Chaotic Modeling and Simulation (CMSIM) 4: 413-423, 2016

Scattering of Rayleigh waves by sinusoidal undulations on the surface of an incompressible monoclinic elastic solid

S. Nair[1] and D. A. Sotiropoulos[2]

[1] Department of Mechanical, Materials and Aerospace Engineering, Illinois Institute of Technology, Chicago, IL 60616, US
(E-mail: `nair@iit.edu`)

[2] Department of Engineering Sciences, Technical University of Crete, Chania 73100, Greece
(E-mail: `dimsotirop@yahoo.com`)

Abstract. Rayleigh waves propagating along the surface of a semi-infinite, monoclinic, incompressible medium are considered. The surface is assumed to be flat with superposed undulations of small amplitude (in relation to the wave length of the Rayleigh waves). Using this smallness, a perturbation approach is used to obtain the zeroth and the first order solutions. In deriving the zeroth order solution, the quartic secular equation for the wave speed for the incompressible case is obtained without taking the limit of a compressible case. The first order corrections to the perfectly-flat surface solutions for stresses and displacements are derived. The results are illustrated with numerical calculations.

Keywords: Rayleigh waves, surface undulations, monoclinic, incompressible solid.

1 Introduction

Rayleigh waves are surface waves propagating in semi-infinite elastic media, [1]. Mal [2] considered the effect of small corrugations on a flat surface in the context of earthquake waves in an isotropic medium. Later, Chandler-Wilde [3] studied scattering of body waves from a rough surface and Markenscoff and Lekoudis [4] considered Love waves in slowly varying layered media. This approach was extended to incompressible, orthotropic media by Nair and Sotiropoulos [6]. Of course, incompressibility itself is an approximation to characterize materials with high values for the bulk modulus compared to the shear modulus. In these studies cited above (except for the case of Love waves) incident waves get reflected from the surface accompanied by two scattered components with distinct wave numbers. From a knowledge of the incident wave number and the scattered wave numbers one may characterize the surface undulations. As a continuation of our previous studies on incompressible, orthotropic materials [7] and on incompressible, monoclinic materials [8], here we consider

Received: 31 October 2016 / Accepted: 24 November 2016

ISSN 2241-0503

Rayleigh waves propagating on a flat surface with sinusoidal, small amplitude undulations in an incompressible, monoclinic material. The small amplitude compared to the wave length allows one to expand the governing equations and the boundary conditions in a perturbation series. Of course, the zeroth order system, which consists of a set of homogeneous equations with homogeneous boundary conditions, describes Rayleigh waves on a flat surface. We directly derive the secular quartic equation for the wave speed explicitly to obtain the original quartic equation obtained by Destrade, et al. [9] as a limiting case of Rayleigh waves in compressible materials. This limiting process has been described by Ting [10] and by Destrade [11]. Rayleigh waves on flat surfaces in the orthotropic, compressible case were studied by Ogden and Pham [12], [13], [14] and Royer and Dieulesaint [15]. The first order equations are, again, homogeneous. However, the first order boundary conditions are inhomogeneous, with the inhomogeneities depending on the zeroth order solutions. Solutions of these equations are given for traction as well a displacement components. The results are illustrated with numerical results for a select range of parameters.

2 Formulation

As shown in Fig. 1, a monoclinic, incompressible medium occupies the domain $-\infty < x_1 < \infty$, $2a \sin \omega x_1 < x_2 < \infty$. Here, the amplitude of the surface of the surface undulation , $2a$, and the spatial frequency, ω, are assumed to be small compared to the wave length and frequency of the Rayleigh wave along a perfectly flat surface. That is, if k represents the Rayleigh wave number, it is necessary to have $\epsilon \equiv ka << 1$ and it is sufficient (not necessary) to have $\omega/k << 1$.

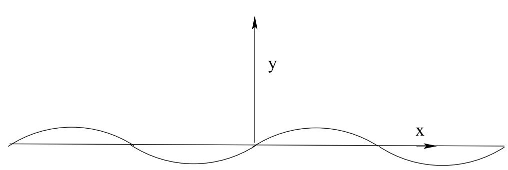

Fig. 1. A semi-infinite medium with an undulating surface.

