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Some aspects of stochastic calculus and approximation in chaotic systems analysis

Gabriel V. Orman[1]

Department of Mathematics and Computer Science, "Transilvania" University of Braşov, 500091 Braşov, Romania.
(E-mail: ogabriel@unitbv.ro)

Abstract. Frequently when we refer to *chaos* and *chaotic and complex systems* to describe the comportment of some natural phenomena, in fact we consider phenomena of the type of a *Brownian motion* which is a more realistic model of such phenomena. Thus one can talk about a passing *from chaotic and complex systems to Brownian motion*. Some aspects regardind the Brownian motion and its Markovian nature will be developed, in short, in this paper; we try also to emphasize their impact for some practical problems.
Keywords: stochastic differential equations, stochastic calculus, Markov processes, Brownian motion..

1 Introduction

It is known that a chaotic perpetual motion of a Brownian particle is the result of the collisions of particle with the molecules of the fluid in which there is.

But this particle is much bigger and also heavier than the molecules of the fluid which it collide, and then each collision has a negligible effect, while the superposition of many small interactions will produce an observable effect.

On the other hand, for a Brownian particle such molecular collisions appear in a very rapid succession, their number being enormous. For a so high frequency, evidently, the small changes in the particle's path, caused by each single impact, are too fine to be observable. For this reason the exact path of the particle can be described only by statistical methods.

We emphasize that L. Bachélier derived the law governing the position of a single grain performing a 1-dimensional Brownian motion starting at $a \in R$ at time $t = 0$; and A. Einstein also derived the same law from statistical mechanical considerations and applied it to the determination of molecular diameters.

Also Paul Lévy found a construction of the Brownian motion and given a profound description of the fine structure of the individual Brownian path. D. Ray obtained some results in the case when the motion is *strict Markov*;

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and W. Feller obtained that the generator of a strict Markovian motion with continuous paths (diffusion) can be expressed as a *differential operator*.

And in the last time we can speak about Markov processes from Kiyosi Itô's perspective (according to D.W. Stroock). The usual class of Markov processes which we consider has many times some restrictions which do not cover many interesting processes. This is the reason for which we try often to obtain some extensions of this notion.

Researches in this direction are due especially to K. Itô and in this context we shall refer below, in short, to some of them.

2 On Markov processes - an extended definition

We start with the concept of a random variable which encodes an experimental outcome as a number, or a vector of real numbers in the multidimensional case. When a random variable has a multidimensional state space, we emphasize that fact by calling it a *random space*.

Let (E, ξ) be a measurable space and $X : (\Omega, \mathcal{K}, P) \to (E, \xi)$ a random variable (i.e. a measurable map).

The image μ of P under X is a probability measure on (E, ξ) called the *law of* X and denoted by $\mathcal{L}(X)$. The events $\{\omega \mid X(\omega) \in A\}$ for $A \in \xi$ form a sub-σ-field of $\mathcal{K}$ called the σ-field *generated* by X and denoted by $\sigma(X)$.

More general, given a family X_α, $\alpha \in I$, of random variables on $(\Omega, \mathcal{K}, P)$ taking values in measurable spaces (E_α, ξ_α), $\alpha \in I$, respectively, the σ-field generated by X_α, $\alpha \in I$, denoted by $\sigma(X_\alpha, \alpha \in I)$, is the smallest sub-$\sigma$-field with respect to which they are all measurable.

They may be situations where it is preferable to view $\{X_\alpha, \alpha \in I\}$ as a single random variable taking values in the product space $\prod E_\alpha$ endowed with the product σ-field $\prod \xi_\alpha$.

If so, this definition reduces to the following:

Definition 21 *Let $(\Omega, \mathcal{K}, P)$ be a probability space and let us denote by E a subset of R^n. A "random variable" X is a function from Ω into E.*

E is referred to as the *state space* of the random variable.

Suppose we have n random variables $X_1(\omega), \cdots, X_n(\omega)$ defined on a probability space.

The random variables $X_1, \cdots, X_n$ are said to be *independent* if the fields (σ-fields) $\mathcal{K}_{X_1}, \cdots, \mathcal{K}_{X_n}$ generated by them are independent.

Definition 22 *A "stochastic process" is a family of real random variables*

$$\{X_t\}_{t \in T}$$

defined on a probability space $(\Omega, \mathcal{K}, P)$, indexed with a time parameter t and assuming values in R^n.

