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Non-relativistic versus Relativistic Low-Speed Momentum Diffusion

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Abstract. Non-relativistic and special-relativistic predictions for low-speed momentum diffusion, which are calculated using the same parameter and the same initially localized Gaussian ensemble of trajectories, are compared for a prototypical Hamiltonian system – the periodically-delta-kicked particle. Contrary to expectation, we show that the agreement between the two predictions can break down after some time.
Keywords: Relativistic Momentum Diffusion, Low Speed, Kicked Particle

1 Introduction

It is conventionally believed [1-3] that the special-relativistic dynamical predictions for a low-speed system are always well-approximated by the Newtonian predictions. However, contrary to conventional belief, numerical study [4,5] of a prototypical Hamiltonian system – the periodically-delta-kicked particle – showed that the Newtonian trajectory does not always agree with the special-relativistic trajectory – the breakdown of agreement between the two single-trajectory predictions is rapid if the trajectories are chaotic, but very slow if the trajectories are non-chaotic. Similar rapid breakdown of agreement between single-trajectory predictions was also found in a model dissipative system [6] and a model scattering system [7]. Recently, we showed [8,9] that the Newtonian and special-relativistic statistical dynamical predictions – position and momentum means and standard deviations, dwell time, transmission and reflection coefficients – for low-speed systems can also rapidly breakdown in agreement.

However, a comparison of the Newtonian and special-relativistic predictions for low-speed momentum diffusion has not been done to ascertain if the special-relativistic prediction is always well-approximated by the Newtonian prediction as conventionally expected. In this paper, we compare the low-speed momentum diffusion predicted by the two theories based on the same parameter and the same ensemble of initial conditions for the periodically-delta-kicked particle. Details of the kicked particle and numerical calculation are presented

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next, followed by the results. The significance of our finding is discussed in the final section.

2 Methods

The periodically-delta-kicked particle is a one-dimensional Hamiltonian system where the delta kicks are due to a sinusoidal potential which is periodically turned on for an instant. The Newtonian equations of motion for the periodically-delta-kicked particle are easily integrated [10] to yield an exact mapping, which is known as the standard map, of the dimensionless scaled position X and dimensionless scaled momentum P from just before the (n-1)th kick to just before the nth kick:

$$P_n = P_{n-1} - \frac{K}{2\pi}\sin(2\pi X_{n-1}) \tag{1}$$

$$X_n = (X_{n-1} + P_n) \bmod 1 \tag{2}$$

where n = 1,2,…, and K is a dimensionless positive parameter. The special-relativistic equations of motion for the periodically-delta-kicked particle are also easily integrated [11,12] to yield an exact mapping for the dimensionless scaled position X and dimensionless scaled momentum P from just before the (n-1)th kick to just before the nth kick:

$$P_n = P_{n-1} - \frac{K}{2\pi}\sin(2\pi X_{n-1}) \tag{3}$$

$$X_n = \left(X_{n-1} + \frac{P_n}{\sqrt{1+\beta^2 P_n^2}} \right) \bmod 1 \tag{4}$$

where n = 1,2,…. In addition to the parameter K, the relativistic standard map [Eqs. (3) and (4)] has another dimensionless positive parameter, β. Since

$$\frac{v}{c} = \frac{\beta P}{\sqrt{1+(\beta P)^2}}\,, \tag{5}$$

$\beta P << 1$ implies $v << c$ (i.e., low speed), where v is the particle speed and c is the speed of light.

The statistical quantity that is typically used to study momentum diffusion is [10,13,14] the mean square momentum displacement (MSMD)

$$\left\langle (\Delta P_n)^2 \right\rangle \equiv \left\langle (P_n - P_0)^2 \right\rangle, \tag{6}$$

where <…> is an average over an ensemble of trajectories. In previous studies [10,13,14] of momentum diffusion in the Newtonian standard map, an initially non-localized semi-uniform ensemble, where semi-uniform means that the initial positions are uniformly distributed but the initial momentums are all the same value, was used in the numerical calculation of the MSMD. These studies [13,14] of the Newtonian standard map have shown that, for parameter K where accelerator mode islands exist, the MSMD has a power law dependence on the

kick n: Dn^{α} where $1 < \alpha < 2$. In this case, the diffusion is termed anomalous. In contrast, for parameter K where there is no accelerator mode island, the diffusion is normal, that is, the MSMD grows linearly [10,13,14].

In our calculations, we use an initially localized ensemble instead where the initial positions and momentums are both Gaussian distributed

$$\rho(X,P,t=0)=\frac{1}{2\pi\sigma_{X_0}\sigma_{P_0}}\exp\left[-\frac{1}{2}\left[\left(\frac{X-\langle X_0\rangle}{\sigma_{X_0}}\right)^2+\left(\frac{P-\langle P_0\rangle}{\sigma_{P_0}}\right)^2\right]\right] \quad (7)$$

with means $<X_0>$ and $<P_0>$, and standard deviations σ_{X_0} and σ_{P_0}. In each theory, the MSMD is first calculated using 10^6 trajectories, where the numerical accuracy is determined by comparing the 30-significant-figure calculation with the 35-significant-figure (quadruple precision) calculation. For example, if the former calculation yields 1.234… and the latter yields 1.235…, the 10^6-calculation is accurate to 1.23 (3 significant figures). The MSMD is then recalculated using 10^7 trajectories with the same accuracy determination. Finally, the accuracy of the MSMD is determined by comparing the 10^6-calculation with the 10^7-calculation. For example, if the 10^6-calculation is accurate to 1.23 and the 10^7-calculation is accurate to 1.24, the MSMD is accurate to 1.2 (2 significant figures).