We assume plane strain condition and stress-strain relations

$$\begin{aligned}\sigma_{11} &= -p + C_{11}u_{1,1} + C_{12}u_{2,2} + C_{16}(u_{1,2} + u_{2,1}),\\ \sigma_{22} &= -p + C_{12}u_{1,1} + C_{22}u_{2,2} + C_{26}(u_{1,2} + u_{2,1}),\\ \sigma_{12} &= C_{66}(u_{1,2} + u_{2,1}) + C_{16}u_{1,1} + C_{26}u_{2,2},\end{aligned} \tag{1}$$

where C_{ij} represent the five material constants, σ_{ij} the stresses, $u_{i,j}$ the displacement gradients and p the hydrostatic pressure.

The equations of motion are

$$\begin{aligned}\sigma_{11,1} + \sigma_{12,2} &= \rho\ddot{u}_1,\\ \sigma_{12,1} + \sigma_{22,2} &= \rho\ddot{u}_2,\end{aligned} \tag{2}$$

where ρ is the density and $\ddot{u}_i = \partial^2 u_i/dt^2$ with t representing time. We supplement the above equations with the incompressibility condition

$$u_{1,1} + u_{2,2} = 0. \tag{3}$$

The boundary conditions are: Components of the traction vector on the wavy surface must be zero.

Using Eq. (3) and eliminating the hydrostatic pressure p, the stress-strain relations, Eq. (1), can be written as

$$\begin{aligned} \sigma_{11} - \sigma_{22} &= (C_{11} + C_{22} - 2C_{12})u_{1,1} + (C_{16} - C_{26})(u_{1,2} + u_{2,1}), \\ \sigma_{12} &= (C_{16} - C_{26})u_{1,1} + C_{66}(u_{1,2} + u_{2,1}). \end{aligned} \tag{4}$$

The equations of motion can be written as

$$\begin{aligned} \sigma_{11,11} + 2\sigma_{12,12} + \sigma_{22,22} &= 0, \\ \sigma_{11,12} + \sigma_{12,11} + \sigma_{12,22} + \sigma_{22,12} &= \rho(u_{1,2} + u_{2,1})^{\cdot\cdot}, \\ \sigma_{11,11} + \sigma_{12,12} &= \rho\ddot{u}_1, \end{aligned} \tag{5}$$

We may introduce the quantities

$$\begin{aligned} s_{ij} = \frac{\sigma_{ij}}{C_{66}}, \quad \bar{\beta} = \frac{C_{11} + C_{22} - 2C_{12}}{C_{66}}, \quad \gamma = \frac{C_{16} - C_{26}}{C_{66}}, \\ q = s_{11} - s_{22}, \quad s = s_{22}, \quad \tau = s_{12}, \quad \alpha = C_{66}, \end{aligned} \tag{6}$$

where the parameter $\bar{\beta}$ is related to the parameter β introduced in our previous work [5, 6, 7] as

$$\beta = \frac{C_{11} + C_{22} - 2C_{12}}{4C_{66}} - 1,$$

in the form

$$\bar{\beta} = 4(\beta + 1). \tag{7}$$

Eqs. (4) can be written as

$$\begin{aligned} q &= \bar{\beta}u_{1,1} + \gamma(u_{1,2} + u_{2,1}), \\ \tau &= \gamma u_{1,1} + u_{1,2} + u_{2,1}. \end{aligned} \tag{8}$$

We require the strain energy, $qu_{1,1} + \tau(u_{1,2} + u_{2,1})$ to be positive definite. Using Eqs. (8), this gives the constraint on the material parameters,

$$\bar{\beta} - \gamma^2 > 0. \tag{9}$$

Inverting the equations in (8), we get

$$\begin{aligned} u_{1,1} &= \frac{1}{\bar{\beta} - \gamma^2}[q - \gamma\tau], \\ u_{1,2} + u_{2,1} &= \frac{1}{\bar{\beta} - \gamma^2}[\bar{\beta}\tau - \gamma q]. \end{aligned} \tag{10}$$

Using Eqs. (10) in the equations of motion (5) and assuming time-harmonic solutions for which the time dependence is expressed using the multiplier $exp(-i\Omega t)$, we have, for any unknown, u,

$$\ddot{u}(x_1, x_2, t) \to -\Omega^2 u(x_1, x_2). \tag{11}$$

The relevant equations are:

$$\begin{aligned} q_{,11} + 2\tau_{,12} + s_{,11} + s_{,22} &= 0, \\ q_{,12} + \tau_{,11} + \tau_{,22} + s_{,22} &= \frac{\rho\Omega^2}{\alpha(\bar{\beta} - \gamma^2)}[\gamma q - \bar{\beta}\tau], \\ q_{,11} + \tau_{,12} + s_{,11} &= \frac{\rho\Omega^2}{\alpha(\bar{\beta} - \gamma^2)}[\gamma\tau - q]. \end{aligned} \tag{12}$$