The parameter space T may be the halfline $[0, +\infty)$, or it may also be an interval $[a, b]$, or the non-negative integers and even subsets of R^n, for $n \geq 1$.

Now, for each $t \in T$ fixed, we have a random variable $\omega \to X_t(\omega),\ \omega \in \Omega$.

A stochastic process will be denoted by $X(t)$.

Now let S be a *state space* and consider a particle which moves in S. Also, suppose that the particle starting at x at the present moment will move into the set $A \subset S$ with probability $p_t(x, A)$ after t units of time, "irrespectively of its past motion", that is to say, this motion is considered to have a *Markovian character*.

The *transition probabilities* of this motion are $\{p_t(x, A)\}_{t,x,A}$ and we considered that the time parameter $t \in T = [0, +\infty)$.

The state space S is assumed to be a *compact Hausdorff space with a countable open base*. The σ-field generated by the open sets (the topological σ-field on S) is denoted by $K(S)$. Therefore, a *Borel set* A is a set in $K(S)$ (i.e. $A \in K(S)$).

The *mean value*

$$m = M(\mu) = \int_R x\,\mu(dx)$$

is used for the center and the scattering degree of a one-dimensional probability measure μ having the second order moment finite, and the *variance* of μ is defined by

$$\sigma^2 = \sigma^2(\mu) = \int_R (x - m)^2 \mu(dx).$$

On the other hand, from the Tchebychev's inequality, for any $t > 0$, we have

$$\mu(m - t\sigma, m + t\sigma) \le \frac{1}{t^2},$$

so that several properties of 1-dimensional probability measures can be derived.

Remark 1. In the case when the considered probability measure has no finite second order moment, σ becomes useless. In such a case one can introduce the central value and the dispersion that will play similar roles as m and σ for general 1-dimensional probability measures.

The *dispersion* δ is defined as follows

$$\delta = \delta(\mu) = -\log \int\int_{R^2} e^{-|x-y|}\mu(dx)\mu(dy).$$

Furthermore it is assumed that the following conditions are satisfied by the transition probabilities $\{p_t(x, A)\}_{t\in T, x\in S, A\in K(S)}$:

1 for t and A fixed,
  a) the transition probabilities are Borel measurable in x;
  b) $p_t(x, A)$ is a probability measure in A;

2 $p_0(x, A) = \delta_x(A)$ (i.e. the δ-measure concentrated at x);

3 $p_t(x, \cdot) \xrightarrow{\text{weak}} p_t(x_0, \cdot)$ as $x \to x_0$ for any t fixed, that is

$$\lim_{x\to x_0} \int f(y)p_t(x, dy) = \int f(y)p_t(x_0, dy)$$

for all continuous functions f on S;

4 $p_t(x, U(x)) \longrightarrow 1$ as $t \searrow 0$, for any neighborhood $U(x)$ of x;
5 the Chapman-Kolmogorov equation holds:

$$p_{s+t}(x, A) = \int_S p_t(x, dy) p_s(y, A).$$

Remark 2. The "transition operators" can be defined in a similar manner. Consider $C = C(S)$ to be the space of all continuous functions (it is a separable Banach space with the supremum norm).

The operators p_t, defined by

$$(p_t f)(x) = \int_S p_t(x, dy) f(y), \quad f \in C$$

are called "transition operators".

Remark 3. Let us consider $R \cup \{\infty\}$ as the one-point compactification of R. Then it can be observed that the conditions (1) – (5) above are satisfied for the "Brownian transition probabilities". One can define

$$\begin{array}{l} p_t(x, dy) = \frac{1}{t\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2t^2}} dy \quad \text{in } R \\ p_t(\infty, A) = \delta_\infty A. \end{array}$$

We can give now the definition of a Markov process as follows:

Definition 23 *A "Markov process" is a system of stochastic processes*

$$\{X_t(\omega), t \in T, \omega \in (\Omega, K, P_a)\}_{a \in S},$$

that is for each $a \in S$, $\{X_t\}_{t \in S}$ is a stochastic process defined on the probability space (Ω, K, P_a).

It can be observed that a definition as it is given above not correspond to many processes that are of a real interest so that it is useful to obtain an extension of this notion. An extended notion has been proposed by K. Itô and it is given below.

Let E be a separable Banach space with real coefficients and norm $||\cdot||$ and let also $L(E, E)$ be the space of all bounded linear operators $E \longrightarrow E$. It can be observed that $L(E, E)$ is a linear space.

