Chaotic Modeling and Simulation (CMSIM) 4: 433–440, 2017

Weighted recurrence networks from chaotic time series

K. P. Harikrishnan[1], R. Jacob[1], R. Misra[2], and G. Ambika[3]

[1] Department of Physics, The Cochin College, Cochin - 682 002, India
(E-mail: *kp_hk2002@yahoo.co.in*)
[2] Inter University Centre for Astronomy and Astrophysics, Pune - 411 007
(E-mail: rmisra@iucaa.ernet.in)
[3] Indian Institute of Science Education and Research, Pune - 411 008, India
(E-mail: g.ambika@iiserpune.ac.in)

Abstract. Recurrence networks are unweighted and undirected complex networks constructed from time series of dynamical systems. In this work, we propose a method to construct *weighted recurrence network* from a time series which can provide a single index to compare the structural complexity of different chaotic attractors. This is because, a specific network measure, the node strength distribution of the weighted recurrence network, from every chaotic attractor follows a common pattern: a power law with exponential cut off at the tail. We show that the power law index is characteristic to the fractal structure of the attractor. For pure noise, this index tends to zero and the distribution becomes exponential.
Keywords: Weighted Recurrence Networks, Time Series Analysis, Node Strength Distribution.

1 Introduction

Nonlinear time series analysis using complex network measures has become an important area of research over the last two decades [1]. This approach has several advantages over the conventional approach, especially in the characterization of the structural properties of the underlying attractor. It involves first transforming the time series into a complex network and analyzing it using the standard network measures. Several methods [2–4] have been proposed in the literature to transform a time series into a complex network, with each of them finding application in particular contexts, to address the complementary features of the time series not obtained from the conventional approach using, mainly, correlation dimension and entropy.

A simple and direct method to convert time series to complex network is using the property of recurrence [5] of every dynamical system and the resulting network is called recurrence network (RN) [6]. To transform the time series into a RN, it is first embedded in a multi-variate state space of dimension M using

Received: 12 July 2017 / Accepted: 10 October 2017

ISSN 2241-0503

the time delay co-ordinates [7]. Every point in the attractor is then identified as a node and a recurrence threshold (ϵ) is set to define the connection between two nodes. Two nodes are considered to be connected if the corresponding points on the attractor are within the limit of this threshold. From the construction, it is clear that the RN is an unweighted and undirected network with the elements of the adjacency matrix A_{ij} either 1 or 0 depending on whether two nodes are connected or not. Once constructed, an array of statistical measures [8] can be defined from the RN that can characterize the structural properties of the attractor underlying the time series [9].

A crucial parameter in the construction of the RN is the recurrence threshold, ϵ, since the characteristic properties of the RN depend on its value. In general, for each embedded attractor from the time series, the value of ϵ has to be determined separately as it varies with the size of the attractor. Two criteria are usually employed [6,8] to select ϵ. The first and the primary one is that there should be a giant component for the resulting RN which sets a lower bound for ϵ. In order to ensure that the network is not overconnected, the upper bound for ϵ is set such that the link density (the ratio of actual connections to all possible connections in a network of N nodes) is only a small fixed fraction of the maximum possible value. This provides a small range $\Delta\epsilon$ of optimum threshold for each system where the resulting network is considered to be a proper network representation of the time series.

Recently, we have proposed a scheme [10] where we tried to fix a small uniform range $\Delta\epsilon$ for choosing the threshold for time series from different chaotic systems. For this, we first transform the time series to a uniform deviate so that the size of the attractor always remains within the unit cube. To find the lower bound of ϵ, we use the standand criterion that the network turns into a single giant component. The upper bound is determined by the condition that the network is not overconnected. However, instead of fixing the link density, we apply a criterion that the characteristic path length (that defines the global connectivity of the network) of RN from chaotic time series is significantly different from that of white noise. The scheme has been effectively applied to compare network measures from different chaotic attractors [10], to study the influence of noise on the structure of chaotic attractors [11] and to propose a new heterogeneity index [12] for complex networks which, in turn, provides a unique measure for each chaotic attrator through RN. We stick to the same criteria for the selection of ϵ in this work.