Considering the small amplitude waviness of the surface, we expand the unknowns, u, in the above equations as

$$u = u_0 + \epsilon u_1 + \cdots \tag{13}$$

The zeroth order system is given by

$$\begin{aligned} q_{0,11} + 2\tau_{0,12} + s_{0,11} + s_{0,22} &= 0, \\ q_{0,12} + \tau_{0,11} + \tau_{0,22} + s_{0,22} &= \frac{\rho\Omega^2}{\alpha(\bar{\beta} - \gamma^2)}[\gamma q_0 - \bar{\beta}\tau_0], \\ q_{0,11} + \tau_{0,12} + s_{0,11} &= \frac{\rho\Omega^2}{\alpha(\bar{\beta} - \gamma^2)}[\gamma\tau_0 - q_0]. \end{aligned} \tag{14}$$

The first order system is of the same form:

$$\begin{aligned} q_{1,11} + 2\tau_{1,12} + s_{1,11} + s_{1,22} &= 0, \\ q_{1,12} + \tau_{1,11} + \tau_{1,22} + s_{1,22} &= \frac{\rho\Omega^2}{\alpha(\bar{\beta} - \gamma^2)}[\gamma q_1 - \bar{\beta}\tau_1], \\ q_{1,11} + \tau_{1,12} + s_{1,11} &= \frac{\rho\Omega^2}{\alpha(\bar{\beta} - \gamma^2)}[\gamma\tau_1 - q_1], \end{aligned} \tag{15}$$

The coupling between the two systems is due to the boundary conditions. For the surface, $x_2 = 2a \sin \omega x_1$, the tangent makes an angle θ with the x_1-axis, given by $\tan\theta = 2a\omega \cos\omega x_1 = 2\epsilon(\omega/k)\cos\omega x_1$. Neglecting quadratic terms in ϵ, we have

$$\sin\theta \sim 2\epsilon(\omega/k)\cos\omega x_1, \quad \cos\theta \sim 1. \tag{16}$$

If T and N represent the non-dimensional tangential and normal traction components on the surface, we have

$$\begin{aligned} T &= s_{12}(\cos^2\theta - \sin^2\theta) + (s_{22} - s_{11})\cos\theta\sin\theta = 0, \\ N &= s_{11}\sin^2\theta + s_{22}\cos^2\theta - 2s_{12}\sin\theta\cos\theta = 0. \end{aligned} \tag{17}$$

Using the approximation of Eq. (16), and the Maclaurin expansions of the form

$$T(x_1, x_2) = T(x_1, 0) + T_{,2}(x_1, 0)x_2 + \cdots, \tag{18}$$

the traction components can be expanded to obtain

$$T_0 = \tau_0 = 0, \quad N_0 = s_0 = 0, \tag{19}$$

$$T_1 = \tau_1 - 2q_0\frac{\omega}{k}\cos\omega x_1 + \frac{2}{k}\tau_{0,2}\sin\omega x_1 = 0, \quad N_1 = s_1 + \frac{2}{k}s_{0,2}\sin\omega x_1 = 0. \tag{20}$$

We now have homogeneous systems of differential equations for the zeroth and first order quantities; but the boundary conditions are homogeneous for the zeroth order and non-homogeneous for the first order.

3 Zeroth order solutions

The zeroth order system (14) can be solved assuming the unknowns in the form

$$u_0(x_1, x_2) = \bar{u}_0 \exp[ik(x_1 + \lambda x_2)], \tag{21}$$

where $\bar{u}_0$ is a constant. The equations for $\bar{q}_0, \bar{\tau}_0$ and $\bar{s}_0$ are:

$$\begin{aligned} (1-\bar{\eta})\bar{q}_0 + (\lambda + \gamma\bar{\eta})\bar{\tau}_0 + \bar{s}_0 = 0, \\ (\lambda + \gamma\bar{\eta})\bar{q}_0 + (1 + \lambda^2 - \bar{\beta}\bar{\eta})\bar{\tau}_0 + 2\lambda\bar{s}_0 = 0, \\ \bar{q}_0 + 2\lambda\tau_0 + (1+\lambda^2)\bar{s}_0 = 0, \end{aligned} \tag{22}$$

where

$$\bar{\eta} = \frac{\rho\Omega^2}{\alpha(\bar{\beta} - \gamma^2)k^2}. \tag{23}$$

We will also use

$$\eta = \bar{\eta}(\bar{\beta} - \gamma^2) = \rho c^2/\alpha, \quad c^2 = \Omega^2/k^2, \tag{24}$$

where c is the speed of the zeroth order wave.