Definition 24 *The collection of stochastic processes*

$$X = \{X_t(\omega) \equiv \omega(t) \in S,\ t \in T,\ \omega \in (\Omega, K, P_a)\}_{a \in S}$$

is called a "Markov process" if the following conditions are satisfied:

1) *the "state space" S is a complete separable metric space and $K(S)$ is a topological σ-algebra on S;*
2) *the "time internal" $T = [0, \infty)$;*
3) *the "space of paths" Ω is the space of all right continuous functions $T \longrightarrow S$ and K is the σ-algebra $K[X_t : t \in T]$ on Ω;*

4) *the probability law of the path starting at a, $P_a(H)$, is a probability measure on (Ω, K) for every $a \in S$ which satisfy the following conditions:*
4a) $P_a(H)$ *is* $K(S)$*-measurable in a for every* $H \in K$*;*
4b) $P_a(X_0 = a) = 1$*;*
4c) $P_a(X_{t_1} \in E_1, \cdots, X_{t_n} \in E_n) =$

$$\int \cdots \int_{a_i \in E_i} P_a(X_{t_1} \in da_1) P_{a_1}(X_{t_2 - t_1} \in da_2) \ldots$$
$$\ldots P_{a_{n-1}}(X_{t_n - t_{n-1}} \in da_n) \qquad \textit{for} \quad 0 < t_1 < t_2 < \ldots < t_n.$$

Remark 4. Evidently there are some differences between this definition and Definition 23 of a Markov process. Thus

i. The space S is not necessary to be compact;
ii. it is not assumed the existence of the left limits of the path;
iii. the transition operator $f \longrightarrow G_t f(\cdot) = E$.
$(f(X_t))$ do not necessarily carry $C(S)$ into $C(S)$ ($C(S)$ being the space of all real-valued bounded continuous functions on S).

3 The Markovian nature of the Brownian path

As we already emphasized the Brownian motion, used especially in Physics, is of ever increasing importance not only in Probability theory but also in classical Analysis. Its fascinating properties and its far-reaching extension of the simplest normal limit theorems to functional limit distributions acted, and continue to act, as a catalyst in random analysis.

It is probable the most important stochastic process.

As some authors remarks too, the Brownian motion reflects a perfection that seems closer to a law of nature than to a human invention.

In 1828 the English botanist Robert Brown observed that pollen grauns suspended in water perform a continual swarming motion. The chaotic motion of such a particle is called *Brownian motion* and a particle performing such a motion is called a *Brownian particle*.

He was not the first to mention this phenomenon and had many predecessors, starting with Leeuwenhoek in the 17^{th} century.

However, Brown's investigation brought it to the attention of the scientific community, hence *Brownian*.

Brownian motion was frequently explained as due to the fact that particles were alive. Poincaré thought that it contradicted the second law of Thermodynamics.

Today we know that this motion is due to the bombardament of the particles by the molecules of the medium. In a liquid, under normal conditions, the order of magnitude of the number of these impacts is of 1020 per second.

It is only in 1905 that kinetic molecular theory led Einstein to the first mathematical model of Brownian motion. He began by deriving its possible existence and then only learned that it had been observed.

Let us imagine a chaotic motion of a particle of colloidal size immersed in a fluid. As we already emphasized such a chaotic motion of a particle is called, usually, *Brownian motion* and the particle which performs such a motion is referred to as a *Brownian particle.* Such a chaotic perpetual motion of a Brownian particle is the result of the collisions of particle with the molecules of the fluid in which there is.

But this particle is much bigger and also heavier than the molecules of the fluid which it collide, and then each collision has a negligible effect, while the superposition of many small interactions will produce an observable effect.

On the other hand, for a Brownian particle such molecular collisions appear in a very rapid succession, their number being enormous. For a so high frequency, evidently, the small changes in the particle's path, caused by each single impact, are too fine to be observable. For this reason the exact path of the particle can be described only by statistical methods.

Thus, the influence of the fluid on the motion of a Brownian particle can be described by the combination of two forces in the following way.

