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Stability and Bifurcation in the Hénon Map and its Generalizations

O. Ozgur Aybar[1], I. Kusbeyzi Aybar[2], and A. S. Hacinliyan[3]

[1] Gebze Institute of Technology, Department of Mathematics, Kocaeli, Turkey
Yeditepe University, Department of Information Systems and Technologies, Istanbul, Turkey
(E-mail: oaybar@yeditepe.edu.tr)
[2] Yeditepe University, Department of Computer Education and Instructional Technology, Istanbul, Turkey
(E-mail: ikusbeyzi@yeditepe.edu.tr)
[3] Yeditepe University, Department of Information Systems and Technologies, Istanbul, Turkey
Yeditepe University, Department of Physics, Istanbul, Turkey
Bogazici University, Department of Physics, Istanbul, Turkey
(E-mail: ahacinliyan@yeditepe.edu.tr)

Abstract. The Hénon map, its higher iterates and generalizations as given in [1] are studied in this work in the sense of stability and bifurcation analysis

$$\begin{aligned} x_{n+1} &= 1 + y_n - a x_n^2 \\ y_{n+1} &= b x_n^k. \end{aligned} \tag{1}$$

Instances of several forms bifurcations are observed. The second iteration of the generalized Hénon map is of interest since period doubling bifurcation is a prominent mechanism as revealed by the bifurcation map. As we proceed to higher iterations, the position of the bifurcations remain essentially unchanged, the nature of the bifurcations change to include saddle node, Hopf, period doubling bifurcations[1–4]. It is also shown that the delayed version of the Hénon map can be reduced to the logistic map if $k = 1$ and bifurcation scenarios in the one dimensional logistic map, such as period doubling are also observed in the Hénon map.
Keywords: Hénon map, Chaos, Stability, Bifurcation.

1 Introduction

Both iterated maps and flows are used as models for chaotic behavior. It is well known that flows have the same equilibrium points with the maps to which they are related by discretization. The classical example is the logistic map. As a differential equation it has a simpler behavior, however when converted to a map it indicates period doubling bifurcation.

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Bifurcation analysis for both the generalized Hénon map and its higher iterations involving the 2^n fold iteration gives rich structures[1]. The generalized Hénon map and the higher iterates as first studied by Skiadas et al. are taken into consideration in this work [2,5]. The relation between the original Hénon map and the logistic map are also studied and the results given are consistent with the bifurcation diagrams of the original Hénon map[5,6].

In our previous work continuous and discrete versions of predator prey models were studied[7,8]. A similar analysis is done in this work for the iterated Hénon map, its higher iterates and the generalized form and bifurcation properties with rich properties [9,10].

The Hénon map and its generalization is given by the system 1 for $k \geq 1$. In the following sections we give explicit results of stability and bifurcation analysis for various values of k and higher iterations of this dynamical system which defines a generalized version of the Hénon map. For the special case that $k = 1$ this system is known as the original Hénon map which sets an example as a chaotic map for given parameter values. We further generalize the y update formula to $y_{n+1} = bx_{n+1}^k$.

2 Stability and bifurcation properties of the first and second iterations of the original Hénon map

The first iteration of the generalized Hénon map introduced in the previous section for $k = 1$ is considered as the original Hénon map given by:

$$\begin{aligned} x_{n+1} &= 1 + y_n - ax_n^2 \\ y_{n+1} &= bx_n. \end{aligned} \tag{2}$$

The equilibrium points of this system are

$$\begin{aligned} (x_1, y_1) &= (\frac{b-1-\beta}{2a}, \frac{b(b-1-\beta)}{2a}) \\ (x_2, y_2) &= (\frac{b-1+\beta}{2a}, \frac{b(b-1+\beta)}{2a}) \end{aligned} \tag{3}$$

and the eigenvalues at these equilibrium points are

$$\lambda_{1,2} = \{\frac{1}{2}(1 - b + \beta \mp \sqrt{4\beta + (1 + b - \beta)^2}\} \tag{4}$$

and

$$\lambda_{3,4} = \{\frac{1}{2}(1 - b - \beta \mp \sqrt{-4\beta + (1 + b + \beta)^2}\}. \tag{5}$$

where $\beta = \sqrt{4a + (b-1)^2}$.