2 Weighted recurrence network and strength distribution

The unweighted RN is constructed first using the selected value of ϵ. In order to convert it into a weighted RN (WRN), one has to assign weight factor to every link in the network. For weighted networks that model any real world system or interaction, the weight factor will be specific to the network. For example, in a transportation network, it may depend on the distance between two nodes while for a communication network, the same may be characterized by the rate of information transfer through the link. Here we introduce a general criterion for assigning the weight factors that can be adopted to any kind of network, but especially useful for RNs.

Assume that the RN has N number of nodes and the $\imath^{th}$ node has a degree k_i. That is, it is connected to k_i other nodes in the network. The weight factor w_{ij} for the link between two nodes $\imath$ and $\jmath$ in the network is defined as:

$$w_{ij} = \frac{\sqrt{k_i k_j}}{k_{max}} \tag{1}$$

where k_{max} is the maximum degree in the network. Note that the maximum possible value of w_{ij} is normalized as 1 and occurs for a link between two nodes which are connected to k_{max} other nodes in the network. For a reference node i in general, it is connected to k_i other nodes with each link having a different weight factor.

From the point of view of a network, the more the number of connections for a node $\imath$, the shorter the path becomes when connected through the node. For example, if a nearly isolated (with $k_i \sim 1$) node is connected to a hub, that connection carries a high weight factor (due to the large degree of the hub) and provides an easy path between the node and any other arbitrary node in the network.

The average of the weight factors associated with a node as determined by its connections is defined as the *strength* of the node, s. For example, for the node $\imath$, we have:

$$s(i) = \sum_{j=1}^{k_i} w_{ij} \tag{2}$$

If all the nodes have equal number of connections $< k >$, the weight factor of each node is approximately the same and the network can be considered as a homogeneous weighted network. As the weight factors among the nodes become more diverse, the network becomes more heterogeneous. The average weight factor associated with the whole network is defined as the weighted link density:

$$\rho_w = \frac{\sum_{i,j} w_{ij}}{N(N-1)} \tag{3}$$

In this work, we use the time series from the standard Lorenz arractor (parameters $\sigma = 10$, $\rho = 8/3$ and $r = 28$) to illustrate the construction of WRN and the utility of its measures. For any unweighted complex network, the degree distribution, denoted by $P(k)$, is a probability distribution representing how many nodes have a given degree k. For random graphs (RG), the degree distribution is Poissonian where as for scale free (SF) networks, it obeys a power law [13]. For the RN from chaotic time series, the degree distribution is characteristic to the structure of the attractor [10]. To generalize the degree distribution for the WRN, we first note that the characteristic property of a node that decides its connectivity in the network is not its degree, but its strength s as defined in Eq. 2 [14]. In other words, the degree distribution has to be replaced by the *strength distribution* of the weighted network which represents the probability $P(s)$ of nodes having a given strength s in a network of N nodes. Even though s varies discretely, it is not an integer like k.

