Abstract. Willamowski-Rossler equation is explored to unfold the population of all the unstable periodic orbits and the corresponding topological structure embedded in the attractor. The linking numbers and torsions of these orbits are manifested in the template so obtained. It is interesting to observe that the templates for the two attractor are similar. on the other hand their exists transition between two attractor as the parameter varies. In the course of our computation we have discovered the shapes and location of periodic windows-continua of parameter values for which solution can be stably periodic embedded in a sea of chaotic solutions. They presents a very beautiful scenario of the manifold and are completely different for the two attractors.

Keywords: Manifold, template, torsions.

1 Introduction

The study of systems described by nonlinear evolution laws has afforded the concept of deterministic chaos, allowing one to look at many phenomena from a new perspective in such disparate fields as fluid mechanics, ecology, economics, etc. Nonlinear dynamical systems(with at least three variables) showing chaos exhibit sensitive dependence on the initial condition and at present universal routes leading to chaos. Usually one describes these systems in terms of a small number of macroscopic variables, supposed to be the most relevant for the problem. In this approach one neglects the microscopic structure of the system, which may induce the presence of fluctuations and correlations that are referred to as internal, intrinsic, or thermodynamic fluctuations(or noise). Very little is known about the relevance of these intrinsic fluctuations in the description of deterministic chaotic systems. However, one can mention the studies by Fox and Keizer[1,2] showing how in some cases the presence of fluctuations...
may induce a breakdown in the usual deterministic description in terms of macroscopic variables. Some other recent studies are those due to Nicolis and coworkers[3,4], who have found, by solving directly the master equation, that a deterministic description in terms of macroscopic variables may still be useful in the study of the behavior of the most probable value (in stead of the mean value). These conclusions find support in the lattice gas cellular automation stimulations performed by Wu and Kapral[5]. The purpose of this work is to show that the inclusion of intrinsic fluctuations may change the observable behavior in some chaotic systems, and more precisely in those systems in which the strange attractor coexists with a locally stable fixed point, such as in the Willamowski-Rossler[6] model of chemical chaos. Some chemical reactions involving auto catalysis are good examples of deterministic chaos[6,7] with the advantage that the evolution differential equations can be written in a simple way if one knows the chemical mechanism of the reaction. On the other hand, the effect of fluctuations in a chemical reaction can be analyzed if one writes the chemical master equation, where the chemical reactions are considered in terms of birth and death processes. The Rossler oscillator was originally introduced as a particularly simple model, which showed chaotic behavior, and surprisingly this needed only a quadratic nonlinearity[1]. For chemists, a major drawback of this model was that the variables could also assume negative values, and it was therefore not suited as prototype for a chemical system exhibiting chaos. Later Willamowski and Rossler introduced another model, which was fully consistent with the basic requirements of chemical systems such as for example only positive values of the variables, detailed balance at equilibrium[2]. Surprisingly only a few studies were dedicated to further investigate the special properties of this model. Aguda and Clarke, who identified the dynamic element governing the behavior of these oscillator and, in particular, suspected a similarity to the Volterra-Lotka(VL) model, thus far made the most detailed investigation[3]. Geysermans and Nicolis applied the master equation to the Willamowski-Rossler (WR) model and established full consistency with thermodynamic principles[4].

Soon after the discovery of the existence of an attractor in the case of a chaotic system people are trying to set up different measures to characterize the properties of such attractors which will give some idea of the dynamics involved in the said non-linear system. The attractor which is actually a set of unstable periodic orbits has an intricate geometry which can be explored with the topological approaches of Gilmore, Leitler, Mindlin et al.[1,2,3] on the other hand very recently it has been found that some nonlinear systems do posses more than one attractor based on parameter values and initial conditions. The Willamowski-Rossler system is one such system studied by some authors but not explored in its totality. In the present communication our motivation is to analyze the topological structure of the multi-attractor Willamowski-Rossler system is one such system studied by some authors but not explored in its totality. Another similar system was discovered by Ray et al.[4,10] from the basic plasma equations and studied from the topological point of view. In this chapter our motivation is to analyze the topological structure of the multi-attractor Willamowski-Rossler system[5,6] and study the differences manifested...
in the structure of linking number, torsion and template for the two attractors. In this connection it may be mentioned that one of the most important characterization of the attractor is also done by Lyapunov exponents. These windows have very interesting geometric shapes and are originally discovered by Gallas[7] and later rediscovered by Lorenz. They were called shrimps. So as a whole we have mapped the full skeleton of the attractors and have constructed the corresponding templates which gives rise to a global information of all the UPOs.

