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Qualitative results for a mixture of Green-Lindsay thermoelastic solids

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Abstract. We study qualitative properties of the solutions of the system of partial differential equations modeling thermomechanical deformations for mixtures of thermoelastic solids when the theory of Green and Lindsay for the heat conduction is considered. Three dissipation mechanisms are proposed in the system: thermal dissipation, viscosity effects on one constituent of the mixture and damping in the relative velocity of the two displacements of both constituents. First, we prove the existence and uniqueness of the solutions. Later we prove the exponential stability of the solutions over the time. We use the semigroup arguments to establish our results.
Keywords: Thermoelastic mixtures, existence, uniqueness, exponential decay, Green-Lindsay heat conduction.

1 Introduction

Thermoelastic mixtures of solids have been an important issue of study for mathematicians and engineers in the last decades (see, e.g., [5–8,12,13,33,34]). Interesting applications of thermoelastic mixtures can be found in several branches of engineering. On the other side, the systems of partial differential equations that arise when different classes of materials are studied drive to challenging problems for mathematicians. In particular, a lot of effort has been done to find qualitative properties of the solutions of these systems. Results concerning existence, uniqueness, continuous dependence and asymptotic stability can be found in the literature [1–3,20,25–27,31,32]. In these contributions the Fourier law is used to describe the heat propagation. However, it is known that the classical Fourier theory gives rise to several paradoxes. Perhaps the

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most known is the infinite velocity of propagation, which is not compatible with the causality principle. Since the decade of the 1960's, other thermomechanic theories that allow heat to propagate as a wave with finite speed have been stated to overcome the aforementioned paradox. These new theories are mainly based on the heat conduction model of Cattaneo and Maxwell [28] (for a deeper knowledge about these theories see Hetnarski and Ignaczack [17,18], the books by Ignaczack and Ostoja-Starzewski [22], Straughan [35] and the references cited therein). In 1972, Green and Lindsay [11] presented another thermoelasticity theory by adding restrictions on the constitutive equations; in fact, they used an entropy production inequality proposed by Green and Laws [10]. In the last decade of the twentieth century, Green and Naghdi [14–16] also proposed a set of three theories which have been deeply studied lately.

In this paper we want to study several qualitative properties of the solutions for the system of partial differential equations that arise for thermoviscoelastic mixtures in the three dimensional case when the heat conduction is modeled using the theory of Green and Lindsay. To be precise, we will analyze the theory proposed by Iesan and Scalia [21] following the works of Green and Lindsay. We want to point out that, recently, the asymptotic behavior of the solutions for the mixtures problem when the Lord-Shulman theory [24] is considered has been studied by Alves *et al.* [4].

The structure of the paper is the following. First of all we recall the field equations, impose the initial and boundary conditions and set the assumptions over the constitutive coefficients. In Section 3 we prove the existence and uniqueness of the solutions using semigroup arguments. In Section 4 we analyze the time behavior of the solutions and we prove their exponential stability following the arguments proposed by Liu and Zheng in his book [23].

2 The system of equations and the basic assumptions

We consider a mixture of two continua and suppose that, at a fixed time, the body occupies a bounded and regular region B of the three-dimensional Euclidean space with boundary smooth enough to apply the divergence theorem.

To write the equations we will use the standard notation conventions: a colon followed by an index i means the partial derivative with respect to the space variable x_i and a dot over the function means the derivative with respect to the time. Summation over repeated indices is assumed.

The field equations for isotropic and homogeneous bodies with a center of symmetry are given by the system (see [21], p.238)

$$\begin{aligned}
&A_1\Delta u_i + A_2 u_{j,ji} + B_1\Delta w_i + B_2 w_{j,ji} - \xi(u_i - w_i) - m(T_{,i} + \alpha\dot{T}_{,i}) \\
&-\xi^*(\dot{u}_i - \dot{w}_i) - b^* T_{,i} + \mu^*\Delta\dot{u}_i + (\lambda^* + \mu^*)\dot{u}_{j,ji} + \tau^*\dot{T}_{,i} = \rho_1\ddot{u}_i \\
&B_1\Delta u_i + B_2 u_{j,ji} + C_1\Delta w_i + C_2 w_{j,ji} + \xi(u_i - w_i) - \beta(T_{,i} + \alpha\dot{T}_{,i}) \\
&+\xi^*(\dot{u}_i - \dot{w}_i) + b^* T_{,i} = \rho_2\ddot{w}_i \\
&k\Delta T - T_0(d\dot{T} + h\ddot{T} + m\dot{u}_{i,i} + \beta\dot{w}_{i,i}) = 0.
\end{aligned} \tag{1}$$

