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Dynamics and Estimates of Star-Shaped Reachable Sets of Nonlinear Control Systems

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Abstract. We consider the problem of estimating reachable sets of nonlinear dynamical control systems with uncertainty in initial states when we assume that we know only the bounding set for initial system positions and any additional statistical information is not available. We study the case when the system nonlinearity is generated by bilinear terms in the right-hand parts of differential equations. In particular, bilinearity may be caused by uncertainty in the matrix elements included in the state velocities of dynamical system. We deal with star-shaped reachable sets and use for its description the Minkowski gauge functions. Using results of the theory of trajectory tubes of control systems and techniques of differential inclusions theory and also results of ellipsoidal calculus we find set-valued estimates of related reachable sets of such nonlinear uncertain control system.
Keywords: Nonlinear control systems, Bilinear dynamics, Estimation approaches, Ellipsoidal calculus, Funnel equations, Trajectory tubes.

1 Introduction

The paper deals with the estimation problems for uncertain systems in the case when a probabilistic description of noise and errors is not available, but only bounds on them are known (Kurzhanski[18], Kurzhanski and Valyi[20], Kurzhanski and Varaiya[21], Schweppe[28]). Such models may be found in many applied areas ranged from engineering problems from physics to economics as well as to biological and ecological modeling when a stochastic nature of the errors is questionable because of limited data or because of complexity of the model (August *et al.*[1], Boscain *et al.*[3], Ceccarelli *et al.*[4], Milanese *et al.*[25]). Related results connected with a so-called bounded-error characterization, also called set-membership approach, has been proposed and intensively developed in the last decades (Bertsekas and Rhodes[2], Chernousko[6], Kurzhanski[18], Kurzhanski and Valyi[20], Kurzhanski and Varaiya[21], Milanese *et al.*[25], Schweppe[28], Walter and Pronzato[29]). For models with linear dynamics under such set-membership uncertainty there are several constructive approaches which allow finding effective estimates of reachable sets

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of control systems under uncertainty (Chernousko[6,7], Gusev[17], Kurzhanski and Valyi[20], Kurzhanski and Varaiya[21], Polyak *et al.*[26]). Among recent challenges in nonlinear set-membership estimation theory is to develop techniques, which produce bounds for the reachable set of unknown states of dynamical systems under uncertainty without being too computationally demanding, some of such approaches may be found e.g. in Brockett[5], Kurzhanski and Filippova[19], Mazurenko[24], Filippova and Lisin[11].

We consider in this paper the nonlinear dynamical system containing both measurable and impulsive controls and assume that the system nonlinearity is generated by the combination of two types of functions in related differential equations, one of which is of bilinear type with ellipsoidal constraints on uncertain matrix parameters and the other one is quadratic. The study continues the previous researches (Filippova and Lisin[11], Filippova, Matviychuk and Makarov[14], Filippova and Matviychuk[15]) and extends the approaches for the case of impulsive control systems with a new class of uncertainties in matrix parameters presented in dynamical equations.

The paper is organized as follows. Section 2 gives the problem statement and introduces the terminology used in the paper. In Section 3 we provide new results on finite difference estimation schemes. Here we present also an algorithm for calculating the external ellipsoidal estimate of reachable sets of the nonlinear system. Finally, Section 4 presents conclusions and the last Section contains acknowledgments.

2 Problem formulation

2.1 Notations and preliminaries

We introduce here the following notations. Let $\mathbb{R}^n$ be the n-dimensional vector space, $\operatorname{comp}\mathbb{R}^n$ be the set of all compact subsets of $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$ stands for the set of all real $n\times m$-matrices, $x'y = (x,y) = \sum_{i=1}^n x_i y_i$ be the usual inner product of $x, y \in \mathbb{R}^n$ with prime as a transpose,