2 Study of the dynamical equations

The Willamowski-Rossler [6] mass action model can be represented by the following chemical equations

\[ \dot{x} = k_1 x - k_{-1}x^2 - k_2 xy + k_2 y^2 - k_4 xz + k_{-4} \]  
\[ \dot{y} = k_2 xy - k_{-2} y^2 - k_3 y + k_{-3} \]  
\[ \dot{z} = -k_4 xz + k_{-4} + k_5 z - k_{-5}z^2 \]

Already in their original paper Willamowski and Rossler noted that some of the rate constant could be set to vary low values without much compromising the dynamics[2]. Indeed a minimal version of this model which still allows chaotic behavior, consists in setting the backward rate constant \( k_{-2} = k_{-3} = k_{-4} = 0 \) to obtain a minimal WR model(MWR).

\[ \dot{x} = k_1 x - k_{-1}x^2 - k_2 xy + k_2 y^2 - k_4 xz \]  
\[ \dot{y} = k_2 xy - k_3 y \]  
\[ \dot{z} = -k_4 xz + k_5 z - k_{-5}z^2 \]

So it is a three dimensional system and \((k_1, k_{-1}, k_2, k_3, k_4, k_5, k_{-5})\) are parameters. The fixed points of the equation are given as

\[(x_0, y_0, z_0) = (0, 0, 0)\]  
\[(x_0, y_0, z_0) = (0, 0, \frac{k_5}{k_{-5}})\]  
\[(x_0, y_0, z_0) = (\frac{k_1}{k_{-1}}, 0, 0)\]  
\[(x_0, y_0, z_0) = (\frac{k_3}{k_2}, \frac{k_{-1}k_3 - k_1k_2}{k_2^2}, 0)\]  
\[(x_0, y_0, z_0) = (\frac{k_{-5}k_1 - k_4k_5}{k_{-5}k_{-1} - k_3^2}, \frac{k_{-1}k_5 - k_4k_3}{k_{-5}k_{-1} - k_2^2}, 0)\]  
\[(x_0, y_0, z_0) = (\frac{k_3}{k_2}, \frac{k_2k_3 \pm \sqrt{k_2^2k_3^2 - 4k_1k_3k_5^2 + 4k_2k_{-1}k_3^2}}{2k_2^2}, 0)\]  
\[(x_0, y_0, z_0) = (\frac{k_3}{k_2}, \frac{k_2k_3 k_{-5} \pm w}{k_2 k_5 - k_3 k_4}, \frac{k_5 k_{-5}}{2ak_2^2})\]

where \(w = \sqrt{k_2^2k_3^2k_5^2 - 4k_2k_{-5}k_4^2k_3^2 + 4k_{-5}k_2^2k_5k_4k_3 - 4k_{-5}k_2^2k_1k_3 + 4k_{-5}k_2k_{-1}k_3^2} \)
The local stability is determined by the eigenvalues of the corresponding jacobian evaluated at the relevant position. The characteristics equation is

\[\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \tag{14}\]