From the first equation above, we find

$$\bar{q}_0 = \frac{1}{\bar{\eta} - 1}[(\lambda + \gamma\bar{\eta})\bar{\tau}_0 + \bar{s}_0]. \tag{25}$$

Eliminating $\bar{q}_0$ from the other two equations, we obtain

$$\begin{aligned} A_{11}(\lambda)\bar{\tau}_0 + A_{12}(\lambda)\bar{s}_0 = 0 \\ A_{21}(\lambda)\bar{\tau}_0 + A_{22}(\lambda)\bar{s}_0 = 0, \end{aligned} \tag{26}$$

where the coefficients, which are functions of λ, are defined as

$$\begin{aligned} A_{11}(\lambda) &= (1-\bar{\eta})(1+\lambda^2 - \bar{\beta}\bar{\eta}) - (\lambda + \gamma\bar{\eta})^2, \\ A_{12}(\lambda) &= A_{21}(\lambda) = \lambda(1 - 2\bar{\eta}) - \gamma\bar{\eta}, \\ A_{22}(\lambda) &= (1-\bar{\eta})(1+\lambda^2) - 1. \end{aligned} \tag{27}$$

Setting the determinant of the system (26), $A_{11}A_{22} - A_{12}A_{21}$, to zero, we obtain the characteristic equation for λ

$$\lambda^4 + 2\gamma\lambda^3 - (2 - \bar{\beta} + \eta)\lambda^2 - 2\gamma\lambda + 1 - \eta = 0. \tag{28}$$

We note that it is η, not $\bar{\eta}$, appearing in the quartic equation, and when $\gamma = 0$ we have an orthotropic material and λ can be obtained by solving a quadratic equation for λ^2. In order to have the disturbance decay in the x_2 direction, we have to choose two values of λ, λ_1 and λ_2, with positive imaginary parts. From the last term in the characteristic equation, this implies that the wave speed parameter $\eta = \rho c^2/\alpha$ has to satisfy the constraint

$$\eta < 1. \tag{29}$$

Following Destrade et al. [9], we may obtain an explicit quartic equation for η in the following way:

Let us separate amplitudes $\bar{\tau}_0$ and $\bar{s}_0$ into two parts—one corresponding to the solution with λ_1 and the other corresponding to λ_2. That is

$$\bar{\tau}_0 = \bar{\tau}_{01} + \bar{\tau}_{02}, \quad \bar{s}_0 = \bar{s}_{01} + \bar{s}_{02}, \tag{30}$$

The system of equations (26) can be satisfied for the two values of $\lambda_i, \quad i = 1, 2,$ in two ways: By choosing

$$\bar{\tau}_{0i} = -C_{0i} A_{12}(\lambda_i), \quad \bar{s}_{0i} = C_{0i} A_{11}(\lambda_i), \tag{31}$$

or by choosing

$$\bar{\tau}_{0i} = -C'_{0i} A_{22}(\lambda_i), \quad \bar{s}_{0i} = C'_{0i} A_{21}(\lambda_i). \tag{32}$$

The boundary conditions of zero traction components mean

$$C_{01} A_{12}(\lambda_1) + C_{02} A_{12}(\lambda_2) = 0, \quad C_{01} A_{11}(\lambda_1) + C_{02} A_{11}(\lambda_2) = 0, \tag{33}$$

or

$$C'_{01} A_{22}(\lambda_1) + C'_{02} A_{22}(\lambda_2) = 0, \quad C'_{01} A_{21}(\lambda_1) + C'_{02} A_{21}(\lambda_2) = 0, \tag{34}$$

The constants C_{01} and C_{02} can be expressed using a single constant D as

$$C_{01} = -D A_{12}(\lambda_2), \quad C_{02} = D A_{12}(\lambda_1). \tag{35}$$

For non-trivial solutions of the two equations, (33) and (34), for C_{0i} and the two for C'_{0i}, we require

$$A_{12}(\lambda_1) A_{11}(\lambda_2) - A_{12}(\lambda_2) A_{11}(\lambda_1) = 0 \tag{36}$$

and

$$A_{22}(\lambda_1) A_{21}(\lambda_2) - A_{22}(\lambda_2) A_{21}(\lambda_1) = 0. \tag{37}$$