$$\|x\| = \|x\|_2 = (x'x)^{1/2}, \quad \|x\|_\infty = \max_{1\le i\le n} |x_i|$$

be vector norms for $x\in\mathbb{R}^n$, $I\in\mathbb{R}^{n\times n}$ be the identity matrix, $\operatorname{tr}(A)$ be the trace of $n\times n$-matrix A (the sum of its diagonal elements). We denote by $B(a,r) = \{x\in\mathbb{R}^n : \|x-a\|\le r\}$ the ball in $\mathbb{R}^n$ with a center $a\in\mathbb{R}^n$ and a radius $r>0$ and by

$$E(a,Q) = \{x\in\mathbb{R}^n:\ (Q^{-1}(x-a),(x-a))\le 1\}$$

the *ellipsoid* in $\mathbb{R}^n$ with a center $a\in\mathbb{R}^n$ and with a symmetric positive definite $n\times n$-matrix Q.

Consider the following system ($t_0\le t\le T$, $x\in\mathbb{R}^n$)

$$\begin{gathered} dx(t) = \big(A(t)x(t) + x'Bx\cdot d + u(t)\big)dt + C dv(t), \\ A(t) = A^0 + A^1(t), \quad A^1(t)\in\mathcal{A}^1, \quad x(t_0-0) = x_0 \in \mathcal{X}_0 = E(a_0,Q_0), \end{gathered} \tag{1}$$

where B and Q_0 are positive definite and symmetric matrices, parameters d, C, a_0 are known n-vectors ($d, C, a_0 \in \mathbb{R}^n$).

Here we assume that the $n\times n$-matrix A^0 is given and the measurable $n\times n$-matrix $A^1(t)$ is unknown but bounded and satisfies the constraint $A^1(t) \in \mathcal{A}^1$ ($t \in [t_0, T]$), therefore we have the inclusion

$$A(t) \in \mathcal{A} = A^0 + \mathcal{A}^1, \quad t_0 \le t \le T. \tag{2}$$

We consider the case when $\mathcal{A}^1$ in (2) has the form

$$\begin{gathered} \mathcal{A}^1 = \Big\{ \ A{=}\{a_{ij}\}{\in}R^{n\times n} : a_{ij} = 0 \ \text{ for } \ i \ne j, \ \text{ and } \\ a_{ii} = a_i, \quad i = 1, \ldots, n, \quad a = (a_1, \ldots, a_n), \quad a'Da \le 1 \ \Big\}, \end{gathered} \tag{3}$$

where $D \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix. This assumption differs from the case considered earlier in Filippova, Matviychuk and Makarov[14], Filippova and Matviychuk[15]) and generalizes the results (Filippova and Lisin[11], Matviychuk[22,23].

We assume here that the impulsive function $v : [t_0, T] \to \mathbb{R}$ in (1) is of bounded variation on $[t_0, T]$, monotonically increasing and right-continuous,

$$\begin{gathered} \mathrm{Var}_{t\in[t_0,T]}\, v(t) = \sup_{\{t_i\}} \sum_{i=1}^{k} |v(t_i) - v(t_{i-1})| \le \mu, \\ \forall t_i : t_0 \le t_1 \le \ldots \le t_k = T, \end{gathered} \tag{4}$$

where $\mu > 0$ is given. Denote the above class of functions $v(\cdot)$ as $\mathcal{V}$.

We assume also $u(t) \in \mathcal{U} = E(\hat{a}, \hat{Q})$ where the center $\hat{a}$ and the matrix $\hat{Q}$ of the ellipsoid U are known.