The original Hénon map can be considered as a quadratic map in one dimension if y_{n+1} is updated first, i.e.:

$$x_{n+1} = bx_n + 1 - ax_n^2. \tag{6}$$

This is an implied delay of one step. A quadratic map can be reduced to the functional form of the logistic map by a linear transformation $x = cy + d$ for appropriate values of the parameters.[11–13]

The Hénon map has two equilibrium points

$$x_{1,2} = \frac{b - 1 \pm \sqrt{(1-b)^2 + 4a}}{2a}, \tag{7}$$

Hence these equilibrium points are real if

$$\sqrt{(1-b)^2 + 4a} > 0. \tag{8}$$

It can also be shown that one of these equilibrium points is stable for the positive sign before the radical, the other one is always unstable.

The parameter values known to exhibit chaotic behavior are $a = 1.4$ and $b = 0.3$ and the two equilibrium points of the system for these parameter values are $(-1.13135, -0.339406)$ and $(0.631354, 0.189406)$. The eigenvalues at the first equilibrium point are $\{2.25982, -1.09203\}$ and the eigenvalues at the second equilibrium point are $\{-2.92374, -0.844054\}$. Hence the first one is a saddle point and the second one is clearly a stable equilibrium point.

Theorem 1. *A quadratic map where the coefficient of the quadratic term is negative can always be reduced to the functional form of the logistic map* $y_{n+1} = \lambda y_n(1 - y_n)$ *by a linear transformation of the form* $x = cy + d, c \neq 0$. *It should be noted that the linear transformation does not respect the unit interval condition of the logistic map, however the Hénon map itself does not stay in* $0 < x < 1,\ 0 < y < 1$.

Proof. After substitution of the linear transformation $x = cy + d$ in the system the constant term should vanish. We have:

$$y_n^2(ac^2) + y_n(2acd - bc) + ad^2 - bd + cy_{n+1} + d - 1 = 0. \tag{9}$$

The condition for vanishing constant term coincides with the condition that gives the equilibrium points, namely:

$$d_{1,2} = \frac{b - 1 \pm \sqrt{(1-b)^2 + 4a}}{2a}. \tag{10}$$

Hence $y_{n+1} = \lambda y_n(1 - y_n)$ is obtained where $\lambda = \pm\sqrt{(1-b)^2 + 4a} + 1$ for both solutions of d.

This theorem is important since it shows that a quadratic map can be converted to the logistic map provided that the logistic map variable remains in the unit interval. Furthermore a quadratic map can also be converted to a tent map where the codimension is incremented by one since the logistic map is reduced to tent map for $\lambda = 4$.

An iterated map and differential equation can be converted to one another by using a specific discretization. However the differential equation obtained by any discretization is invertible by the implicit function theorem while the corresponding map is usually non-invertible. This of course implies codimension is decreased by one upon conversion to map[14,15].

The formula for x_{n+1} with $y_n = bx_n$ substituted in the same way as the step leading to $x_{n+1} = 1 + bx_n - ax_n^2$ of the first iteration can be factorized as follows

$$x_n - f^2(x_n) = (x_n - f(x_n))(a^2x_n^2 - abx_n - ax_n - a + 1) \tag{11}$$

as expected where $f^2(x_n) = f(f(x_n))$. Theorem 1 then guarantees that both factors can be transformed into either the logistic map or its version with the reversed sign $x_{n+1} = \lambda(x_n^2 - x_n)$.

In the numerical analysis literature, one of the possible variations for successive iteration is the commonly known Jacobi iteration, and the second one is the Gauss-Seidel iteration. The difference lies in the fact that whether all variables are updated at the end of an iteration or the newer values for a variable are immediately used in later equations of the same iteration. The two-dimensional map

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{aligned} \tag{12}$$

is an instance of the Jacobi variant. The Gauss-Seidel variant uses $y_{n+1} = g(x_{n+1}, y_{n+1})$ for the second term.