Here $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are the displacements of each constituent, T is the temperature, A_1, A_2, B_1, B_2, C_1, C_2, ξ, m, ξ^*, b^*, μ^*, λ^*,

β, k, d, h, τ^*, ρ_1, ρ_2, α and T_0 are the constitutive coefficients. T_0 is usually the temperature at the reference configuration and, from now on and without loose of generality, we will assume that $T_0 = 1$. Coefficient α is a relaxation parameter in the temperature which is typical for the Green-Lindsay theory. We use Δ to denote the Laplace operator.

We consider three different dissipation mechanisms in the system: thermal dissipation, viscosity effects on the first constituent and damping in the relative velocity.

To have a well posed problem we need to impose initial and boundary conditions to the unknowns of system (1). As initial conditions we take

$$\begin{aligned} &u_i(x,0) = u_i^0(x),\ \ \dot{u}_i(x,0) = v_i^0(x) \ \text{ in } B, \\ &w_i(x,0) = w_i^0(x),\ \dot{w}_i(x,0) = z_i^0(x) \text{ in } B, \\ &T(x,0) = T^0(x),\ \ \dot{T}(x,0) = \theta^0(x) \ \text{ in } B, \end{aligned} \tag{2}$$

for some given functions. And as boundary conditions we consider homogeneous Dirichlet conditions:

$$u_i(x,t) = w_i(x,t) = T(x,t) = 0, \text{ in } \partial B. \tag{3}$$

In order to obtain results of existence and uniqueness for the solutions of the problem determined by system (1) with initial conditions (2) and boundary conditions (3) we need some basic assumptions over the coefficients.

First of all, we impose that the internal mechanical energy of the system has to be positive. To this end, we will suppose that the matrices

$$\begin{pmatrix} A_1 & B_1 \\ B_1 & C_1 \end{pmatrix} \text{ and } \begin{pmatrix} A_2 & B_2 \\ B_2 & C_2 \end{pmatrix} \tag{4}$$

are definite positive and that $\xi > 0$.

We will also assume that

$$\rho_1 > 0,\, \rho_2 > 0,\, d\alpha - h > 0,\, h > 0, \tag{5}$$

conditions given by the entropy production law.

Secondly, we want the dissipation to be also positive and, therefore, we impose that the following inequalities hold:

$$\begin{aligned} 3\lambda^* + 2\mu^* \geq 0,\, \mu^* \geq 0,\, 4(d\alpha - h)(3\lambda^* + 2\mu^*) - \tau^* \geq 0, \\ k > 0, \text{ and } 4k\xi^* - (b^*)^2 \geq 0. \end{aligned} \tag{6}$$

3 Existence and uniqueness of solutions

The aim of this section is to prove the existence and the uniqueness of solutions for problem (1)–(3). It is worth noting that for the problem arising when the Fourier heat conduction law is used, existence and uniqueness of the solutions have been already proved [31]. The arguments we use here are similar to the ones used there. Nevertheless, we think suitable to sketch at least the more important features.

For the system coefficients we assume the conditions proposed at (4), (5) and (6).

With the usual notation, we introduce the Hilbert spaces (see [9]) $L^2(B)$, $H_0^1(B)$, $H^2(B)$ and $H^{-1}(B)$ acting on a bounded domain B. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the L^2-inner product and the L^2-norm, respectively.

In the case of Dirichlet thermal boundary conditions, let us consider the Hilbert space

$$\mathcal{H} = \mathbf{H}_0^1 \times \mathbf{H}_0^1 \times H_0^1 \times \mathbf{L}^2 \times \mathbf{L}^2 \times L^2.$$

We will denote the elements of $\mathcal{H}$ by $U = (\mathbf{u}, \mathbf{w}, T, \mathbf{v}, \mathbf{z}, \theta)$, with $\mathbf{v} = \dot{\mathbf{u}}$, $\mathbf{z} = \dot{\mathbf{w}}$ and $\theta = \dot{T}$.