1. The considered particle is much larger than the particle of the fluid so that the cumulated effect of the interaction between the Brownian particle and the fluid may be taken as having a hydrodynamical character. Thus, the first of the forces acting on the Brownian particle may be considered to be the forces of *dynamical friction.* It is known that the frictional force exerted by the fluid on a small sphere immersed in it is determined from the Stockes's law: *the drag force per unit mass acting on a spherical particle of radius a is given by $-\beta\mathbf{v}$, with $\beta = \dfrac{6\pi a\eta}{m}$*, where m is the mass of the particle, η is the coefficient of dynamical viscosity of the fluid, and $\mathbf{v}$ is the velocity of particle.
2. The other force acting on the Brownian particle is caused by the individual collisions with the particles of the fluid in which there is. This force produces instantaneous changes in the acceleration of the particle. Furthermore, this force is *random both in direction and in magnitude*, and one can say that it is a *fluctuating force.* It will be denoted by $\mathbf{f}(\mathbf{t})$. For $\mathbf{f}(\mathbf{t})$ the following assumptions are made:
   - *i* The function $\mathbf{f}(\mathbf{t})$ is statistically independent of $\mathbf{v}(t)$.
   - *ii* $\mathbf{f}(\mathbf{t})$ has variations much more frequent than the variations in $\mathbf{v}(t)$.
   - *iii* $\mathbf{f}(\mathbf{t})$ has the average equal to zero.

A completely different origin of mathematical Brownian motion is a game theoretic model for fluctuations of stock prices due to L. Bachélier from 1900.

In his doctoral thesis L. Bachélier hinted that it could apply to physical Brownian motion.

Therein, and in his subsequent works, he used the heat equation and, proceeding by analogy with *heat propagation* he found, albeit formally, distributions of various functionals of mathematical Brownian motion.

Heat equations and related parabolic type equations have been used rigorously by Kolmogorov, Petrovsky, Khintchine.

Bachélier, L. *Théorie de la spéculation.* Ann. Sci. École Norm. Sup., 17, 1900, 21-86.

L. Bachélier derived the law governing the position of a single grain performing a 1-dimensional Brownian motion starting at $a \in \mathbf{R^1}$ at time $t = 0$:

$$P_a[x(t) \in db] = g(t, a, b)db \qquad (t, a, b) \in (0, +\infty) \times \mathbf{R^2}, \tag{1}$$

where g is the source (Green) function

$$g(t, a, b) = \frac{e^{-\frac{(b-a)^2}{2t}}}{\sqrt{2\pi t}} \tag{2}$$

of the problem of heat flow:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial a^2} \qquad (t > 0). \tag{3}$$

Bachélier also pointed out the Markovian nature of the Brownian path expressed in

$$\begin{aligned} &P_a[a_1 \le x(t_1) < b_1,\, a_2 \le x(t_2) < b_2,\, \cdots,\, a_n \le x(t_n) < b_n] = \\ &= \int_{a_1}^{b_1}\int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} g(t_1,\, a,\, \xi_1)\, g(t_2 - t_1, \xi_1, \xi_2) \cdots \\ &\cdots g(t_n - t_{n-1},\, \xi_{n-1},\, \xi_n)\, d\xi_1\, d\xi_2 \cdots d\xi_n, \quad 0 < t_1 < t_2 < \cdots t_n \end{aligned} \tag{4}$$

and used it to establish the law of maximum displacement

$$P_0\left[\max_{s \le t} x(s) \le b\right] = 2 \int_0^b \frac{e^{-\frac{a^2}{2t}}}{\sqrt{2\pi t}}\, da \qquad t > 0,\ b \ge 0. \tag{5}$$

It is very interesting that A. Einstein, in 1905, also derived (1) from statistical mechanical considerations and applied it to the determination of molecular diameters.

We emphasize again that a rigorous definition and study of Brownian motion requires measure theory.

But as soon as the ideas of Borel, Lebesgue and Daniell appeared, it was possible to put the Brownian motion on a firm mathematical foundation and this was achived in 1923 by N. Wiener.

Consider the space of *continuous* path $w : t \in [0, +\infty) \to R$ with coordinates $x(t) = w(t)$ and let $\mathbf{B}$ be the smallest Borel algebra of subsets B of this path space which includes all the simple events $B = (w : a \le x(t) < b)$, $(t \ge 0,\, a < b)$. Wiener established the existence of nonnegative Borel measures $P_a(B)$, $(a \in R,\, B \in \mathbf{B})$ for which (4) holds. Among other things, this result attaches a precise meaning to Bachélier's statement that *the Brownian path is continuous*.

As we already emphasized at the beginning, Paul Lévy found another construction of the Brownian motion and gives a profound description of the fine structure of the individual Brownian path.

The standard Brownian motion can be now defined.