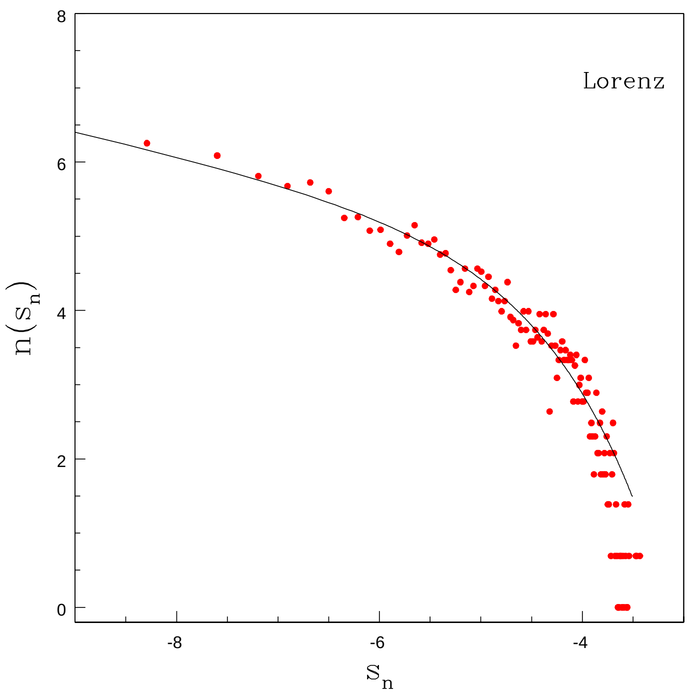

Fig. 1. Distribution of the normalized node strength in the weighted RN constructed from the standard Lorenz attractor time series (red solid circles) in log scale. The solid line is the power law fit before the exponantial cut off (see text). The embedding dimension used for constructing the network is $M = 3$ and the number of nodes in the network, $N = 5000$.

Instead of using the strength distribution directly, we compute a *normalized strength distribution* that reveals the utility of WRN. Since s varies discretely, we can write

$$\sum_s P(s)\Delta s = 1 \tag{4}$$

We now find the number of nodes $n(s_n) \equiv NP(s)$ (rather than the probability of nodes) corresponding to a normalized strength $s_n = \frac{\Delta s}{N}$ and the above equation can be re-written as

$$\sum_{s_n} n(s_n) s_n = 1 \tag{5}$$

Here $n(s_n)$ is the number of nodes having strength around the normalized value s_n, which varies in the range $[0, 1]$.

In Fig. 1, we show the normalized node strength distribution of the WRN from the Lorenz attractor (solid circles) in the $\log - \log$ scale. Log scale is taken to show that $n(s_n)$ decreases with s_n as a power law initially with an exponantial cut off at the tail. In fact, we have found that the variation can be represented using the following functional fit:

$$n(s_n) \propto s_n^{-\gamma} e^{-s_n/c} \tag{6}$$

with the parameters γ and c depending on the particular system. This functional fit is also shown in Fig. 1 (solid line). The crucial parameter here is the power law index γ indicating a scale free character for the distribution initially. The average value of γ from 10 different simulations is found to be 0.33 ± 0.05 for WRN from the Lorenz attractor.

We now show that this distribution is a characteristic property of every chaotic attractor and is independent of changes in parameters, such as, embedding dimension M and the number of nodes in the network N. This is illustrated using WRN from Lorenz attractor in Fig. 2 using two N values with fixed M (top panel) and vice versa (bottom panel). The result implies that the power law index γ is a characteristic index for a chaotic attractor.

We have repeated the calcuation for several other chaotic attractors and have found that there is a unique power law index γ for the WRN from each chaotic attractor. In Fig. 3, we show the strength distribution from the standard Rössler attractor ($a = 0.2$, $b = 0.2$ and $c = 7.8$) and the Henon attractor ($a = 1.4$ and $b = 0.3$). Finally, we consider the WRN from a pure white noise and show that the distribution is qualitatively different from that of chaotic time series. It is found that the power law part is absent in this case and the variation is purely exponential:

$$n(s_n) \propto e^{-s_n/c} \tag{7}$$

The distribution is shown in Fig. 4 for two different N values with fixed M (top panel) and vice versa (bottom panel). Contrary to tha case of chaotic time series, the distribution changes with M for white noise. The reason is that there is no structure for the attractor and the trajectory tends to fill the available state space volume. This changes the WRN and hence the distribution.

3 Conclusion

Analysis of time series data by converting into recurrence networks is an active area of research with several practical applications. The recurrence networks considered so far in the literature are unweighted with binary connections. Here we propose a specific method to construct weighted recurrence network from time series. The degree distribution gets generalised to the node strength distribution which we consider in detail in this work. We show that the strength distribution of WRN from all chaotic attractors follow a common pattern having a power law variation with an exponential cut off at the tail. The power law index γ is characteristic to the specific attractor and $\gamma \to 0$ as the time