where

\[
\begin{align*}
a_2 &= k_4x - e + 2k_{-5}z - k_2x + k_3 - a + 2xk_{-1} + k_2y + k_4z \\
a_1 &= -2k_{-5}k_2xz + 4k_{-5}k_{-1}xz + 2k_2k_{-1}yz - k_2k_4xz + k_2k_4xy - k_2k_5y \\
&+ 2k_{-5}k_4z^2 - 2k_{-5}k_1z + 2k_{-5}k_3z + 2k_4k_{-1}x^2 - k_2k_4x^2 + k_3k_4z - 2k_5k_{-1}x \\
&+ k_1k_2x - 2k_2k_{-1}x^2 + k_2k_3x - k_4k_5z + 2k_3k_{-1}x + k_2k_3y - k_1k_4x \\
&+ k_3k_4x - k_3k_5 + k_1k_5 - k_1k_3 - 2k_2^2y^2 \\
a_0 &= k_1k_3k_5 + 2k_2^2k_5y^2 - 2k_3k_4xy^2 + k_1k_2k_4x^2 - 2k_{-1}k_2k_4x \\
&- k_1k_3k_4x + 2k_3k_4k_{-1}x^2 - k_1k_2k_5x + 2k_2k_5k_{-1}x^2 - 2k_3k_5k_{-1}x \\
&- k_2k_5k_4y - k_5k_5k_3z - 4k_{-5}k_2^2y^2z - 2k_{-5}k_5k_3z + 2k_{-5}k_4k_3^2 + k_4k_3k_2xy \\
&+ k_5k_2k_4xz + 2k_{-5}k_2k_1xz - 4k_{-5}k_2k_{-1}zx^2 - 2k_{-5}k_2k_4z^2x + 4k_{-1}k_3k_5xz \\
&+ 2k_{-5}k_3k_2yz
\end{align*}
\]

The stability of fixed points are obtained from using Routh-Hurwitz criterion \(a_2 > 0, \ a_2a_1 - a_0 > 0\) and \(a_0 > 0\). Here \((x, y, z)\) in expressions of \(a_2, a_1\) and \(a_0\) stands for a general fixed point. To start with if we consider the origin \((0, 0, 0)\) then Routh-Hurwitz criterion reduces to following inequalities

\[
\begin{align*}
k_3 - k_5 + k_1 &> 0 \tag{15} \\
(k_3 - k_5 + k_1)(k_5k_1 - k_5k_1 - k_1k_3) &> k_1k_3k_5 \tag{16} \\
k_1k_3k_5 &> 0 \tag{17}
\end{align*}
\]

For the fixed points given by Eq. (8), Routh-Hurwitz criterion reduces to

\[
\begin{align*}
k_5 + k_3 - k_1 + k_5^2k_4 \frac{k_3}{k_{-5}} &> 0 \tag{18} \\
(k_5 + k_3 - k_1 + k_5^2k_4 \frac{k_3}{k_{-5}})(k_5^2k_4 \frac{k_3}{k_{-5}} + k_5k_3 - k_1k_5) \\
&+ k_3k_4k_5 \frac{k_3}{k_{-5}} - k_1k_3) &> k_1k_3k_5 \tag{19} \\
k_5^2k_3k_4 \frac{k_3}{k_{-5}} - k_1k_3k_5 &> 0 \tag{20}
\end{align*}
\]

Similarly, relations for other fixed points can also be obtained using similar procedure. To proceed with the exploration, we first consider the structure in the parameter space of \((k_{-1}, k_4)\) where other parameter values are kept at \((k_1 = 30, k_2 = 1.0, k_3 = 10, k_5 = 0.25 \ \text{and} \ k_{-5} = 0.5)\). This is shown in Fig. (1). Here we have plotted the maximum Lyapunov exponents with simultaneous variation of \(k_{-1}\) and \(k_4\) in a \((512 \times 512)\) grid space. Here two
Fig. 1. Shrimp structure for Willamowski and Rossler when other parameters are kept at \((k_1 = 30, k_2 = 1.0, k_3 = 10, k_5 = 0.25 \text{ and } k_{-5} = 0.5)\)

distinct islands, represented with variation from yellow to red color in Fig. (1a), represent two attractor regions. In the space, in between two regions the attractors gets destroyed. In latter part of chapter, we will try to determine if these two attractors are same or different in nature. Chaotic oscillations (i.e. positive exponents) are represented using colors while periodic oscillations (negative exponents) are shown using a blue scale. The color scale is linear on both sides of zero but is not uniform from minimum to maximum exponent. In other words, the scales of colors and of gray shadings are linear on both sides of zero but are independent from each other. Furthermore, the scales of each individual phase diagram were always renormalized to reflect the corresponding minima and maxima of the diagram. That is why colors of identical structures may look slightly different from panel to panel, when enlarged. Different Hubs are clearly indicated in Figs. (1c) and (1d). Individual spirals originating/terminating at the focal hub are characterized by specific families of periodic oscillations embedded in the chaotic phase. Spirals and the spiral nesting are truly codimension-two phenomena: they may be only fully unfolded by tuning two or more parameters simultaneously. There are also white spirals of chaos. In Fig. (2), variation of periodic region with the variation of parameter \((k_4, k_{11}, \text{and } k_{55})\) are indicated through bifurcation diagrams with corresponding parameters. In first two pictures, two different region of attractors are evident. Destruction of one attractor through boundary crises leads to another attractor. This is the reason for choosing \((k_4, k_{11})\) as a candidate for shrimp structure plot on previous occasion. So it appears that both the attractors existed in two different parts of parameter space. Figure of two such attractors are shown in Fig. (3). Corresponding parameter values are given in figure captions. Though they look similar, we would try to distinguish them from its manifold calculations and topological structures. In Fig. (4), we have plotted the manifold structures of two different attractors. If we compare
Fig. (4a) with Fig. (4c) and Fig. (4b) with Fig. (4d) the distinctions become evident. The topological structures will be seen when we will be exploring the nature of unstable periodic orbit individually.