The reachable set $\mathcal{X}(t)$ of the system (1)-(4) at time t ($t_0 < t \le T$) is defined as the cross-section of the related trajectory tube (Kurzhanski and Filippova[19])

$$\begin{gathered} \mathcal{X}(t) = \mathcal{X}(t;\ t_0, X_0) = \{\ x \in \mathbb{R}^n :\ \exists\, x_0 \in X_0,\ \exists\, u(\cdot) \in \mathcal{U},\ \exists\, v(\cdot) \in \mathcal{V}, \\ \exists\, A(\cdot) \in \mathcal{A}^1 \ \text{ such that } \ x = x(t) = x(t;\, u(\cdot),\, v(\cdot),\, x_0,\, A(\cdot))\ \}. \end{gathered} \tag{5}$$

The main problem studied here is the following

Problem. Find the external ellipsoidal estimate $E(a^+(t), Q^+(t))$ (with respect to the inclusion of sets) of the reachable set $\mathcal{X}(t)$ ($t_0 < t \le T$) by using the analysis of a special type of nonlinear control system (1)-(4) with uncertain initial data.

2.2 Auxiliary results

Bilinear dynamic systems constitute a special class of nonlinear systems representing a variety of important physical processes. A great number of results related to control problems for such systems has been developed over past decades, among them we mention here Boscain *et al.*[3], Brockett[5], Chernousko[7], Filippova and Lisin[11], Matviychuk[22,23], Mazurenko[24], Polyak *et al.*[26]. Reachable sets of bilinear systems in general are not convex, but have special properties (for example, may be star-shaped (Kurzhanski and Filippova[19], Filippova and Lisin[11]) We, however, consider here the guaranteed state estimation problem and use ellipsoidal calculus for the construction of external estimates of reachable sets of such systems.

Consider first the following control system of bilinear type which is simpler than the system (1) because it does not contain quadratic terms

$$\dot{x} = A(t)\, x + u(t), \quad t_0 \leq t \leq T, \quad x_0 \in \mathcal{X}_0 = E(a_0, Q_0), \tag{6}$$

where $x, a_0 \in R^n$, with a matrix Q_0 being symmetric and positive definite. We will assume that

$$u(t) \in \mathcal{U} = E(\hat{a}, \hat{Q}). \tag{7}$$

The bilinearity of the system (6) is due to the fact that the measurable matrix function $A(t) \in R^{n \times n}$ is not known but satisfies the constraint

$$A(t) \in \mathcal{A}, \quad t_0 \leq t \leq T, \tag{8}$$

where

$$\mathcal{A} = \{A \in R^{n \times n} : A = \operatorname{diag} a, \quad a \in A_0\}, \tag{9}$$

$$A_0 = \{\ a \in \mathbb{R}^n : \quad \sum_{i=1}^{n} |a_i|^2 \leq 1\ \}. \tag{10}$$

Assumption 1 *We will assume further that* $0 \in \mathcal{X}_0$ *and* $0 \in \mathcal{U}$*.*

With this assumption the reachable sets $X(t)$ of the system (6) are compact and star-shaped [11] and the following equality is true for Minkowski (gauge) functional [8,11],

$$h_M(z) = \inf\{t > 0 : z \in tM, x \in \mathbb{R}^n\},$$

namely the next theorem is true.

Theorem 1 ([22]) *The following external estimate is true*

$$\mathcal{X}(t_0 + \sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)) + o(\sigma)B(0,1), \quad \lim_{\sigma \to +0} \sigma^{-1} o(\sigma) = 0, \tag{11}$$

where

$$\begin{gathered} a^+(\sigma) = a_0 + \sigma\hat{a}, \quad Q^+(\sigma) = (p^{-1} + 1)Q_1(\sigma) + (p+1)\sigma^2\hat{Q}, \\ Q_1(\sigma) = \operatorname{diag}\ (p^{-1} + 1)\sigma^2 a_{0i}^2 + (p+1)r^2(\sigma) \mid i = 1, \ldots, n\,, \\ r(\sigma) = \max_z ||z|| \cdot (h_{(I+\sigma\mathcal{A})\mathcal{X}_0}(z))^{-1}, \end{gathered} \tag{12}$$

and p *is the unique positive root of the equation* $\displaystyle\sum_{i=1}^{n} \frac{1}{p + \alpha_i} = \frac{n}{p(p+1)}$ *with* $\alpha_i \geq 0$ $(i = 1, ..., n)$ *being the roots of the following equation* $|Q_1(\sigma) - \alpha\sigma^2\hat{Q}| = 0$.