According to the Jacobi variant the second iteration of the original Hénon map is given by:

$$\begin{aligned} x_{n+1} &= -a^3x_n^4 + 2a^2x_n^2y_n + 2a^2x_n^2 - ay_n^2 - 2ay_n - a + bx_n + 1 \\ y_{n+1} &= b(-ax_n^2 + y_n + 1). \end{aligned} \tag{13}$$

The system has four equilibrium points two of which are inherited from the original Hénon map and the other equilibrium points are

$$(-(\frac{b-1}{2a}) \pm \sqrt{(\frac{1}{a} - 3(\frac{b-1}{2a})^2)},\ -b(\frac{b-1}{2a} \pm \sqrt{(\frac{1}{a} - 3(\frac{b-1}{2a})^2)})). \tag{14}$$

The eigenvalues that are inherited from the original Hénon map are

$$\{\pm\beta(b-1) \pm \sqrt{2\alpha} + 2a + b^2 - b + 1\}, \tag{15}$$

and the eigenvalues of the third and fourth equilibrium points are

$$\{\pm 2\sqrt{a^2 - 2ab^2 + 3ab - 2a + b^4 - 3b^3 + 4b^2 - 3b + 1} - 2a + 2b^2 - 3b + 2\}$$

where $\beta = \sqrt{4a + (b-1)^2}$ and

$$\alpha = ((2ab - 2a + b^3 - 2b^2 + 2b - 1)\beta + 2a^2 + 4ab^2 - 6ab + 4a + b^4 - 3b^3 + 4b^2 - 3b + 1).$$

We proceed by giving the detailed stability results for the original Hénon map. We recall that the original Hénon map has two real equilibrium points for $a > -\frac{(b-1)^2}{4}$ [3,4].

Lemma 1. *For $a = -\frac{(b-1)^2}{4}$ and $\beta = 0$ the original Hénon map has a unique equilibrium point at $(\frac{-2}{b-1}, \frac{-2b}{b-1})$ with the eigenvalues $\{1, -b\}$ indicating saddle node bifurcation[3,4].*

Proof. By substituting $a = -\frac{(b-1)^2}{4}$ and $\beta = 0$ in Equation 3 the two equilibrium points are found to overlap each other at $(\frac{-2}{b-1}, \frac{-2b}{b-1})$. The Jacobian of the system at the equilibrium point is

$$\begin{pmatrix} 1-b & 1 \\ b & 0 \end{pmatrix}$$

and hence the eigenvalues are $\{1, -b\}$.

We consider the saddle node bifurcation with a numerical example. For $b = 0.3$ and $a = -0.1225$ the equilibrium point of the system is $(2.8571, 0.8571)$ indicating a saddle node bifurcation with a stable and an unstable branch shown in Figure 1. The range for a along the stable branch is between $-0.1225 \leq a \leq 0.3675$.

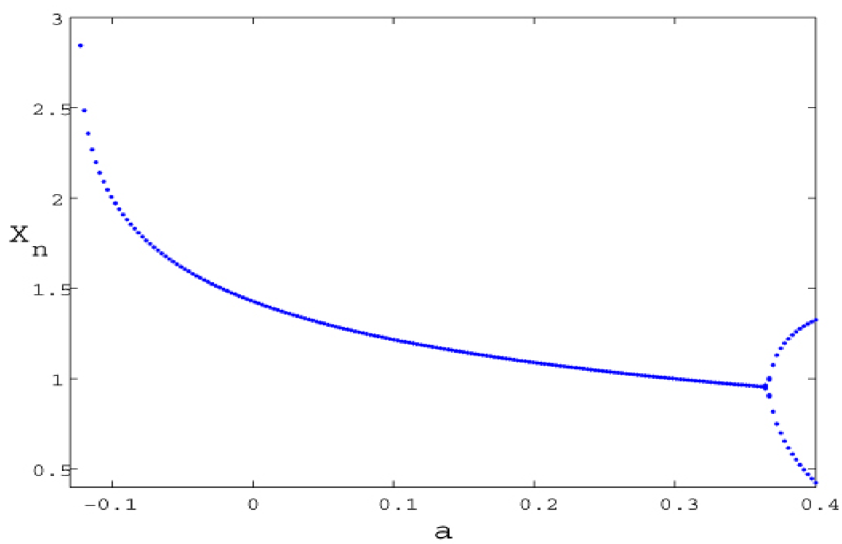

Fig. 1. Bifurcation diagram of the original Hénon map for $-0.15 \leq a \leq 0.4$ and $b = 0.3$.

For the special case that $a = \frac{3(b-1)^2}{4}$ period doubling bifurcation is observed for the original Hénon map. At this period doubling point the original Hénon map and the second iteration of the original Hénon map have completely overlapped equilibrium points at $(\frac{2}{b-1}, \frac{2b}{b-1})$. The eigenvalues for the original Hénon map are $\{-1, b\}$ and $\{\frac{\pm\sqrt{9b^2-14b+9}-3(b-1)}{2}\}$ and the eigenvalues for the second iteration of the original Hénon map are $\{1, b^2\}$ and $\{\frac{\pm 3\sqrt{9b^2-14b+9}|b-1|+9b^2-16b+9}{2}\}$.