We define an inner product in $\mathcal{H}$ by

$$\langle(\mathbf{u}, \mathbf{w}, T, \mathbf{v}, \mathbf{z}, \theta), (\widetilde{\mathbf{u}}, \widetilde{\mathbf{w}}, \widetilde{T}, \widetilde{\mathbf{v}}, \widetilde{\mathbf{z}}, \widetilde{\theta})\rangle_{\mathcal{H}} = \frac{1}{2}\int_B \mathit{\Pi}\, dV, \tag{7}$$

where

$$\begin{aligned}\mathit{\Pi} = A_1 u_{i,j}\overline{\widetilde{u}_{i,j}} + A_2 u_{i,i}\overline{\widetilde{u}_{j,j}} + B_1(u_{i,j}\overline{\widetilde{w}_{i,j}} + w_{i,j}\overline{\widetilde{u}_{i,j}}) + B_2(u_{i,i}\overline{\widetilde{w}_{j,j}} + w_{i,i}\overline{\widetilde{u}_{j,j}}) +\\ C_1 w_{i,j}\overline{\widetilde{w}_{i,j}} + C_2 w_{i,i}\overline{\widetilde{w}_{j,j}} + \rho_1 v_i\overline{\widetilde{v}_i} + \rho_2 z_i\overline{\widetilde{z}_i} + \xi(u_i - w_i)(\overline{\widetilde{u}_i - \widetilde{w}_i}) +\\ \frac{h}{\alpha}(T + \alpha\theta)(\overline{\widetilde{T} + \alpha\widetilde{\theta}}) + (d - \frac{h}{\alpha})T\overline{\widetilde{T}} + \alpha k T_{,i}\overline{\widetilde{T}_{,i}}.\end{aligned} \tag{8}$$

Here, and from now on, a bar over a variable denotes its complex conjugate.

The corresponding norm is given by

$$\|(\mathbf{u}, \mathbf{w}, T, \mathbf{v}, \mathbf{z}, \theta)\|_{\mathcal{H}}^2 = \frac{1}{2}\int_B \mathit{\Pi}^* dV, \tag{9}$$

where

$$\begin{aligned}\mathit{\Pi}^* = A_1 u_{i,j}\overline{u_{i,j}} + A_2 u_{i,i}\overline{u_{j,j}} + B_1(u_{i,j}\overline{w_{i,j}} + w_{i,j}\overline{u_{i,j}}) + B_2(u_{i,i}\overline{w_{j,j}} + w_{i,i}\overline{u_{j,j}}) +\\ C_1 w_{i,j}\overline{w_{i,j}} + C_2 w_{i,i}\overline{w_{j,j}} + \rho_1 v_i\overline{v_i} + \rho_2 z_i\overline{z_i} + \xi(u_i - w_i)(\overline{u_i - w_i}) +\\ \frac{h}{\alpha}(T + \alpha\theta)(\overline{T + \alpha\theta}) + (d - \frac{h}{\alpha})T\overline{T} + \alpha k T_{,i}\overline{T_{,i}}.\end{aligned}$$

In view of the assumptions on the constitutive coefficients we conclude that there exists a positive constant c such that the inequality

$$\|(\mathbf{u}, \mathbf{w}, T, \mathbf{v}, \mathbf{z}, \theta)\|_{\mathcal{H}}^2 \geq c\left(\|\nabla\mathbf{u}\|^2 + \|\nabla\mathbf{w}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{z}\|^2 + \|\theta\|^2 + \|\nabla T\|^2\right),$$

is satisfied. For the sake of simplicity, here and in what follows we will employ the same symbol c for different constants, even in the same formula. Notice that the inner product proposed here is equivalent to the usual one in the Hilbert space $\mathcal{H}$

We will rewrite system (1) in matricial terms and, afterwards, we will use the technique of contractive semigroups to prove the existence and uniqueness of the solutions.