Substituting the explicit expressions for the functions A_{ij} from Eqs. (27), we get Eqs. (36) and (37) in the forms

$$(2\bar{\eta}-1)\bar{\eta}\lambda_1\lambda_2 + \gamma\bar{\eta}^2(\lambda_1+\lambda_2) - (1-2\bar{\eta})[(1-\bar{\beta}\bar{\eta})(1-\bar{\eta}) - \gamma^2\bar{\eta}^2] + 2\gamma^2\bar{\eta}^2 = 0, \tag{38}$$

and

$$(1-\bar{\eta})(2\bar{\eta}-1)\lambda_1\lambda_2 + (1-\bar{\eta})\gamma\bar{\eta}(\lambda_1+\lambda_2)(2\bar{\eta}-1)\bar{\eta} = 0. \tag{39}$$

Multiplying Eq. (38) by $(1-\bar{\eta})$ and Eq. (39) by $\bar{\eta}$, we can eliminate the terms containing λ_i and the resulting secular equation for the wave speed parameter η is

$$2(\bar{\beta}-\gamma^2)\bar{\eta}^4 - 5(\bar{\beta}-\gamma^2)\bar{\eta}^3 + (4+4\bar{\beta}-3\gamma^2)\bar{\eta}^2 - (4+\bar{\beta})\bar{\eta} + 1 = 0. \tag{40}$$

Solutions of this secular equation for selected values of β and γ can be found in our previous paper [5].

In Table 1 we have listed the values of $\eta = \rho c^2/\alpha$ which yield two exponents λ_1 and λ_2 with positive imaginary parts for various values of $\bar{\beta}$ and γ. When the material is isotropic, $\bar{\beta} = 4$ and $\gamma = 0$. We had mentioned the necessary

Table 1. Dependence of η, λ_1, and λ_2 on the material parameters $\bar{\beta}$ and γ.

$\bar{\beta}$	γ	η	λ_1	λ_2
1.0	0.0	0.43016	-0.85731+0.14106i	0.85731+0.14106i
	0.2	0.38944	-0.97267+0.26369i	0.77267+0.06442i
2.0	0.0	0.70440	-0.66929+0.30943i	0.66929+0.30943i
	0.2	0.66789	-0.79830+0.46497i	0.59830+0.17652i
3.0	0.0	0.84583	-0.39722+0.48463i	0.39722+0.48463i
	0.2	0.81926	-0.58016+0.71683i	0.38016+0.26079i
4.0	0.0	0.91262	0.0+0.29560i	0.0+1.0i
	0.2	0.89378	-0.37851+1.07292i	0.17851+0.22404i
5.0	0.0	0.94560	0.0+0.16379i	0.0+1.42393i
	0.2	0.93170	-0.29731+1.43663i	0.09731+0.14923i
6.0	0.0	0.96341	0.0+0.10999i	0.0+1.73911i
	0.2	0.95262	-0.26578+1.74109i	0.06578+0.10462i
7.0	0.0	0.97390	0.0+0.08061i	0.0+2.00490i
	0.2	0.96517	-0.24959+2.00337i	0.04959+0.07080i

condition, $\eta < 1$, for the existence of a pair of complex conjugate solutions for λ. When $\bar{\beta}$ is less than 4.00, there are cases where two values of $\eta < 1$ exist. However, only one of these gives two exponentially decaying solutions in the x_2-direction.

The zeroth order solutions are

$$\begin{aligned} \bar{\tau}_{01} &= -\bar{\tau}_{02} = DA_{12}(\lambda_2)A_{12}(\lambda_1), \\ \bar{s}_{01} &= -\bar{s}_{02} = -DA_{12}(\lambda_2)A_{11}(\lambda_1), \\ \bar{q}_{01} &= \frac{1}{\bar{\eta}-1}[(\lambda_1+\gamma\bar{\eta})\bar{\tau}_{01}+s_{01}], \quad q_{02} = \frac{1}{\bar{\eta}-1}[(\lambda_2+\gamma\bar{\eta})\bar{\tau}_{02}+s_{02}]. \end{aligned} \tag{41}$$

Using the second equation in (2), the vertical displacement $\bar{u}_{20}$ on the surface can be written as

$$\bar{u}_{20} = -\frac{i}{k\eta}[\lambda_1\bar{s}_{01}+\lambda_2\bar{s}_{02}] \tag{42}$$

Table 1 shows the values of η, λ_1 and λ_2 for selected values of $\bar{\beta}$ and γ.