A. Einstein, *Investigations on the theory of the Brownian movement*, New York, 1956.
N. Wiener, *Differential space.* J. Math. Phys. 2, 1923, 131-174.

Definition 31 *A continuous-time stochastic process $\{B_t \mid 0 \leq t \leq T\}$ is called a "standard Brownian motion" on $[0,T)$ if it has the following four properties:*

i $B_0 = 0$.
ii *The increments of B_t are independent; that is, for any finite set of times $0 \leq t_1 < t_2 < \cdots < t_n < T$, the random variables*

$$B_{t_2} - B_{t_1},\, B_{t_3} - B_{t_2},\, \cdots, B_{t_n} - B_{t_{n-1}}$$

are independent.
iii *For any $0 \leq s \leq t < T$ the increment $B_t - B_s$ has the normal distribution with mean 0 and variance $t - s$.*
iv *For all ω in a set of probability one, $B_t(\omega)$ is a continuous function of t.*

The Brownian motion can be represented as a random sum of integrals of orthogonal functions. Such a representation satisfies the theoretician's need to prove the existence of a process with the four defining properties of Brownian motion, but it also serves more concrete demands. Especially, the series representation can be used to derive almost all of the most important analytical properties of Brownian motion. It can also give a powerful numerical method for generating the Brownian motion paths that are required in computer simulation.

Remark 5. Let us consider $R \cup \{\infty\}$. Then one can define

$$p_t(x, dy) = \frac{1}{t\sqrt{2\pi}}\, e^{-\frac{(y-x)^2}{2t^2}}\, dy \quad \text{in } R$$
$$p_t(\infty, A) = \delta_\infty A.$$

Let us observe that the conditions 1b) and 2-5 assumed on the transition probabilities $\{p_t(x,A)\}_{t\in T, x\in S, A\in K(S)}$, given in Section 2, are satisfied in this case for "Brownian transition probabilities" where $R \cup \{\infty\}$ is considered as the one-point compactification of R.

Finally we shall give an interesting result regarding to a 3-dimensional Brownian motion.

Let X be a Markov process in a generalized sense as it is given in Definition 24. Let us denote by $\mathbf{B}(S)$ the space of all bounded real $K(S)$-measurable functions and let us consider a function $f \in \mathbf{B}(S)$.

It is supposed that

$$E_a\Big(\int_0^\infty |f(X_t)| dt\Big) \tag{6}$$

is bounded in a. Then, the following

$$Uf(a) = E_a\Big(\int_0^\infty f(X_t) dt\Big) \tag{7}$$

is well-defined and is a bounded $K(S)$-measurable function of $a \in S$.

The Uf is called *the potential* of f with respect to X. Having in view that $Uf = \lim_{\alpha \downarrow 0} R_\alpha f$, it is reasonable to write R_0 instead of U. Based on this fact, $R_\alpha f$ will be called *the potential of order* α of f.

Remark 6. It is useful to retain that $R_\alpha f \in \mathbf{B}(S)$ for $\alpha > 0$; and generally $f \in \mathbf{B}(S)$ while $R_0 f (= Uf) \in \mathbf{B}(S)$ under the condition (6).

Now the name *potential* is justified by the following theorem on *the 3-dimensional Brownian motion*

Theorem 31 *(K. Itô). Let X be the 3-dimensional Brownian motion. If $f \in \mathbf{B}(S)$ has compact support, then f satisfies (6) and*

$$Uf(a) = \frac{1}{2\pi} \int\limits_{R^3} \frac{f(b)db}{|b-a|} = \frac{1}{2\pi} \times Newtonian\ potential\ of\ t. \tag{8}$$

Remark 7. Many other details, proofs and related problems can be found in [1], [2], [3], [4], [14], [6], [13], [12], [9].

Conclusion 31 *The Brownian motion can be represented as a random sum of integrals of orthogonal functions. Such a representation satisfies the theoretician's need to prove the existence of a process with the four defining properties of Brownian motion, but it also serves more concrete demands, one of the most important being the "chaotic and complex systems analysis".*

Especially, the series representation can be used to derive almost all of the most important analytical properties of Brownian motion.

It can also give a powerful numerical method for generating the Brownian motion paths that are required in computer simulation.

At the same time, as we have said at the beginning, we think that when, in various problems, we say "chaos" or "chaotic and complex systems" or we use another similar expression to define the comportment of some natural phenomena, in fact we imagine phenomena similarly to a Brownian motion which is a more realistic model of such phenomena. And this opinion lie at the basis of this paper.

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