3 Results

Here we will present a representative example to illustrate our findings. In this example, the parameter β in the relativistic standard map [Eqs. (3) and (4)] is small, 10^{-7}, and so the mean particle speed is low, at most about 0.001% of the speed of light. The parameter K is 6.9115, and the ensemble is initially Gaussian localized in phase space with means $<X_0> = 0.5$ and $<P_0> = 99.9$, and standard deviations $\sigma_{X_0} = \sigma_{P_0} = 10^{-12}$.

Fig. 1 shows that the Newtonian and special-relativistic predictions for the MSMD are very close and fluctuating for the first 16 kicks, but, from kick 17 onwards, the MSMD predicted by the two theories disagree with each other completely. For example, at kick 17, the Newtonian and special-relativistic MSMD are, respectively, 0.19505 (accurate to 5 significant figures) and 1.523049 (accurate to 7 significant figures), where the numerical accuracies were determined using the method described in the previous section and so there is no numerical artifact in the results.

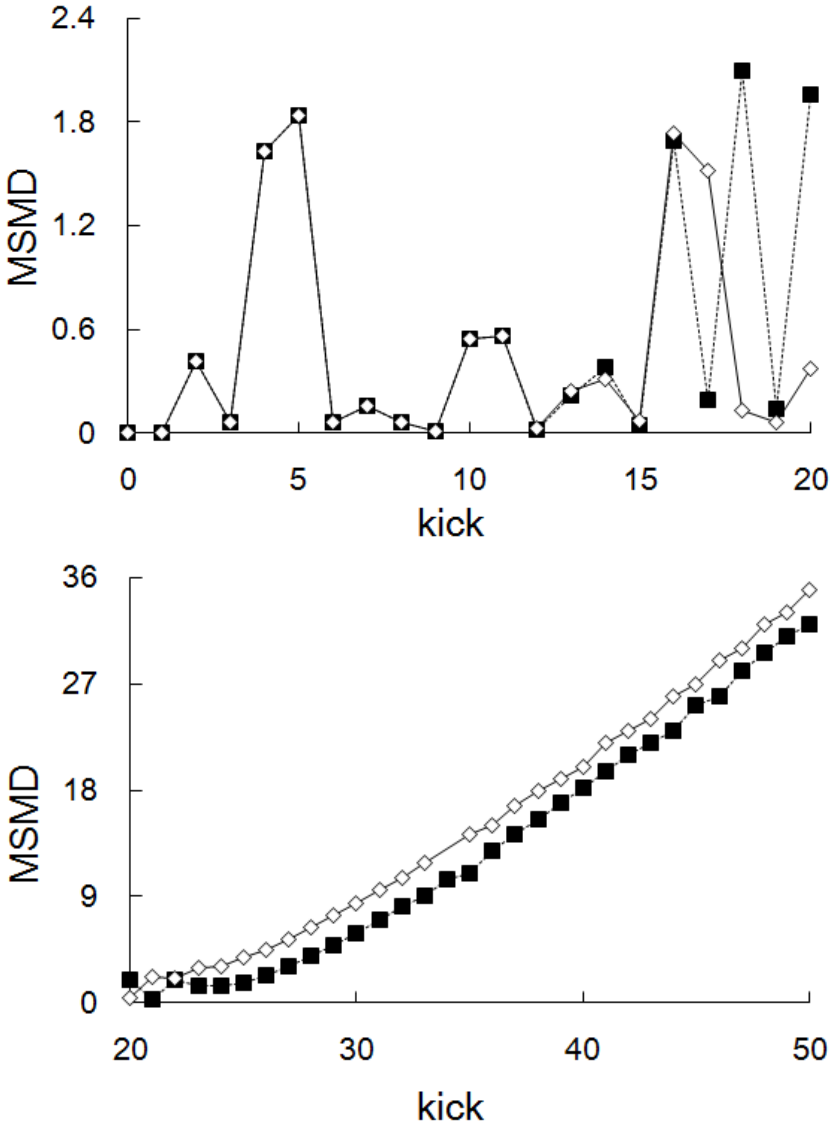

Fig. 1. Newtonian (squares) and special-relativistic (diamonds) MSMD for the first 20 kicks (top) and from kick 20 to kick 50 (bottom). MSMD which cannot be resolved in accuracy is not plotted.

4 Conclusions

In summary, we have shown that the Newtonian and special-relativistic predictions for low-speed momentum diffusion in a Hamiltonian system can break down after some time. Which of the two different predictions is empirically correct if a test were to be conducted? One would expect the special-relativistic prediction to be empirically correct since special relativity continues to be successfully verified [15-17]. If so, our finding shows that Newtonian mechanics does not always yield empirically correct predictions for *low-speed* momentum diffusion as would be expected conventionally.

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