Lemma 2. *The first and second iterations of the original Hénon map show Hopf bifurcation for $a = -0.982051$ and $a = 2.48205$ while $b = -1$.*

Proof. For $b = -1$ and $a = -0.982051$ the equilibrium points of the original Hénon map are $(1.1546, -1.1546)$ and $(0.88185, -0.88185)$ and the eigenvalues are $\{1.6686, 0.5992\}$ and $\{\frac{\sqrt{3}}{2} \pm \frac{i}{2}\}$. The equilibrium points of the second iteration of the original Hénon map are the same of those of the original Hénon map and the eigenvalues are $\{0.35913, 2.7844\}$ and $\{\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\}$.

For $b = -1$ and $a = 2.48205$ the equilibrium points of the original Hénon map are $(0.3489, -0.3489)$ and $(-1.1547, -1.1547)$ and the eigenvalues are $\{0.18011, 5.5519\}$ and $\{\frac{-\sqrt{3}}{2} \pm \frac{i}{2}\}$. The equilibrium points of the second iteration of the original Hénon map are the same of those of the original Hénon map and the eigenvalues are $\{0.03244, 30.8239\}$ and $\{\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\}$.

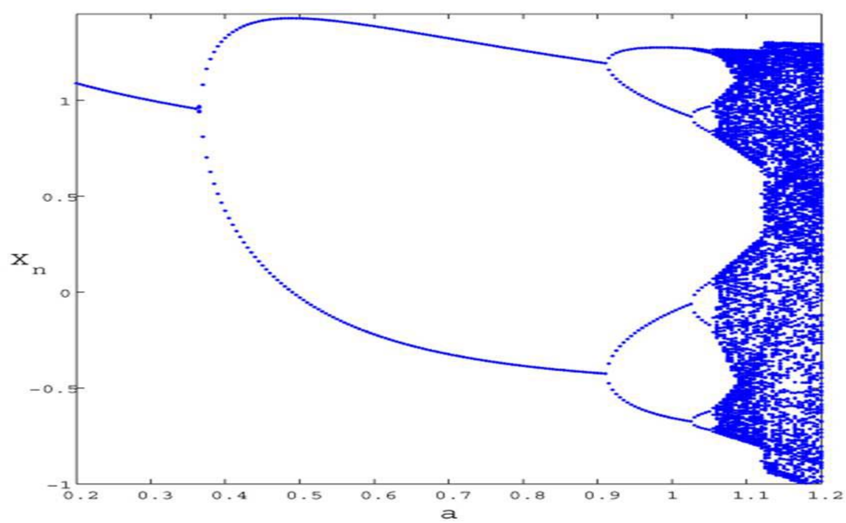

Fig. 2. Bifurcation diagram of the original Hénon map for $0.2 \leq a \leq 1.2$ and $b = 0.3$.

The first iteration of the original Hénon map exhibits a period doubling bifurcation and transition to chaos about $a = 0.2$ as shown in Figure 2 demonstrating an example of stable branch.Further stability analysis for the parameter value $a = 0.2$ gives the two equilibrium points $(1.08945, 0.326836)$ and $(-4.58945, -1.376836)$ and the eigenvalues $\{0.371580, -0.80736213\}$ and $\{1.986779, -0.1509981\}$ respectively. The first equilibrium point is a stable sink and the second equilibrium point is a saddle point. For the same parameter val-