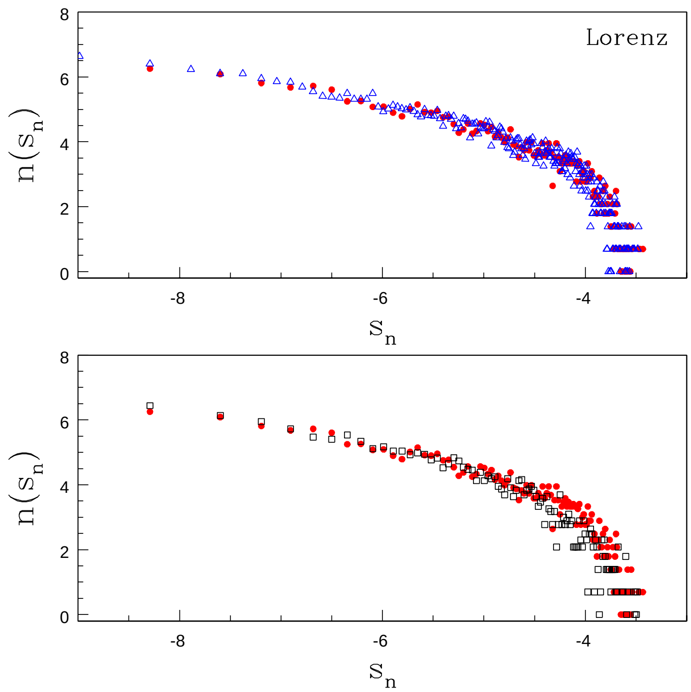

Fig. 2. Top panel shows the normalised strength distributions for the RN from the standard Lorenz attractor for number of nodes $N = 5000$ (red solid circles) and $N = 10000$ (blue open triangles), with embedding dimension $M = 3$. Bottom panel shows the same for two embedding dimensions $M = 3$ (red solid circles) and $M = 4$ (black open squares), with $N = 5000$.

series tends to pure white noise. Our analysis here is only preliminary using only a single statistical measure from the WRN. The results indicate that the WRN can be a potential tool in nonlinear time series analysis. More detailed analysis using other net theoretic measures are currently underway and will be presented elsewhere.

References

1. R. Albert and A. L. Barabasi, Statistical mechanics of complex networks, *Rev. Mod. Phys*, **74**, (2002)
2. J. Zhang and M. Small, Complex networks from pseudoperiodic time series: topology versus dynamics, *Phys. Rev. Lett.*, **96**, 238701 (2006)
3. X. Xu, J. Zhang and M. Small, Super family phenomena and motifs of networks induced from time series, *Proc. Natl. Acad. Sci. USA*, **105**, 19601 (2008)
4. N. Marwan, J. F. Donges, Y. Zou, R. V. Donner and J. Kurths, Complex network approach for recurrence analysis of time series, *Phys. Lett. A*, **373**, 4246 (2009)
5. J. P. Eckmann, S. O. Kamphorst and D. Ruelle, Recurrence plot of dynamical systems, *Europhys. Letters*, **5**, 973 (1987)

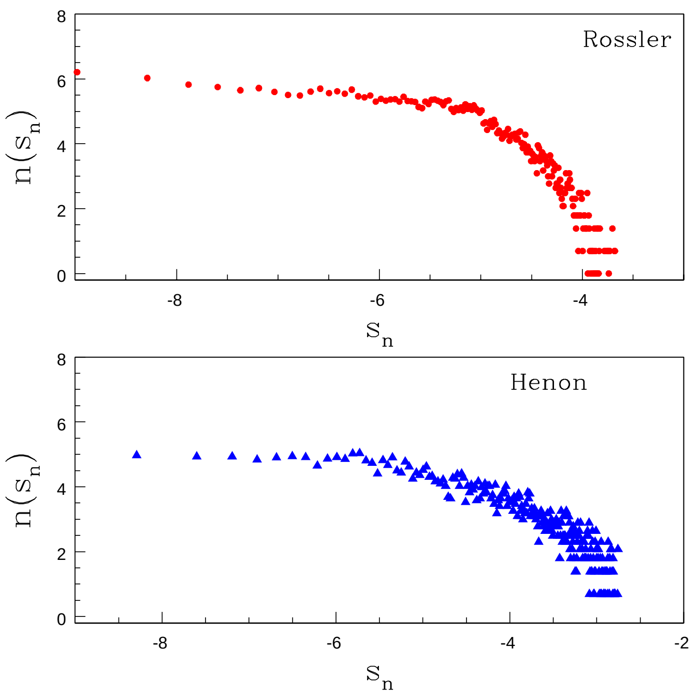

Fig. 3. Top panel shows the node strength distribution of the weighted RN constructed from the Rössler attractor. Bottom panel shows the same for the RN from the Henon attractor. In both cases, $N = 5000$ and $M = 3$.