Table 2 shows the values of the nondimensional zeroth order stresses $\tau_{01} = -\tau_{02}$, $s_{01} = -s_{02}$, and the vertical displacement measure ku_{20} for $\bar{\beta} = 3.0$ and $\gamma = 0.2$. The constant D in Eq. (34) is set to unity to normalize the solutions. This constant is a measure of the amplitude of the zeroth order Rayleigh wave.

Table 2. Nondimensional zeroth order stresses and the vertical displacement at the surface for $\bar{\beta} = 3$ and $\gamma = 0.2$.

Quantity	Value
η	0.8193
λ_1	-0.5802+0.7168i
λ_2	0.3802+0.2608i
$\tau_{01} = -\tau_{02}$	-0.0732
$s_{01} = -s_{02}$	-0.0091-0.0444i
ku_{20}	0.0470-0.0353i

4 First order solutions

From the form of the boundary conditions (19), we need solutions for the first order equations which depend on x_1 as a linear combinations of $e^{i(k+\omega)x_1}$ and as $e^{i(k-\omega)x_1}$. Let

$$\begin{aligned} q_{1+} &= \bar{q}_+ e^{ik[(1+\bar{\omega})x_1+\mu x_2]}, \quad q_{1-} = \bar{q}_- e^{ik[(1-\bar{\omega})x_1+\mu x_2]}, \\ \tau_{1+} &= \bar{\tau}_+ e^{ik[(1+\bar{\omega})x_1+\mu x_2]}, \quad \tau_{1-} = \bar{\tau}_- e^{ik[(1-\bar{\omega})x_1+\mu x_2]}, \\ s_{1+} &= \bar{s}_+ e^{ik[(1+\bar{\omega})x_1+\mu x_2]}, \quad s_{1-} = \bar{s}_- e^{ik[(1-\bar{\omega})x_1+\mu x_2]}, \end{aligned} \tag{43}$$

where $\bar{q}_+$, etc. are constants and

$$\bar{\omega} = \omega/k.$$

Substituting these in Eqs. (15), we find

$$\begin{aligned} \bar{q}_* + 2\mu_*\bar{\tau}_* + (1+\mu_*^2)\bar{s}_* &= 0, \\ (\mu_* + \gamma\eta_*)\bar{q}_* + (1+\mu_*^2 - \bar{\beta}\bar{\eta}_*)\bar{\tau}_* + 2\mu_*\bar{s}_* &= 0, \\ (1-\bar{\eta}_*)\bar{q}_* + (\mu_* + \gamma\bar{\eta}_*)\bar{\tau}_* + \bar{s}_* &= 0, \end{aligned} \tag{44}$$

where $*$ stands for $\pm$ and

$$\bar{\eta}_* = \bar{\eta}_\pm = \bar{\eta}/(1\pm\bar{\omega})^2. \tag{45}$$

Using

$$\bar{q}_* = \frac{1}{\bar{\eta}_* - 1}[(\mu_* + \gamma\bar{\eta}_*)\bar{\tau}_* + \bar{s}_*], \tag{46}$$

in the first two equations in (44), we have

$$\begin{aligned} \hat{A}_{11}\bar{\tau}_* + \hat{A}_{12}\bar{s}_* &= 0, \\ \hat{A}_{21}\bar{\tau}_* + \hat{A}_{22}\bar{s}_* &= 0, \end{aligned} \tag{47}$$

where

$$\begin{aligned} \hat{A}_{11} &= (1-\bar{\eta}_*)(1+\mu_*^2 - \bar{\beta}\eta_*) - (\mu_* + \gamma\bar{\eta}_*)^2, \\ \hat{A}_{12} &= \hat{A}_{21} = \mu_*(1-2\bar{\eta}_*) - \gamma\bar{\eta}_*, \\ \hat{A}_{22} &= (1-\bar{\eta}_*)(1+\mu_*^2) - 1. \end{aligned} \tag{48}$$