In order to obtain a written synthetic expression for our problem, we introduce the following operators:

$$\mathbf{A}_1\mathbf{u} = \frac{1}{\rho_1}(A_1 u_{i,jj} + A_2 u_{j,ji} - \xi u_i), \qquad \mathbf{A}_2\mathbf{w} = \frac{1}{\rho_1}(B_1 w_{i,jj} + B_2 w_{j,ji} + \xi w_i),$$
$$\mathbf{A}_3\mathbf{v} = \frac{1}{\rho_1}(\mu^* v_{i,jj} + (\lambda^* + \mu^*) v_{j,ji} - \xi^* v_i), \quad \mathbf{A}_4\mathbf{z} = \frac{1}{\rho_1}(\xi^* z_i),$$
$$\mathbf{A}_5 T = \frac{1}{\rho_1}(-mT_{,i} - b^* T_{,i}), \qquad \mathbf{A}_6\theta = -\frac{1}{\rho_1}((m\alpha - \tau^*)\theta_{,i}),$$
$$\mathbf{A}_7\mathbf{u} = \frac{1}{\rho_2}(B_1 u_{i,jj} + B_2 u_{j,ji} + \xi u_i), \qquad \mathbf{A}_8\mathbf{w} = \frac{1}{\rho_2}(C_1 w_{i,jj} + C_2 w_{j,ji} - \xi w_i),$$
$$\mathbf{A}_9\mathbf{v} = \frac{1}{\rho_2}(\xi^* v_i), \qquad \mathbf{A}_{10}\mathbf{z} = -\frac{1}{\rho_2}(\xi^* z_i),$$
$$\mathbf{A}_{11} T = \frac{1}{\rho_2}(-\beta T_{,i} + b^* T_{,i}), \qquad \mathbf{A}_{12}\theta = -\frac{1}{\rho_2}(\beta\alpha\theta_{,i}),$$
$$A_{13}\mathbf{v} = \frac{1}{h}(-m v_{i,i}), \qquad A_{14}\mathbf{z} = -\frac{1}{h}(\beta z_{i,i}),$$
$$A_{15} T = \frac{1}{h}(k \triangle T), \qquad A_{16}\theta = -\frac{d}{h}\theta.$$

Therefore, system (1) with initial conditions (2) and boundary conditions (3) can be written as

$$\frac{d}{dt}U(t) = \mathcal{A}U(t), \quad U(0) = U_0, \tag{10}$$

where $U_0 = \left(\mathbf{u}^0, \mathbf{w}^0, T^0, \mathbf{v}^0, \mathbf{z}^0, \theta^0\right)$, and $\mathcal{A}$ is the following matrix operator

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & \mathcal{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_5 & \mathbf{A}_3 & \mathbf{A}_4 & \mathbf{A}_6 \\ \mathbf{A}_7 & \mathbf{A}_8 & \mathbf{A}_{11} & \mathbf{A}_9 & \mathbf{A}_{10} & \mathbf{A}_{12} \\ 0 & 0 & A_{15} & A_{13} & A_{14} & A_{16} \end{pmatrix}. \tag{11}$$

The domain of the operator $\mathcal{A}$ is $D(\mathcal{A}) = \{U \in \mathcal{H} : \quad \mathcal{A}U \in \mathcal{H}\}$. It is clear that it contains a dense subspace of $\mathcal{H}$ and, therefore, $D(\mathcal{A})$ is dense in the Hilbert space $\mathcal{H}$.

Lemma 1. *The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions, denoted by $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$.*

Proof. We will show that $\mathcal{A}$ is a dissipative operator and that 0 belongs to the resolvent of $\mathcal{A}$ (in short, $0 \in \rho(\mathcal{A})$). Then our conclusion will follow by using the Lumer-Phillips theorem (see, e.g., [29]).

If $U \in D(\mathcal{A})$ then, direct calculations give

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\frac{1}{2}\int_B D^+ dV, \tag{12}$$

where

$$D^+ = \mu^* v_{i,j}\overline{v_{i,j}} + (\lambda^* + \mu^*) v_{i,i}\overline{v_{j,j}} + \xi^*(v_i - z_i)(\overline{v_i} - \overline{z_i}) + \tau^* \,\mathrm{Re}\,\theta \overline{v}_{i,i}$$

$$+b^* \operatorname{Re} T_{,i}(\overline{v_i} - \overline{z_i}) + (d\alpha - h)\theta\overline{\theta} + kT_{,i}\overline{T_{,i}}.$$

In view of the conditions imposed over the constitutive coefficients we conclude that

$$\operatorname{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq 0,$$

and, therefore, the operator $\mathcal{A}$ is dissipative.