Fig. 2. Bifurcation for Willamowski-Rossler system. (a) Represents bifurcation with respect to $k_4$, (b) Represents the bifurcation with respect to $k_{11}$ and (c) represents the bifurcation with respect to $k_{55}$. In first two figures, presence of two different attractors are evident.

3 Topological properties

Topology of a strange attractor\cite{10,11,12,13} is actually a result of simultaneous stretching and folding of various trajectories inside the attractor. Topological information gives an idea of the organization and interconnection of the UPO’s, which can be characterized by the corresponding linking numbers between them. Further more these linking numbers themselves can be deduced from the template. Few of the unstable periodic orbits obtained from attractors at $(k_{-1} = 0.3, k_4 = 1.15)$ while all other parameter values are kept same as Fig. (1), is shown in Fig. (5). A detailed discussion on various topological properties has already been given in the last chapter. The relevant template structures are given below. For attractor at $(k_{-1} = 0.3, k_4 = 1.15)$ while all other parameter values are kept same as Fig. (1), is
Fig. 3. Two different attractors for Willamowski-Rossler system. (a) \((k_{-1} = 0.6, k_4 = 1.075)\), (b) \((k_{-1} = 0.3, k_4 = 1.15)\). All other parameter values are kept same as Fig. (1)

Fig. 4. Two different manifolds with attractors for Willamowski-Rossler system. (a) & (b) Indicates two projections of manifold at \((k_{-1} = 0.6, k_4 = 1.075)\). (c) & (d) Indicates two projections of manifold at \((k_{-1} = 0.3, k_4 = 1.15)\). All other parameter values are kept same as Fig. (1)
Fig. 5. Different period for attractors of Willamowski-Rossler system at \((k_{-1} = 0.6, k_4 = 1.075)\). All other parameter values are kept same as Fig. (1)

\[
\begin{pmatrix}
0 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
0 & -1
\end{pmatrix}
\]

Similarly, attractor at \((k_{-1} = 0.6, k_4 = 1.075)\) while all other parameter values are kept same as Fig. (1), is

\[
\begin{pmatrix}
0 & 0 \\
0 & -2
\end{pmatrix}
\begin{pmatrix}
0 & -1
\end{pmatrix}
\]

From the template structures, it is evident the two attractors are not same. Clearly there is an extra torsion present in the attractor at \((k_{-1} = 0.6, k_4 = 1.075)\) from that at attractor at \((k_{-1} = 0.3, k_4 = 1.15)\).

The corresponding template structures are shown in Fig. (6). Here the arrow denotes the direction of the flow.

4 Conclusion

Our topological analysis detailed above tries to throw some light on the geometrical structure of the attractor of Willamowski-Rössler system. This treatment can give some hints about the global dynamics underneath. It can be ascertained that only such an analysis can give us a holistic idea about the
Fig. 6. Two different attractors for Willamowski-Rossler system. (a) \((k_1 = 0.6, k_4 = 1.075)\), (b) \((k_1 = 0.3, k_4 = 1.15)\). All other parameter values are kept same as Fig. (1)

dynamical unfolding going on. We have obtained the homoclinic orbit, linking matrix, stable and unstable manifold and last but not the least the template. We have seen a recent approach of flow curvature manifold initiated by Ginoux can give a very elegant view of the attractor in relation to the various fixed point and their corresponding null-lines.

5 References