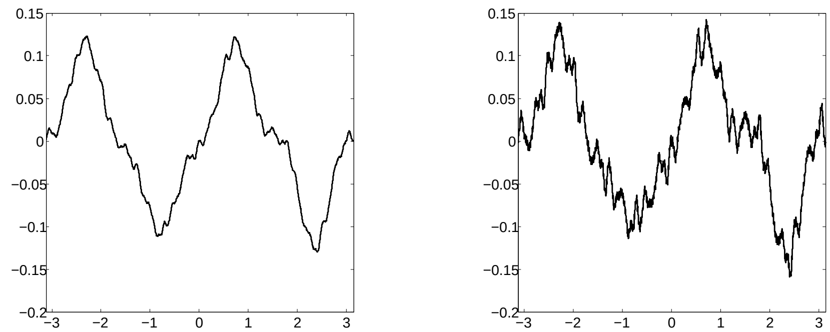

Fig. 3. Approximant $D^{\alpha}_{m,\gamma}(f)$ of a sampled signal for $m = 12$, $\gamma = 5$ and in the interval $[-\pi, \pi]$ for different values of α.

Now

$$\begin{aligned} |D^{\alpha}_{m,\gamma}(f)(x) - D_{m,\gamma}(f)(x)| &= |H_{m,\gamma} \sum_{i=1}^{2m} f(x_i)(P^{\alpha}_{m,i,\gamma})(x) - P_{m,i,\gamma}(x)| \\ &\leq H_{m,\gamma} \sum_{i=1}^{2m} |f(x_i)||(P^{\alpha}_{m,i,\gamma})(x) - P_{m,i,\gamma}(x)| \\ &\leq H_{m,\gamma} 2m \|f\|_\infty \sum_{i=1}^{2m} |(P^{\alpha}_{m,i,\gamma})(x) - P_{m,i,\gamma}(x)| \\ &\leq (\frac{\pi}{2})^{\gamma} m \|f\|_\infty \max_{1\leq i\leq 2m} \|P^{\alpha}_{m,i,\gamma} - P_{m,i,\gamma}\|_\infty. \end{aligned} \tag{27}$$

Finally, using (4), (26) and (27)the proof follows. ◇

The next result is the refined version of the above theorem.

Theorem 8. *Let $f \in \mathcal{C}[-\pi, \pi]$ be Hölder continuous such that for $x, x' \in [-\pi, \pi]$,*

$$|f(x) - f(x')| \leq K|x - x'|^q,\ 0 < q \leq 1.$$

Then for $\gamma > q + 1$,

$$\|D_{m,\gamma}(f) - f\|_\infty \leq \tfrac{K}{2}(\tfrac{\pi}{2})^{\gamma}(\tfrac{\pi}{2m})^q\Big(1 + 2^q + 2^{\gamma}\big(\tfrac{1}{\gamma-(q+1)} + \tfrac{1}{\gamma-1}\big)\Big)$$
$$+ m(\tfrac{\pi}{2})^{\gamma}\|f\|_\infty \tfrac{|\alpha|_\infty}{1-|\alpha|_\infty} \max_{1\leq i\leq 2m} \|P^{\alpha}_{m,i,\gamma} - b_{m,i,\gamma}\|_\infty,$$

where $\alpha, b_{m,i,\gamma}$ *are the suitable scaling vector and basis function used to construct the fractal perturbation of* $P_{m,i,\gamma}$.

Proof. The result follows from Theorem 6 and the triangle inequality

$$\|D^{\alpha}_{m,\gamma}(f) - f\|_{\infty} \leq \|D^{\alpha}_{m,\gamma}(f) - D_{m,\gamma}(f)\|_{\infty} + \|D_{m,\gamma}(f) - f\|_{\infty}.$$

Corollary 1. *If* $f \in \mathcal{C}[-\pi, \pi]$ *is Hölder continuous with exponent* q $(0 < q \leq 1)$ *and* $\gamma > q + 1$ *and if we select scaling factors* α *as* $|\alpha_i| \leq \frac{1}{m^{q+1}}$, *for* $i = 1, 2, \ldots N - 1$ *where* N *is the number of partition of* $[-\pi, \pi]$, *then* $D^{\alpha}_{m,\gamma}(f)$ *converges uniformly to* f *as* m *tends to infinity. The order of convergence* $\mathcal{O}(m^{-q})$ *does not depend on* γ.

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