Now, we show that $0 \in \rho(\mathcal{A})$. Given $F = (\mathbf{f}, \mathbf{g}, h, \mathbf{p}, \mathbf{q}, r) \in \mathcal{H}$, we must show that there exists a unique $U = (\mathbf{u}, \mathbf{w}, T, \mathbf{v}, \mathbf{z}, \theta)$ in $D(\mathcal{A})$ such that $\mathcal{A}U = F$, that is

$$\begin{aligned}
&\mathbf{v} = \mathbf{f} && \text{in } \mathbf{H}_0^1,\\
&\mathbf{z} = \mathbf{g} && \text{in } \mathbf{H}_0^1,\\
&\theta = h && \text{in } H_0^1,\\
&\mathbf{A_1u} + \mathbf{A_2w} + \mathbf{A_5}T + \mathbf{A_3v} + \mathbf{A_4z} + \mathbf{A_6}\theta = \mathbf{p} && \text{in } \mathbf{L}^2,\\
&\mathbf{A_7u} + \mathbf{A_8w} + \mathbf{A_{11}}T + \mathbf{A_9v} + \mathbf{A_{10}z} + \mathbf{A_{12}}\theta = \mathbf{q} && \text{in } \mathbf{L}^2,\\
&A_{15}T + A_{13}\mathbf{v} + A_{14}\mathbf{z} + A_{16}\theta = r && \text{in } L^2.
\end{aligned} \tag{13}$$

Using $(13)_1$-$(13)_3$ in $(13)_6$ we have

$$A_{15}T = r - (A_{13}\mathbf{f} + A_{14}\mathbf{g} + A_{16}h). \tag{14}$$

Thus, the existence of a unique $T \in H^2$ satisfying (14) is clear. Now, from $(13)_4$ and $(13)_5$, we obtain the following system of equations with unknowns $\mathbf{u}$ and $\mathbf{w}$.

$$\begin{cases} \mathbf{A_1u} + \mathbf{A_2w} = \mathbf{p} - \mathbf{A}_5T - \mathbf{A}_3\mathbf{f} - \mathbf{A}_4\mathbf{g} - \mathbf{A}_6h & \text{in } \mathbf{H}^{-1,2},\\ \mathbf{A_7u} + \mathbf{A_8w} = \mathbf{q} - \mathbf{A}_{11}T - \mathbf{A}_9\mathbf{f} - \mathbf{A}_{10}\mathbf{g} - \mathbf{A}_{12}h & \text{in } \mathbf{H}^{-1,2}. \end{cases} \tag{15}$$

Taking into account the conditions assumed for the material constants, the sesquilinear form $\mathbf{B} : H_0^1 \times H_0^1 \to \mathbb{C}$ given by

$$\mathbf{B}[(\mathbf{u}, \mathbf{w}), (\widetilde{\mathbf{u}}, \widetilde{\mathbf{w}})] = \langle \mathbf{A_1u} + \mathbf{A_2w}, \overline{\widetilde{\mathbf{u}}}\rangle + \langle \mathbf{A_7u} + \mathbf{A_8w}, \overline{\widetilde{\mathbf{w}}}\rangle$$

is continuous and coercive. As the right hand side of (15) is in the dual, using the Lax-Milgram theorem (see, e.g., [9]), it follows that there exists a unique vector $(\mathbf{u}, \mathbf{w})$ satisfying system (15).

Therefore, there exists also a unique vector U satisfying (13). It is easy to show that $\|U\|_{\mathcal{H}} \leq c\|F\|_{\mathcal{H}}$, for a positive constant c. Hence, we conclude that $0 \in \rho(\mathcal{A})$.

As a consequence of the above established results, we can state the following theorem.

Theorem 1. *The operator given by matrix $\mathcal{A}$ generates a contraction semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$, and for $U_0 \in D(\mathcal{A})$ there exists a unique solution $U(t) \in \mathcal{C}^1([0,\infty), \mathcal{H}) \cap \mathcal{C}^0([0,\infty), D(\mathcal{A}))$ of system (1) with initial conditions (2) and boundary conditions (3).*

Remark 1. The existence and uniqueness result can be extended to the same system but with supply terms. In fact, in this case we can obtain continuous dependence on the initial conditions and the supply terms. That means that, under the conditions proposed for the constitutive constants, the problem is well posed in the sense of Hadamard.