The characteristic equation for μ_* is obtained, similar to the zeroth order system, by setting $\hat{A}_{11}\hat{A}_{22} - \hat{A}_{12}\hat{A}_{21} = 0$, as

$$\mu_*^4 + \mu_*^2(\bar{\beta} - 2 - \eta_*) + 2\gamma\mu^*(\mu_*^2 - 1) + 1 - \eta_* = 0, \tag{49}$$

where

$$\eta_* = (\bar{\beta} - \gamma^2)\bar{\eta}_*. \tag{50}$$

Eq. (49) gives two solutions for μ_*, namely, μ_{*1} and μ_{*2} with positive imaginary parts for a certain range of values for $\bar{\omega}$. Unlike the zeroth order system, in Eq. (49) η_* is known for a given ω and η. We have a constraint on ω to keep the value of η_-, which is greater than η_+, below unity for exponentially decaying solutions, that is

$$\eta < (1 - \bar{\omega})^2. \tag{51}$$

Also, the nondimensional wave speeds of the scattered waves are given by

$$\eta_+ = \eta/(1 + \bar{\omega})^2, \quad \eta_- = \eta/(1 - \bar{\omega})^2. \tag{52}$$

Eqs. (47) can be solved in the form

$$\bar{\tau}_{*j} = -\hat{C}_{*j}\hat{A}_{12}(\mu_{*j}), \quad \bar{s}_{*j} = \hat{C}_{*j}\hat{A}_{11}(\mu_{*j}), \quad j = 1, 2, \tag{53}$$

where the constants $\hat{C}_{*j}, j = 1, 2$ have to be found using the boundary conditions. Corresponding to the two acceptable values of μ_*, we assume

$$\bar{\tau}_* = \bar{\tau}_{*1} + \bar{\tau}_{*2}, \quad \text{etc.} \tag{54}$$

The boundary conditions, Eqs. (19), become

$$\begin{aligned} \bar{\tau}_{*1} + \bar{\tau}_{*2} &= D_{*1} \equiv (\lambda_1 - \lambda_2)\left[\frac{\bar{\omega}}{\bar{\eta} - 1} - *1\right]\bar{\tau}_{01}, \\ \bar{s}_{*1} + \bar{s}_{*2} &= D_{*2} \equiv - * (\lambda_1 - \lambda_2)\bar{s}_{01}, \end{aligned} \tag{55}$$

where, again, "*" stands for "±".

Using the zeroth order solutions in the boundary conditions, we have

$$\begin{aligned} \hat{C}_{*1} &= -\frac{1}{\Delta_*}[\hat{A}_{11}(\mu_{*2})D_{*1} + \hat{A}_{12}(\mu_{*2})D_{*2}], \\ \hat{C}_{*2} &= \frac{1}{\Delta_*}[\hat{A}_{11}(\mu_{*1})D_{*1} + \hat{A}_{12}(\mu_{*1})D_{*2}], \end{aligned} \tag{56}$$

where

$$\Delta_* = \hat{A}_{11}(\mu_{*2})\hat{A}_{12}(\mu_{*1}) - \hat{A}_{11}(\mu_{*1})\hat{A}_{12}(\mu_{*2}). \tag{57}$$

Once, the constants $\hat{C}_{*j}$ are known, the first order solutions for $\bar{\tau}_{*j}$ and $\bar{s}_{*j}$ can be computed using Eq. (53).

Finally, the first order correction to the vertical displacement is

$$\begin{aligned} u_{21} = -\frac{i}{\eta k}&\left[(1 + \bar{\omega})(\bar{\tau}_{+1} + \bar{\tau}_{+2} + \mu_{+1}\bar{s}_{+1} + \mu_{+2}\bar{s}_{+2})\right. \\ &\left. + (1 - \bar{\omega})(\bar{\tau}_{-1} + \bar{\tau}_{-2} + \mu_{-1}\bar{s}_{-1} + \mu_{-2}\bar{s}_{-2})\right]. \end{aligned} \tag{58}$$

Table 3. Nondimensional first order stresses and the vertical displacement at the surface for $\bar{\beta} = 3.0$ and $\gamma = 0.2$ for two values of $\bar{\omega}$.

Quantity	$\bar{\omega} = 0.02$	$\bar{\omega} = 0.04$
$\bar{\eta}_+$	0.7874	0.7574
μ_{+1}	-0.5874+0.7326i	-0.5935+0.7470i
μ_{+2}	0.3874+0.3016i	0.3935+0.3341i
$\bar{\tau}_{+1}$	-0.0559+0.0544i	-0.0557+0.0562i
$\bar{\tau}_{+2}$	-0.0164-0.0201i	-0.0185-0.0210i
s_{+1}	-0.0416-0.0263i	-0.0442-0.0251i
s_{+2}	0.0127-0.0122i	0.0153-0.0133i
$\bar{\eta}_-$	0.8530	0.8889
μ_{-1}	-0.5714+0.6994i	-0.5606+0.6801i
μ_{-2}	0.3714+0.2055i	0.3606+0.1136i
$\bar{\tau}_{-1}$	0.0575-0.0503i	0.0601-0.0470i
$\bar{\tau}_{-2}$	0.0108+0.0178i	0.0063+0.0154i
$\bar{s}_{-1}$	0.0356+0294i	0.0315+0.03187i
$\bar{s}_{-2}$	-0.0066+0.0091i	-0.0025+0.0066i
ku_{21}	-0.0039-0.0028i	-0.0088-0.0061i