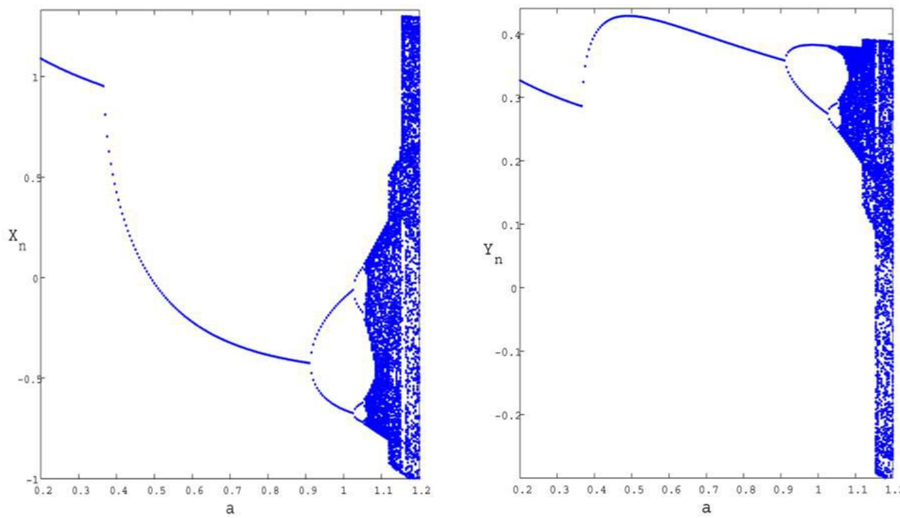

Fig. 3. Bifurcation diagram of the second iteration of the original Hénon map for $0.2 \leq a \leq 1.2$ and $b = 0.3$ for both x and y.

ues the second iteration of the original Hénon map has four equilibrium points, two of them are complex conjugates, i.e. $(1.75 \pm 2.046338i, 0.525 \mp 0.6139i)$ and the others are inherited from the original Hénon map respectively. The eigenvalues of the complex conjugate equilibrium points are $\{1.7072, 0.05271\}$, the eigenvalues of the third equilibrium point are {0.138072, 0.6518336} and the eigenvalues of the fourth equilibrium point are {0.0228, 3.9472}. The complex conjugate equilibrium points are saddles, the third equilibrium point is a stable sink and the fourth equilibrium point is a saddle as expected and the bifurcation diagram is shown in Figure 3. When a approaches 0.3675, a period-2 orbit is observed. The equilibrium points at $a = 0.3675$ are $(0.95238, 0.285714)$ and $(-2.857142, -0.857142)$ and the eigenvalues corresponding to these equilibrium points are {0.3, -1} and {2.234271, -0.134271}. Hence the first equilibrium point gives the beginning of the period doubling bifurcation and the second equilibrium point is a saddle point[16].

In the second iteration period doubling bifurcation occurs when the equilibrium points of the first iterate $f(x_n)$ lose their stability but the second iteration $f^2(x_n)$ develops a pair of new stable equilibrium points $x_{n\pm}$ such that $f^2(x_{n\pm}) = x_{n\pm}$ while f forms a period-2 attractor $f(x_{n\pm}) = x_{n\mp}$. The second iteration can not move $x_{n\pm}$ to $x_{n\mp}$ hence one of the branches for $f(x_n)$ becomes invisible for $f^2(x_n)$. Furthermore the special form of the Hénon map where y_n is calculated as the iterated value of x_n causes x_n and y_n to switch branches for f^2. This fact applies for $0.3625 \leq a \leq 0.9125$. At $a = 0.9125$ a period-4 attractor occurs as shown in Figure 3.

For $a = 1$ the equilibrium points are (0.70948, 0.21284) and (-1.40948, -0.42284) and the eigenvalues are {0.186824, -1.60578} and {2.921643, -0.102681} respectively. Both of the equilibrium points are saddle points. The bifurcation

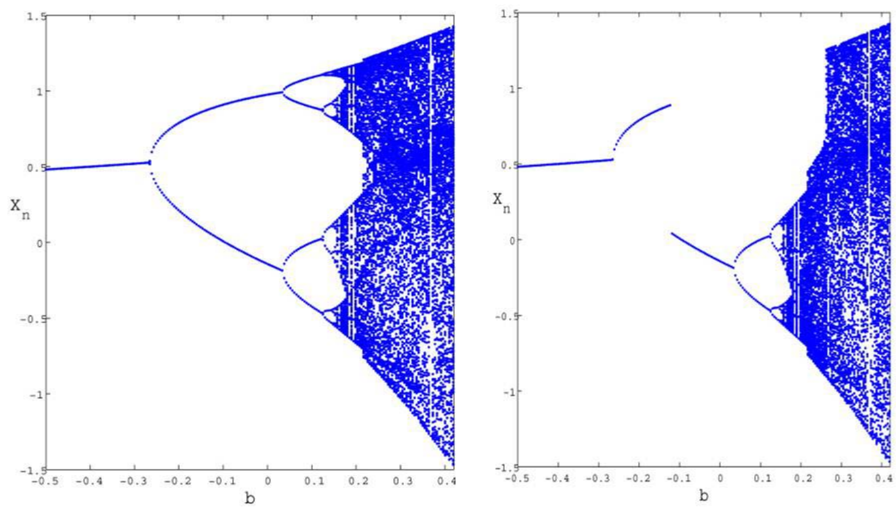

Fig. 4. Bifurcation diagram of the first and second iterations of the original Hénon map for $-0.5 \leq b \leq 0.42$ and $a = 1.2$ for x.