Note that the result presented in Theorem 1 is more convenient for computational implementations than the estimates given in Chernousko[7] and Filippova and Matviychuk[16] for systems with separate constraints on elements of unknown matrix which defines bilinear terms in the system dynamics.

In this paper we extend and modify this approach to the case when the impulsive control system contains nonlinearities defined by a quadratic form in the right-hand sides of differential equations and also nonlinearities defined by uncertainty in coefficients of linear terms of the systems with quadratic constraints.

3 Main result

Consider the general case of the system dynamics (1)-(4) and here we take $\mathcal{X}_0 = E(a_0, Q_0) = E(a_0, k^2 B^{-1})$ with $k > 0$ (the same result will be valid also if we assume instead that $\mathcal{X}_0 = E(a_0, Q_0) \subseteq E(a_0, k^2 B^{-1})$ with some $k > 0$, see also related discussions in Filippova *et al.*[14], Filippova and Matviychuk[15]), therefore we consider the impulsive system

$$
\begin{gathered}
dx(t) = \big(A(t)x(t) + x'Bx \cdot d + u(t)\big)dt + Cdv(t),\\
A(t) = A^0 + A^1(t), \quad A^1(t) \in \mathcal{A}^1,\\
x(t_0 - 0) = x_0 \in \mathcal{X}_0 = E(a_0, k^2 B^{-1}),
\end{gathered}
\tag{13}
$$

with constraints

$$
A(t) \in \mathcal{A} = A^0 + \mathcal{A}^1, \quad t_0 \le t \le T, \tag{14}
$$

$$
\begin{gathered}
\mathcal{A}^1 = \big\{ \ A{=}\{a_{ij}\}{\in}R^{n\times n} : a_{ij} = 0 \ \text{ for } \ i \ne j, \ \text{ and}\\
a_{ii} = a_i, \quad i = 1, \ldots, n, \quad a = (a_1, \ldots, a_n), \quad a'Da \le 1 \ \big\},
\end{gathered}
\tag{15}
$$

where $D \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix.

Let us use the approach of Rishel[27] and introduce a new time variable:

$$
\eta(t) = t + \int_{t_0}^{t} dv(s),
$$

and a new state coordinate

$$
\tau(\eta) = \inf\{t \mid \eta(t) \ge \eta\}.
$$

Consider the following inclusion

$$
\begin{gathered}
\frac{d}{d\eta} \begin{pmatrix} z \\ \tau \end{pmatrix} \in H(\tau, z),\\
z(t_0) = x_0 \in \mathcal{X}_0, \quad \tau(t_0) = t_0, \quad t_0 \le \eta \le T + \mu,\\
H(\tau, z) = \bigcup_{0\le\nu\le1} \left\{ \nu \begin{pmatrix} C \\ 0 \end{pmatrix} + (1-\nu) \begin{pmatrix} Az + z'Bz \cdot d + E(\hat{a}, \hat{Q}) \\ 1 \end{pmatrix} \right\}.
\end{gathered}
\tag{16}
$$

Denote as $w = \{z, \tau\}$ the extended state vector of the system (16) and the reachable set of the system (16) as $\mathcal{W}(\eta) = \mathcal{W}(\eta; t_0, w_0, \mathcal{A}, \mathcal{X}_0 \times \{t_0\})$ $(t_0 \le \eta \le T + \mu)$.