Table 3 shows the first order corrections to the stresses and the vertical displacement for two values of the frequency of undulations, $\bar{\omega} = 0.02$ and $\bar{\omega} = 0.04$. It was observed that when $\bar{\omega} = 0.05$, two exponentially decaying solutions in the vertical direction ceased to exist. The quartic equation (28) for λ gives a pair of complex conjugate and two real roots for this case.

5 Conclusion

We consider an incompressible, monoclinic solid in plane strain with a free surface in the form of a low amplitude sine wave. Rayleigh waves propagating along the surface is analyzed using perturbations in two stages: the first stage deals with a Rayleigh wave along a flat surface and the second stage deals with corrections due to the small amplitude sinusoidal perturbations to the flat surface. We derive the secular quartic equation for the wave speed directly to obtain the previous result obtained by Destrade et al. [9] using a limiting case of a compressible solid. We also obtain expressions for the first order corrections in the form of scattered waves with the corresponding shear and normal stresses. Numerical values are given in nondimensional form to illustrate the computational procedure for two different spatial frequencies of the undulations.

References

1. Lord Rayleigh. On waves propagated along plane surface of an elastic solid, *Proceedings of the London Mathematical Society*, 17: 4-11, 1885.

2.A. K. Mal. Dispersion of Love waves in spherical earth with corrugated surface, *Pure and Applied Geophysics*, 53: 25-44, 1962.
3.S. N. Chandler-Wilde, et al. Scattering by infinite one-dimensional rough surfaces, *Proc. R. Soc. Lond. A*, 455: 3767-3787, 1999.
4.X. Markenscoff and S. G. Lekoudis. Love waves in slowly varying layered media, *Pure and Applied Geophysics*, 114: 805-810, 1976.
5.S. Nair and D. A. Sotiropoulos. Interfacial waves in incompressible monoclinic materials with an interlayer, *Mech. Materials*, 31: 225-233, 1999.
6.S. Nair and D. A. Sotiropoulos. Elastic wave scattering from a perturbed flat interface, *Mechanical Waves for Composite Structures Characterization*, edited by D. A. Sotiropoulos,(Kluwer Academic Publishers, Dordrecht, The Netherlands), 151-168, 2001.
7.S. Nair and D. A. Sotiropoulos. Elastic waves in orthotropic incompressible materials and reflection from an interface, *J. Acoust. Soc. of Am.*, 102: 102-107, 1997.
8.D. A. Sotiropoulos and S. Nair. Elastic waves in monoclinic incompressible materials and reflection from an inteface, *J. Acoust. Soc. of Am.*, 105: 2981-2983, 1999.
9.M. Destrade, P. A. Martin and T. C. T. Ting. The incompressible limit in linear anisotropic elasticity, with applications to surface waves and elastostatics, *Journal of the Mechanics and Physics of Solids*, 50: 1453-1468, 2002.
10.T. C. T. Ting. A unified formalism for elastostatics or steady state motion of compressible or incompressible anisotropic elastic materials, *Int. J. Solids Struct.*, 39: 5427-5445, 2002.
11.M. Destrade. Rayleigh waves in symmetry planes of crystals: explicit secular equations and some explicit wave speeds, *Mech. Materials*, 35: 931-939, 2003.
12.R. W. Ogden and C. V. Pham. On Rayleigh waves in incompressible orthotropic elastic solids, *J. Acoust. Soc. of Am.*, 115: 530-533, 2004.
13.C. V. Pham and R. W. Ogden. Formulas for the Rayleigh wave speed in orthotropic elastic solids, *Arch. Mech.*, 56: 247-265, 2004.
14.C. V. Pham and R. W. Ogden. On the Rayleigh wave speed in orthotropic elastic solids, *Meccanica*, 40: 147-161, 2005.
15.D. Royer and E. Dieulesaint. Rayleigh wave velocity and displacement in orthorhombic, tetragonal, hexagonal, and cubic crystals, *J. Acoust. Soc. of Am.*, 76: 1438-1444, 1984.