diagram of the first and second iterations of the original Hénon map for $-0.5 \leq b \leq 0.419$ and $a = 1.2$ is given in Figures 4 and 5. For $a = 1.2$ and $b = -0.1$ the equilibrium points are $(0.563137, -0.0563137)$ and $(-1.4798, 0.14798)$ and the eigenvalues are $\{-0.0785562, -1.2729739\}$ and $\{0.0283838, 3.523146\}$ respectively. Both of the equilibrium points are saddles. About the first equilibrium point period-2 orbit is observed. Considering the second iteration of the original Hénon map with same conditions there are four equilibrium points of which two are the same as the first iteration. The eigenvalues are $\{1.620462, 0.0061710\}$ and $\{12.41256, 0.000805\}$. Again both equilibrium points are saddle points. The system hides its periodic behavior that is observed for the first iteration and the increase in the iteration of the system makes the system a more chaotic and complex one instead of a multiperiodic one.

3 Conclusion

In this paper we investigated the bifurcation analysis and stability structure of the generalized Hénon map and its higher iterations[17,18]. The second iteration of the generalized Hénon map is of interest since period doubling bifurcation is a prominent mechanism as revealed by the bifurcation map. The

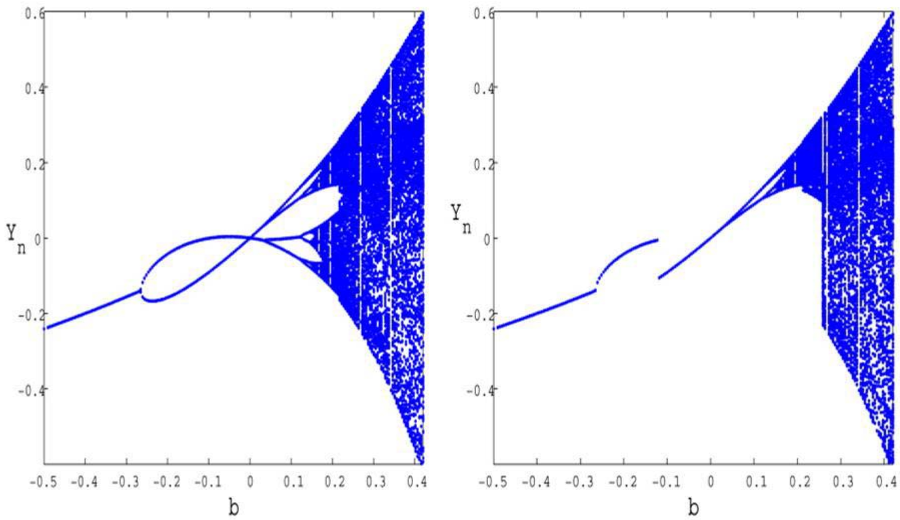

Fig. 5. Bifurcation diagram of the first and second iterations of the original Hénon map for $-0.5 \leq b \leq 0.42$ and $a = 1.2$ for y.

second iteration can either be done on the original Hénon map or on the delayed version as proposed by Skiadas[2]. The two options correspond to the Jacobi and Gauss-Seidel iterations in numerical analysis where all variables are either updated following a complete iteration or each updated value is used for the subsequent equations in the same iterations. As we proceed to higher iterations, the position of the bifurcations remain essentially unchanged, the nature of the bifurcations change to include virtually all kinds of bifurcations. The generalization of the y_n updating formula to the form x_n^2 does not qualitatively change the nature. The bifurcation scenario is not sensitive to k. When we increase the value of k, we notice that the bifurcation diagrams do not change their general properties. The stability analysis related to these results are also investigated. Generally magnitude of one eigenvalue is less than 1 and the other is greater than 1. Therefore we have a saddle for these kinds of equilibrium points. On the other hand, we have stable sinks for other type of eigenvalues. As we look generally, we do not observe any bifurcation out of the range $0.2 \leq a \leq 1.875$ so that we can obtain similar results according to the Gauss-Seidel iteration.

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