Theorem 2 *The following inclusion holds true for $\sigma > 0$*

$$\mathcal{W}(t_0 + \sigma) \subseteq \bigcup_{0 \le \nu \le 1} \mathcal{W}(t_0, \sigma, \nu) + o(\sigma)B(0,1), \quad \lim_{\sigma \to +0} \sigma^{-1} o(\sigma) = 0, \tag{17}$$

where

$$\begin{gathered} \mathcal{W}(t_0, \sigma, \nu) = \begin{pmatrix} E\big(a^*(\sigma,\nu), Q^*(\sigma,\nu)\big) \\ t_0 + \sigma(1-\nu) \end{pmatrix}, \\ a^*(\sigma,\nu) = a_0 + \sigma((1-\nu)(a_0{}' B a_0 \cdot d + k^2 d + \hat{a}) + \nu C), \\ Q^*(\sigma,\nu) = (p^{-1}+1)\tilde{Q}(\sigma,\nu) + (p+1)\sigma^2 \hat{Q}^*_\nu, \end{gathered} \tag{18}$$

with $E(\hat{a}_\nu, \hat{Q}^*_\nu)$ being the ellipsoid with minimal volume such that

$$\nu C + (1-\nu)E(\hat{a}, \hat{Q}) + 2(1-\nu)d \cdot a_0' B \cdot E(0, k^2 B^{-1}) \subseteq E(\hat{a}_\nu, \hat{Q}^*_\nu),$$

$$\hat{a}_\nu = \nu C + (1-\nu)\hat{a},$$

and with function

$$\tilde{Q}(\sigma,\nu) = \mathrm{diag}\{\, (p^{-1}+1)\sigma^2 a_{0i}^2 + (p+1) r^2(\sigma) \mid i = 1, \ldots, n \,\}, \tag{19}$$

$$r(\sigma) = \max_z ||z|| \cdot (h_{(I+\sigma\mathcal{A}) * \mathcal{X}_0}(z))^{-1}, \tag{20}$$

Here $p = p(\sigma,\nu)$ is the unique positive root of the equation

$$\sum_{i=1}^n \frac{1}{p + \lambda_i} = \frac{n}{p(p+1)}$$

and $\lambda_i = \lambda_i(\sigma,\nu) \ge 0$ $(i = 1, \ldots, n)$ satisfy the equation $|\tilde{Q}(\sigma,\nu) - \lambda\sigma^2 \hat{Q}^*_\nu| = 0$.

Proof. The above generalization is based on a combination of the techniques described in the previous section and the results of Filippova and Matviychuk[14,15].

The following estimate may be easily derived from the above theorem.

Corollary 1 *The reachable set $\mathcal{X}(T)$ is the projection of $\mathcal{W}(T+\mu)$ at the subspace of variable z: $\mathcal{X}(T) = \pi_z \mathcal{W}(T+\mu)$.*

Remark 1. To determinate simpler estimate of the reachable set $\mathcal{W}(t_0+\sigma)$ we introduce small parameter $\varepsilon > 0$ and embed the degenerate ellipsoid $\mathcal{W}(t_0,\sigma,\nu)$ in nondegenerate ellipsoid $E_\varepsilon\big(w(t_0,\sigma,\nu), O_\varepsilon(t_0,\sigma,\nu)\big)$:

$$\mathcal{W}(t_0,\sigma,\nu) \subseteq E_\varepsilon\big(w(t_0,\sigma,\nu), O_\varepsilon(t_0,\sigma,\nu)\big),$$

$$w(t_0,\sigma,\nu) = \begin{pmatrix} a^*(\sigma,\nu) \\ t_0 + \sigma(1-\nu) \end{pmatrix}, \quad O_\varepsilon(t_0,\sigma,\nu) = \begin{pmatrix} Q^*(\sigma,\nu) & 0 \\ 0 & \varepsilon^2 \end{pmatrix}.$$

Thus, for all small $\varepsilon > 0$ we get

$$\mathcal{W}(t_0,\sigma) \subset \mathcal{W}_\varepsilon(t_0,\sigma) = \bigcup_{0\le\nu\le 1} E_\varepsilon\big(w(t_0,\sigma,\nu), O_\varepsilon(t_0,\sigma,\nu)\big)$$
$$\subset E_\varepsilon(w^+(\sigma), O^+(\sigma)), \quad \lim_{\varepsilon\to+0} h(\mathcal{W}(t_0,\sigma), \mathcal{W}_\varepsilon(t_0,\sigma)) = 0.$$