6. R. V. Donner, Y. Zou, J. F. Donges, N. Marwan and J. Kurths, Recurrence networks: A novel paradigm for nonlinear time series analysis, *New J. Phys.*, **12**, 033025 (2010)
7. P. Grassberger and I. Procaccia, Measuring the strangeness of strange attractors, *Physica D*, **9**, 189 (1983)
8. R. V. Donner, M. Small, J. F. Donges, N. Marwan, Y. Zou, R. Xiang and J. Kurths, Recurrence based time series analysis by means of complex network methods, *Int. J. Bif. Chaos*, **21**, 1019 (2011)
9. R. V. Donner, J. Heitzig, J. F. Donges, Y. Zou, N. Marwan and J. Kurths, The geometry of chaotic dynamics - A complex network perspective, *Eur. Phys. J. B*, **84**, 653 (2011)
10. R. Jacob, K. P. Harikrishnan, R. Misra and G. Ambika, Uniform framework for the recurrence-network analysis of chaotic time series, *Phys. Rev. E*, **93**, 012202 (2016)
11. R. Jacob, K. P. Harikrishnan, R. Misra and G. Ambika, Characterization of chaotic attractors under noise: A recurrence network perspective, *Comm. Nonlinear Sci. Num. Simul.*, **41**, 32 (2016)
12. R. Jacob, K. P. Harikrishnan, R. Misra and G. Ambika, Measure for degree heterogeneity in complex networks and its application to recurrence network analysis, *Royal Soc. Open Sci.*, **4**, 160757 (2017)
13. M. E. J. Newman, The structure and function of complex networks, *SIAM Rev.*, **45**, 167 (2003)

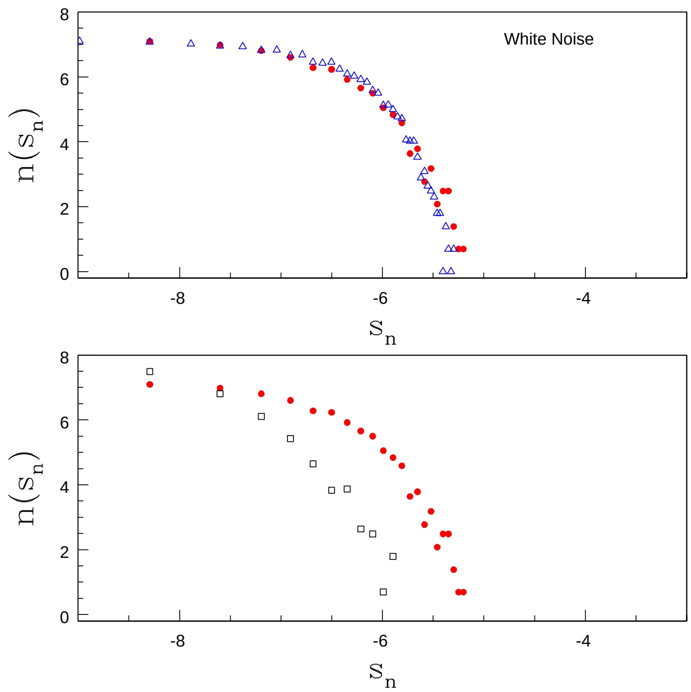

Fig. 4. A comparison of the normalized strength distributions of the WRN from white noise for two N values (top panel) with fixed M and vice versa (bottom panel). In the top panel, solid circles correspond to $N = 5000$ and open triangles to $N = 10000$ with M fixed at 3, whereas, in the bottom panel, solid circles correspond to $M = 3$ and open squares to $M = 4$ with N fixed at 5000.

14. J. D. Noh and H. Reiger, Stability of shortest paths in complex networks with random edge weights, *Phys. Rev E*, **66**, 066127 (2002)