4 Exponential stability

In this section we analyze the asymptotic behavior of the solutions with respect to the time variable. We consider thermal dissipation, viscosity effects on the first constituent of the mixture and damping effects on the relative velocity of the two displacements of both constituents. We prove that, with the above considerations, the solutions are exponentially stable.

To enforce the dissipation mechanisms act we have to assume that

$$3\lambda^*+2\mu^* > 0,\ \mu^* > 0,\ 4(d\alpha-h)(3\lambda^*+2\mu^*)-\tau^* > 0,\ k > 0 \text{ and } 4k\xi^*-(b^*)^2 > 0. \tag{16}$$

Besides the assumptions for the coefficients given by (16), we also assume that

$$\begin{aligned}&\int_B (B_1 w_{i,j}w_{i,j} + B_2 w_{i,i}w_{j,j})\,dV \geq C\int_B w_{i,j}w_{i,j}\,dV \text{ or}\\&\int_B (B_1 w_{i,j}w_{i,j} + B_2 w_{i,i}w_{j,j})\,dV \leq -C\int_B w_{i,j}w_{i,j}\,dV.\end{aligned} \tag{17}$$

It is useful to recall the following known result (see [19], [23], [30]):

Theorem 2. *Let $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if the following two conditions are satisfied:*

(i) $i\mathbb{R} \subset \rho(\mathcal{A})$,
(ii) $\overline{\lim}_{|\lambda|\to\infty} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Lemma 2. *The operator $\mathcal{A}$ defined at (11) satisfies that $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. Following the arguments given by Liu and Zheng [23], the proof consists of the following three steps:

(i) Since 0 is in the resolvent of $\mathcal{A}$, using the contraction mapping theorem, we have that for any real λ such that $|\lambda| < ||\mathcal{A}^{-1}||^{-1}$, the operator $i\lambda\mathcal{I} - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - \mathcal{I})$ is invertible. Moreover, $||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1}, ||\mathcal{A}^{-1}||^{-1})$.

(ii) If $\sup\{||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||, |\lambda| < ||\mathcal{A}^{-1}||^{-1}\} = M < \infty$, then by the contraction theorem, the operator

$$i\lambda\mathcal{I} - \mathcal{A} = (i\lambda_0\mathcal{I} - \mathcal{A})\Big(\mathcal{I} + i(\lambda-\lambda_0)(i\lambda_0\mathcal{I} - \mathcal{A})^{-1}\Big),$$

is invertible for $|\lambda - \lambda_0| < M^{-1}$. It turns out that, by choosing λ_0 as close to $||\mathcal{A}^{-1}||^{-1}$ as we can, the set $\{\lambda, |\lambda| < ||\mathcal{A}^{-1}||^{-1} + M^{-1}\}$ is contained in the resolvent of $\mathcal{A}$ and $||(i\lambda\mathcal{I}-\mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1}-M^{-1}, ||\mathcal{A}^{-1}||^{-1}+M^{-1})$.

(iii) Let us assume that the intersection of the imaginary axis and the spectrum is not empty, then there exists a real number ϖ with $||\mathcal{A}^{-1}||^{-1} \leq |\varpi| < \infty$ such that the set $\{i\lambda, |\lambda| < |\varpi|\}$ is in the resolvent of $\mathcal{A}$ and $\sup\{||(i\lambda\mathcal{I}-\mathcal{A})^{-1}||, |\lambda| < |\varpi|\} = \infty$. Therefore, there exists a sequence of real numbers λ_n with $\lambda_n \to \varpi$, $|\lambda_n| < |\varpi|$ and a sequence of vectors $U_n = (\mathbf{u}_n, \mathbf{w}_n T_n, \mathbf{v}_n, \mathbf{z}_n, \theta_n)$ in the domain of the operator $\mathcal{A}$ and with unit norm such that

$$\|(i\lambda_n\mathcal{I}-\mathcal{A})U_n\| \to 0. \tag{18}$$

If we write (18) in components, we obtain the following conditions:

$$i\lambda_n\mathbf{u}_n - \mathbf{v}_n \to \mathbf{0}, \text{ in } \mathbf{H}^1 \tag{19}$$

$$i\lambda_n\mathbf{w}_n - \mathbf{z}_n \to \mathbf{0}, \text{ in } \mathbf{H}^1 \tag{20}$$

$$i\lambda_n T_n - \theta_n \to 0, \text{ in } H^1 \tag{21}$$

$$i\lambda_n\mathbf{v}_n - \mathbf{A}_1\mathbf{u}_n - \mathbf{A}_2\mathbf{w}_n - \mathbf{A}_3\mathbf{v}_n - \mathbf{A}_5 T_n - \mathbf{A}_6\theta_n \to \mathbf{0}, \text{ in } \mathbf{L}^2 \tag{22}$$

$$i\lambda_n\mathbf{z}_n - \mathbf{A}_7\mathbf{u}_n - \mathbf{A}_8\mathbf{w}_n - \mathbf{A}_{11} T_n - \mathbf{A}_{12}\theta_n \to \mathbf{0}, \text{ in } \mathbf{L}^2 \tag{23}$$

$$i\lambda_n\theta_n - A_{15}T_n - A_{13}\mathbf{v}_n - A_{14}\mathbf{z}_n - A_{16}\theta_n \to 0, \text{ in } L^2. \tag{24}$$

In view of the dissipative term for the operator, we see that

$$\theta_n, \nabla\mathbf{v}_n, \mathbf{v}_n - \mathbf{z}_n, \nabla T_n \to 0. \tag{25}$$

From (19) we also have that $\nabla\mathbf{u}_n \to 0$.

If we multiply (22) by $\mathbf{w}_n$ we obtain that

$$\int_B (B_1 w_{i,j}w_{i,j} + B_2 w_{i,i}w_{j,j})\, dV \to 0,$$

and, from the assumptions (17), we conclude that $\nabla\mathbf{w}_n$ tends to zero. Therefore, $\mathbf{z}_n$ also tends to zero and, in consequence, we have seen that the imaginary axis in contained in the resolvent of $\mathcal{A}$.

Lemma 3. *The operator $\mathcal{A}$ defined at (11) satisfies that*

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof. Let us assume the existence of $\lambda_n \to \infty$ and a sequence of unit norm vectors U_n such that the relations (19) – (24) hold. Again, our aim is to prove that U_n tends to zero. We proceed as in the previous case. In fact, we obtain again that θ_n, $\nabla\mathbf{v}_n$, $\mathbf{v}_n - \mathbf{z}_n$ and $\nabla T_n \to 0$. And, hence $\nabla\mathbf{u} \to 0$ because of (19). Now, from (22), we see that

$$\lambda_n^{-1}(\mathbf{A}_1\mathbf{u}_n + \mathbf{A}_2\mathbf{w}_n + \mathbf{A}_3\mathbf{v}_n + \mathbf{A}_6\theta_n) \to 0.$$

From (20) we see that $\lambda_n \mathbf{w}_n$ is bounded. Then, we multiply the above expression by $\lambda_n \mathbf{w}_n$ and, using an argument similar to the one proposed in the proof of Lemma 2, we obtain that $\nabla \mathbf{w}_n$ tends to zero.

Finally, multiplying (23) by $\mathbf{w}_n$ we obtain

$$\langle i\lambda_n \mathbf{z}_n, \frac{\mathbf{z}_n}{i\lambda_n}\rangle - \langle \mathbf{A}_7 \mathbf{u}_n, \mathbf{w}_n\rangle - \langle \mathbf{A}_8 \mathbf{w}_n, \mathbf{w}_n\rangle - \langle \mathbf{A}_{11} T_n, \mathbf{w}_n\rangle - \langle \mathbf{A}_{12} \theta_n, \mathbf{w}_n\rangle \to 0.$$

Applying the integration by parts, we see that $\mathbf{z}_n$ also tends to zero. So, the lemma is proved.

Theorem 3. *The C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t \geq 0}$ is exponentially stable.*

The proof is a direct consequence of Lemma 2, Lemma 3 and Theorem 2.

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