The passage to the family of nondegenerate ellipsoids enables one to use the algorithms developed in Filippova and Matviychuk[12] and to construct the external estimate $E_\varepsilon(w^+(\sigma), O^+(\sigma))$ of the *union of ellipsoids* $\mathcal{W}_\varepsilon(t_0,\sigma)$. Therefore applying the above mention procedure we will get the ellipsoidal estimate of the reachable set $\mathcal{W}(t_0+\sigma)$, namely we will have the inclusion

$$\mathcal{W}(t_0+\sigma) \subset E_\varepsilon(w^+(\sigma), O^+(\sigma)) + o(\sigma)B(0,1).$$

The following algorithm is based on Theorem 2 and may be used to produce the external ellipsoidal estimates for the reachable sets of the system (1).

Algorithm. Subdivide the time segment $[t_0, T+\mu]$ into subsegments $[t_i, t_{i+1}]$, where $t_i = t_0 + ih$ $(i = 1,\ldots,m)$, $h = (T+\mu-t_0)/m$, $t_m = T+\mu$. Subdivide the segment $[0,1]$ into subsegments $[\nu_j, \nu_{j+1}]$, where $\nu_i = ih_*$, $h_* = 1/m$, $\nu_0 = 0$, $\nu_m = 1$.

1. Repeated steps begin with Step 1:
   - Take $\sigma = h$ and for given $\mathcal{X}_0 = E(a_0, k^2B^{-1})$ define by Theorem 4 the sets $\mathcal{W}(t_0,\sigma,\nu_i)$ $(i = 0,\ldots,m)$.
   - Find ellipsoid $E_\varepsilon(w_1(\sigma), O_1(\sigma))$ in $\mathbb{R}^{n+1}$ such that
   $$\mathcal{W}(t_0,\sigma,\nu_i) \subseteq E_\varepsilon(w_1(\sigma), O_1(\sigma)) \quad (i = 0,\ldots,m).$$
   At this step we find the ellipsoidal estimate for the union of a finite family of ellipsoids [12].
   - Find the projection $E(a_1, Q_1) = \pi_z E_\varepsilon(w_1(\sigma), O_1(\sigma))$ by Lemma 2.
   - Find the smallest $k_1 > 0$ such that $E(a_1, Q_1) \subseteq E(a_1, k_1^2B^{-1})$ (k_1^2 is the maximal eigenvalue of the matrix $B^{1/2}Q_1B^{1/2}$).
   - Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_1, k_1^2B^{-1})$ as the initial ellipsoid at instant t_1.
2. The next step repeats the previous iteration beginning with new initial data.

At the end of the process we will get the external estimate $E(a^+(T), Q^+(T))$ of the reachable set of the system (1).

The algorithm presented above uses the the special structure of nonlinearity of studied control system and is based on techniques of ellipsoidal calculus. However we need to underline here that one of the subsequent steps of the Algorithm contains the projection operator on the subspace of state variables which is a significant complication of the whole estimation procedure, but it is caused by the presence of the impulsive components in the control functions.

Remark 2. State estimation approaches presented in this paper use the special structure of nonlinearity and uncertainty of the control system. The results presented here use and develop previous investigations of ellipsoidal estimation procedures for uncertain systems.

4 Conclusion

The paper deals with the problems of state estimation for uncertain dynamical control systems for which we assume that the initial state is unknown but bounded with given constraints and the matrix in the linear part of state velocities is also unknown but bounded. We study here the case when the system nonlinearity is generated by the combination of two different quadratic functions in related differential equations.

The modified estimation techniques were elaborated here to efficiently describe the reachable sets of control systems with studied nonlinear dynamics and under uncertainty in initial states, including some classes of nonlinear control systems of impulsive type. We present here procedures and numerical schemes for the construction of external estimates of reachable sets of the systems under consideration.

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